Rainbow and Orthogonal Paths in Factorizations of K_n

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Abstract: **For n even, a factorization of a complete graph K_n is a partition of the edges into $n-1$ perfect matchings, called the factors of the factorization. With respect to a factorization, a path is called *rainbow* if its edges are from distinct factors. A rainbow Hamiltonian path takes exactly one edge from each factor and is called *orthogonal* to the factorization. It is known that not all factorizations have orthogonal paths. Assisted by a simple edge-switching algorithm, here we show that for $n \geq 8$, the rotational factorization of K_n, GK_n has orthogonal paths. We prove that this algorithm finds a rainbow path with at least $(2n+1)/3$ vertices in *any factorization* of K_n (in fact, in any proper coloring of K_n). We also give some problems and conjectures about the properties of the algorithm.** © 2010 Wiley Periodicals, Inc. J Combin Designs 18: 167–176, 2010

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1. INTRODUCTION

For n even, a *factorization* of a complete graph K_n is a partition of the edges into $n-1$ perfect matchings, called the *factors* of the factorization (see [4]—VII/5, [9, 12]). With respect to a factorization, a path is called a *rainbow path* or, simply, R-path if its edges are from distinct factors. A rainbow Hamilton path is called *orthogonal to the factorization* because it has exactly one edge from each factor of the factorization.

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There exist factorizations with many orthogonal Hamilton paths. Horton [6] proved that if $n/2$ is odd and not divisible by 3 and 5, then there exists a factorization of K_n and a decomposition of K_n into $n/2$ Hamilton paths such that *every path* of the decomposition is orthogonal to the factorization (a tournament design).

On the other hand, consider the following factorization, called *geometric factorization* in [12], defined for $n=2^k$. The vertices of K_n are the binary sequences of length k and the factors are defined by the vertex pairs with the same binary sum in bitwise addition. It follows from a simple counting argument ([8, 3, 1]) that this factorization has no orthogonal Hamilton paths at all. Nevertheless, it is widely believed, in fact conjectured in [3, 1], that this is a "worst case", i.e., for *every* factorization of K_n there is a rainbow path with $n-1$ vertices. However, it seems that presently finding a rainbow path even with $n-o(n)$ vertices is out of reach, in contrast to the situation concerning Ryser's conjecture.

Ryser [11] conjectured that every Latin square of order n has a partial transversal of length n for odd n (and length $n-1$ for even n). Now the existence of partial transversals of length $n-o(n)$ is known, here we just cite two important papers [13, 5] from the history of the advances. It would follow from Ryser's conjecture (when n is even) that in every factorization of K_n there is a "partial rainbow 2-factor", a rainbow subgraph that is the union of vertex-disjoint cycles covering all but one vertex of K_n. However, it might happen that *all cycles are short in the rainbow* 2*-factor above*. Therefore, Ryser's conjecture does not imply the existence of long rainbow paths in factorizations. It is worth noting that the proof method of [13] is used in [2] to show that there is a Hamilton path in every factorization of K_n that has edges from $n-o(n)$ distinct factors.

Using an edge-switching technique of Pósa [10] that proved to be useful in extremal graph theory, it is shown in [2] that every factorization of K_n has a rainbow cycle of order at least $(\frac{4}{7}-o(1))n$ (improving an earlier result $n/2$ in [1]). With a simple edge-switching algorithm, here we show that there is a slightly longer rainbow path, namely one with at least $(2n+1)/3$ vertices (Theorem 3) in every factorization of K_n.

In the light of the above remarks it makes sense to determine the length of the longest rainbow path in special factorizations. We find that it is not obvious even for the most well-known factorization either, the rotational factorization of K_n. Although this factorization obviously contains many rainbow paths (and cycles) on $n-1$ vertices, a counting argument, similar to the geometric factorization mentioned above, may prevent the existence of an orthogonal Hamiltonian path. In fact, there *is* a counting argument, see Lemma 1, that prevents the existence of orthogonal Hamilton decompositions mentioned in the first paragraph. However, our main results, Theorems 1, 2 show that there exist orthogonal Hamilton paths for $n \geq 8$ in the rotational factorization of K_n. In some cases (for $n=4k+2$) the patterns of the orthogonal Hamiltonian paths we found have been suggested by a computer program applying the edge-switching algorithm. It is a challenge to find more aesthetic, and/or simpler, constructions. We also present some problems, conjectures and statistics about the properties of the algorithm.

2. HAMILTON PATHS ORTHOGONAL TO THE ROTATIONAL FACTORIZATION

The *Rotational factorization*, GK_n, is the following well-known factorization of K_n (arithmetic is modulo $n-1$). The vertex set is $V=\{0,1,\ldots,n-2\}\cup\{\infty\}$ and for

$m=0,1,\ldots,n-2$, factor F_m has the edge set $\{(m-x,m+x):x\in[1,(n-1)/2]\}\cup(\infty,m)$. We say that an edge (x,y) has color $c(x,y)=m$ if it belongs to factor F_m. The definition implies that $c(x,y)$, the color of edge (x,y), can be computed by $x+y=2c(x,y)$ if $\infty\notin\{x,y\}$, otherwise, $c(x,y)=min\{x,y\}$.

Notice that there is an R-path (in fact an R-cycle) with $n-1$ vertices in the standard factorization: $0,1,2,\ldots,n-2$. However, this R-path (in fact any R-path through all vertices of $\{0,1,\ldots,n-2\}$) cannot be extended to an orthogonal path. This follows from the following lemma.

Lemma 1. *No orthogonal path starts at* ∞ *in* GK_n.

Proof. Suppose there is an orthogonal path $\infty,x_1,\ldots,x_{n-1}$, then $x_i+x_{i+1}=2c(x_i,x_{i+1})$ for $i=1,\ldots,n-2$. Since $c(\infty,x_1)=x_1$, adding up these equations yields

$$2\sum_{i=1}^{n-1}x_i-(x_1+x_{n-1})=2\sum_{i=2}^{n-1}x_i$$

leading to $x_1=x_{n-1}$, which is a contradiction. □

Since no orthogonal path can start at vertex ∞, from symmetry, we may suppose that the orthogonal path we are looking for enters ∞ from 0. It is probably not necessary, but we shall always use the edge $(\infty,n/2)$ to continue it. The solution we provide for $n=4k$ in Theorem 1 has a relatively easy pattern and was discovered without computers. We shall mostly use two types of path in the constructions, one is called *increasing* path, or simply i-path, the other one is a *jumping* or simply, j-path. The i-path takes consecutive vertices $a,a+1,\ldots$ of $[0,n-2]$ and the j-path takes vertices $a,a+3,a+2,\ldots$ of $[0,n-2]$ i.e. uses the sequence $+3,-1,+3,-1,\ldots,-1$ to define the next element. These paths are useful because along i- and j-paths colors increase one by one. We shall also use reverse i- and j-paths by reversing the order on the i- and j-paths, respectively.

Theorem 1. *The following path* (*see Fig.* 1) *is an orthogonal path from* 1 *to* $2k-1$ *in* GK_n *for* $n=4k, k\geq 2$. *Take an i-path* $1,2,\ldots,2k-3,2k-2$; *continue with* $2k-2,2k,\infty,0,4k-3$; *finish with the reverse j-path* $4k-3,4k-2,4k-5,4k-4,\ldots,2k+32k+4,2k+12k+2,2k-1$.

Proof. The i-path defines colors increasing one by one from $2k+1$ to $4k-3$ because

$$1+2=4k+2=2(2k+1),\quad 2k-3+2k-2=2(4k-3).$$

The reverse j-path defines colors decreasing one by one from $2k-2$ to 1 because

$$4k-3+4k-2=2(2k-2),\quad 2k+2+2k-1=2\times 1.$$

Path $2k-2,2k,\infty,0,4k-3$ gives the four missing colors $2k-1,2k,0,4k-2$. □

For $n=4k+2$, we have a bit more complicated orthogonal paths, in fact, we have used a computer program to find solutions in the following way. The program starts with an R-path, Q_n on $n-1$ vertices (see below), and repeatedly applies exchanges until it ends with a path orthogonal to GK_n. Although the program has always ended with a solution so far, we could not prove so, as stated in Conjecture 1 later. We used

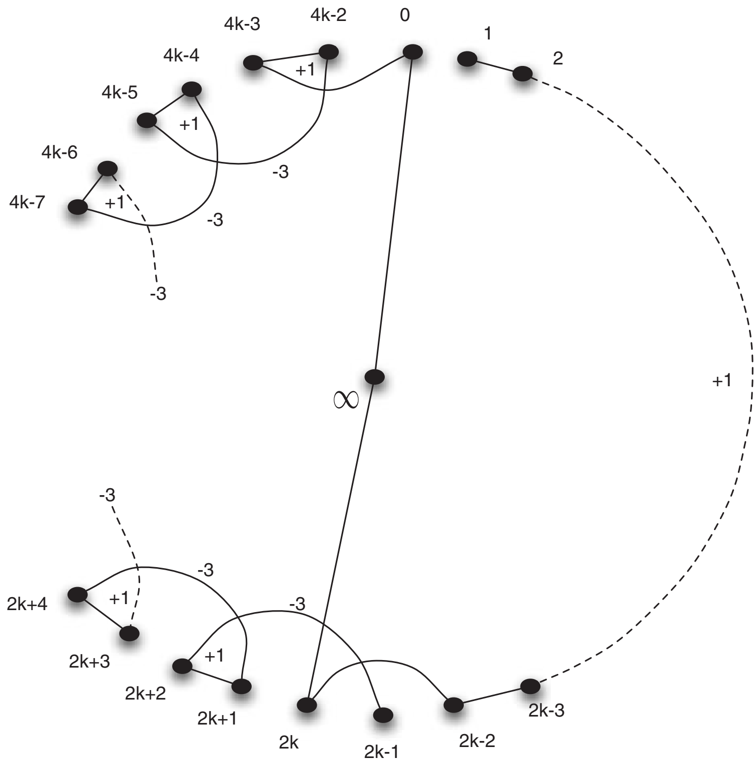

FIGURE 1. An orthogonal path for $n=4k$.

the program to get solutions for $10 \le n \le 46$ and tried to guess how to generalize their patterns for general $n=4k+2$. We ended up with several possibilities, but not with an outstandingly aesthetic one.

Theorem 2. *There exist orthogonal paths in GK_n for $n=4k+2, k \ge 2$.*

Proof.
Case 1: $n=8k+2$. Start with i-path $6k+2, 6k+3, \ldots, 8k, 0$; continue with $0, \infty, 4k+1, 1$; take i-path $1, 2, \ldots, 4k-1, 4k$; finish with j-path $4k, 4k+3, 4k+2, \ldots, 6k+1, 6k$ (see Fig. 2). The i-paths have colors $2k+2, \ldots, 4k-1, 4k$ and $4k+2, \ldots, 8k$ and the j-path gives the color sequence $1, 2, \ldots, 2k$. Path $0, \infty, 4k+1, 1$ brings in the missing colors $0, 4k+1, 2k+1$.

Case 2: $n=8k+6$. Start with j-path $Q=6k+6, 6k+5, 6k+8, \ldots, 8k+4, 8k+3, 1$; then take i-path $R=1, 2, \ldots, 4k+1, 4k+2$. Paths Q, R have colors $2k+3, 2k+4, \ldots, 4k+1, 4k+2$ and $4k+4, 4k+5, \ldots, 8k+4$. The last part of the path, P, varies slightly according to the parity of k:

Subcase 2.1. *k is odd*. Here, $P=P_1 \cup P_2$, where $P_1=4k+2, 4k+5, 4k+6, 4k+3, \infty, 0, 4k+4$, then we repeatedly jump with jumping sequence $+5, +1, -3, +1$ to get $P_2=4k+4, 4k+9, 4k+10, 4k+7, 4k+8, \ldots, 6k-2, 6k+3, 6k+4, 6k+1, 6k+2$ (for $k=1$, we do not take P_2), see Figure 3. Path P_1 has colors $1, 3, 2, 4k+3, 0, 2k+2$, then

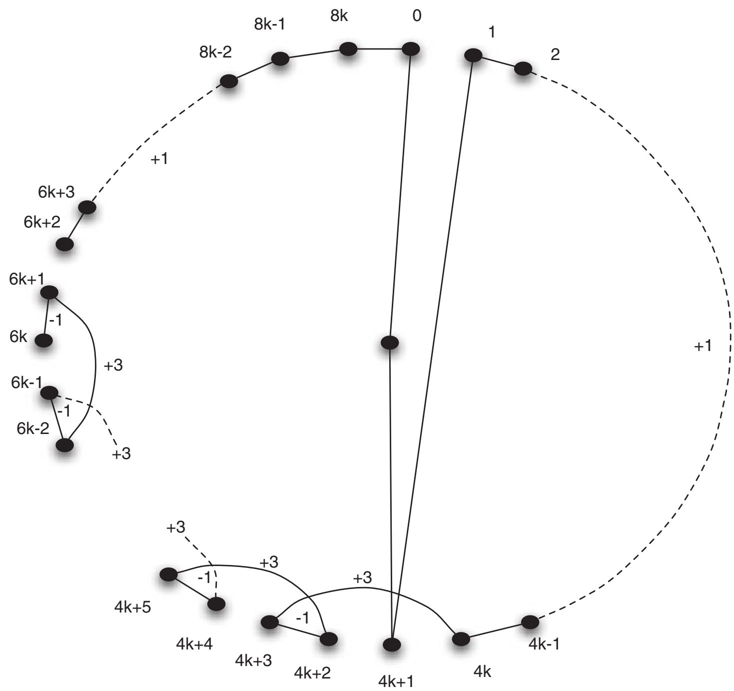

FIGURE 2. An orthogonal path for $n = 8k+2$.

P_2 brings in colors $4, 7, 6, 5, \ldots, 2k+1, 2k, 2k-1$, together with the colors of Q, R, we have everything in $[0, n-2] = [0, 8k+4]$.

Subcase 2.2. *k is even*. Here, $P = P_1 \cup P_2$, where $P_1 = 4k+2, 4k+5, 4k+8, 4k+7, 4k+4, 0, \infty, 4k+3, 4k+6$, then we repeatedly jump with jumping sequence $+5, +1, -3, +1$ to get $P_2 = 4k+6, 4k+11, 4k+12, 4k+9, 4k+10, \ldots, 6k-2, 6k+3, 6k+4, 6k+1, 6k+2$ (for $k = 2$ we do not take P_2), see Figure 4. Path P_1 has colors $1, 4, 5, 3, 2k+2, 0, 4k+3, 2$, then P_2 brings in colors $6, 9, 8, 7, \ldots, 2k+1, 2k, 2k-1$, together with the colors of Q, R, we have everything in $[0, n-2] = [0, 8k+4]$. □

3. AN EDGE SWITCHING PROCEDURE GENERATING RAINBOW PATHS

Next, we describe a procedure we used for generating paths orthogonal to GK_n to guess patterns that generalize if n is in certain residue classes. It repeatedly "adds color" to R-paths and resembles to the idea used by Pósa ([10], see also in [7] 10.20) to find long paths in graphs with a certain expansion property. Similar methods were used in [13, 2] to find long R-paths in factorizations.

Procedure "add color"

Assume we have an R-path $P_i = x, \ldots, y$ and a color c is missing from P_i. We define an R-path $P_{i+1} = y, \ldots, z$ that will use color c as follows. Let p be the vertex for which

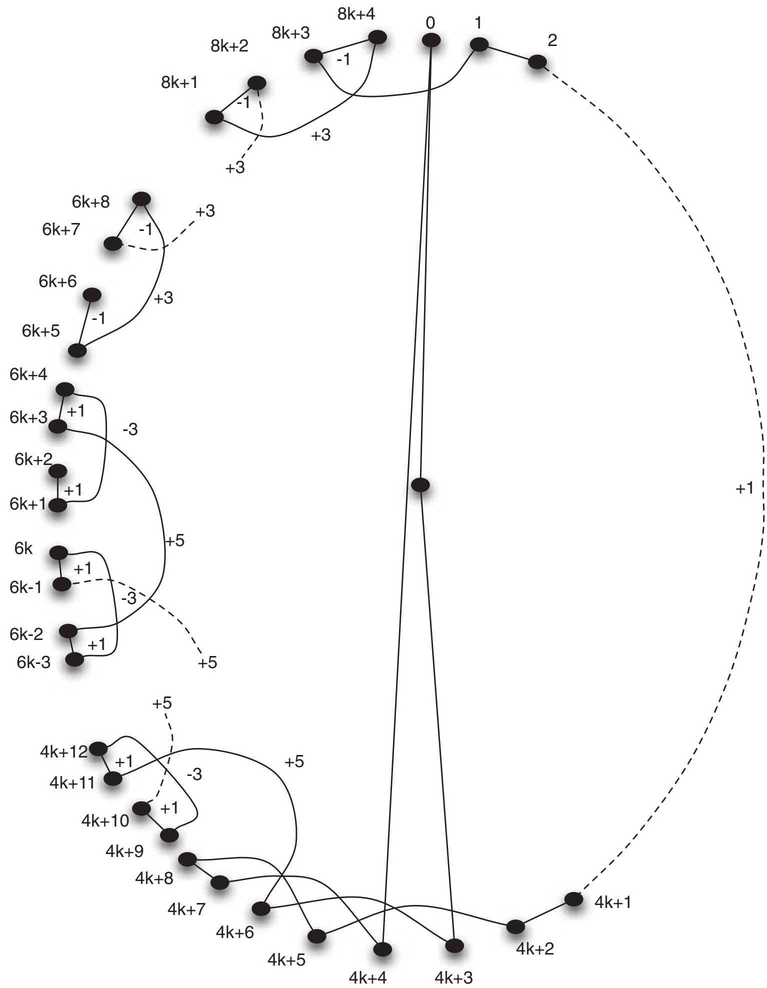

FIGURE 3. An orthogonal path for $n = 8k+6$ with k odd.

xp has color c.

- *Step 1*. If $p \notin V(P_i)$, then set $z = p$, $P_{i+1} = y, \ldots, x, z$. (Note that we get a longer R-path here.)
- *Step 2*. If $p \in V(P_i)$, then P_{i+1} is defined by starting at y, following P_i until p, then jumping to x and following P_i again and stopping at the predecessor z of p. Briefly, one may say that $P_{i+1} = y, \ldots, z$ is obtained by an "edge-switching": deleting the edge zp and adding the edge xp (see Fig. 5). (Note that we do not increase the length of the R-path here.)

One can make an algorithm from this procedure by selecting an R-path and a factorization as the input, then iterating the procedure (if adding color used an edge-switching, then a natural choice of new missing color is the color removed from the path). One may notice that the *add color* procedure induces an orientation of the path, as it selects

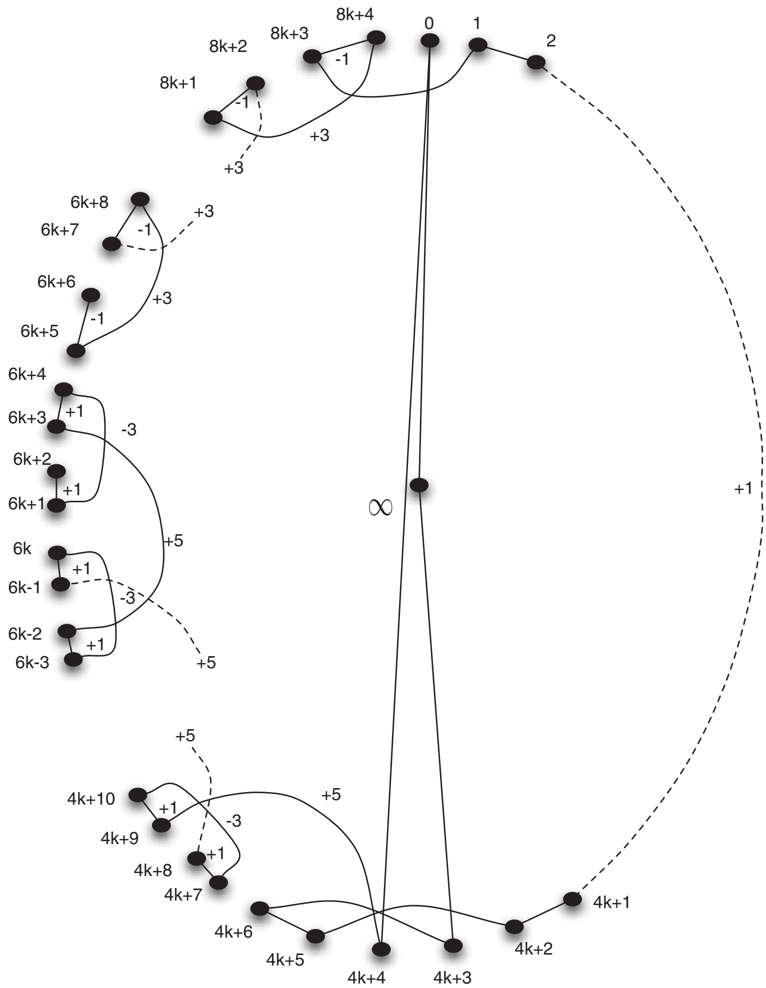

FIGURE 4. An orthogonal path for $n = 8k+6$ with k even.

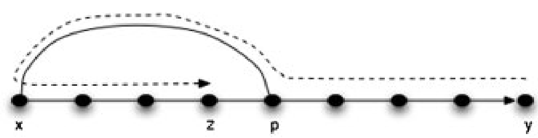

FIGURE 5. Edge-switching step.

the endpoint x in both Step 1 and Step 2. In fact it adds a missing color using the first vertex of a path. Similarly, one can define an *add color* procedure using the end vertex of a path, though it is easy to see that *edge-switching* using the last vertex is the reverse of *edge-switching* using the first vertex: indeed, starting from a path $P_i = x \ldots y$ and a missing color $c(x, p)$, the standard *edge-switching* produces path $P_{i+1} = y, \ldots, z$ with a

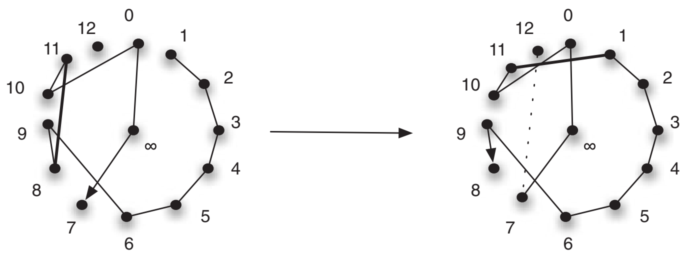

FIGURE 6. Edge switching for $n=14$.

missing color $c(z, p)$. Then *edge-switching* using the endpoint z and the missing color $c(z, p)$ produces back the original path P_i.

Note that it is possible that the algorithm never ends because Step 2 can be repeated endlessly.

Mostly we tried the "add color" procedure with factorization GK_n, $n \geq 8$, and with an "almost orthogonal" input R-path, Q_n with $n-1$ vertices, defined as follows: Q_n starts with i-path $1, 2, \ldots, n/2-1$, then continues with j-path $n/2-1, n/2+2, n/2+1, \ldots, n-6, n-3, n-4, 0$, and finally, $0, \infty, n/2$. We used Step 2 with alternating endpoints until Step 1 is reached, i.e. the extension to an orthogonal path is found.

For $n=14$ (see Fig. 6), for example, we have $x=1, y=7, z=12$,

$$P_1 = 1, 2, 3, 4, 5, 6, 9, 8, 11, 10, 0, \infty, 7$$

with missing color $m=6$. Then, $p=11$ from $1+p=2\times 6$ and

$$P_2 = 7, \infty, 0, 10, 11, 1, 2, 3, 4, 5, 6, 9, 8 = x.$$

The missing color is $c(8, 11)=3$ and $p=12$ because $7+p=2\times 3$. Since $z=p$, the procedure stops, $12 \cup P_2$ is the orthogonal path.

For $n=18$, we have $x=1, y=9, z=16$,

$$P_1 = 1, 2, 3, 4, 5, 6, 7, 8, 11, 10, 13, 12, 15, 14, 0, \infty, 9; \quad m=8, p=15,$$

$$P_2 = 9, \infty, 0, 14, 15, 1, 2, 3, 4, 5, 6, 7, 8, 11, 10, 13, 12; \quad m=5, p=1,$$

$$P_3 = 12, 13, 10, 11, 8, 7, 6, 5, 4, 3, 2, 1, 9, \infty, 0, 14, 15; \quad m=8, p=4,$$

$$P_4 = 15, 14, 0, \infty, 9, 1, 2, 3, 4, 12, 13, 10, 11, 8, 7, 6, 5; \quad m=13, p=11,$$

$$P_5 = 5, 6, 7, 8, 11, 15, 14, 0, \infty, 9, 1, 2, 3, 4, 12, 13, 10; \quad m=2, p=16,$$

and the procedure stops with orthogonal path $16 \cup P_5$.

Conjecture 1. *The procedure described above produces an orthogonal path for every* $n \geq 8$ *if* $P_1 = Q_n$.

The following table represents the number of edge switches done before finding an increasing path starting from Q_n. The part "one-direction" gives the data where the algorithm runs as described, and in part "both directions" we try both endpoints run in parallel. Column *maxtrn* is the maximal number of edge switches, column *avgtrn* is the average number of edge switches and column *dir1* is the percentage of the cases where adding the missing color at the first vertex found an orthogonal path faster.

In all the cases the procedure produced an orthogonal path, though the orbits of the almost orthogonal paths by edge-switchings (number of almost orthogonal paths reachable by edge-switchings) seem to increase quickly. We also noticed that using both directions increases the speed of the algorithm significantly.

	Both directions			One direction	
Interval	dir1 (%)	maxtrn	avgtrn	maxtrn	avgtrn
10–498	46,34	1278	174	2822	415
10–998	45,16	5281	404	384,720	4336
10–1498	44,5	10,502	610	29,482,774	257,673
10–1998	44,58	11,724	841	2,110,203,018	3,300,528
10–9998	44,27	187,575	5373		

We conclude the paper by questions and a result about the effectiveness of the "greedy" edge-switching procedure we used. What is the longest R-path in the standard factorization starting from an arbitrary edge in the switching procedure? In particular, is it possible that the procedure should stop with a non-extendible R-path with less than $n-1$ vertices? One may ask the same question about the procedure if the input is an *arbitrary* factorization of K_n. The next result gives a lower bound valid also for proper colorings of K_n, i.e. for colorings where incident edges are colored with different colors.

Theorem 3. *In every proper coloring of K_n, the edge-switching procedure finds an R-path with at least $(2n+1)/3$ vertices.*

Proof. Suppose that $P=x_1,x_2,\dots,x_k$ is an R-path non-extendible by edge-switching, colored with color set C, $|C|=k-1$. Let A denote the set of vertices not covered by P. Since P is not extendible from x_k, only the colors of C can be present on the edge set $E=\{x_k y | y\in A\}$. Applying the same argument to x_1 and using that there are $n-1$ distinct colors on the edge set incident to x_1, we get that at least $(n-1)-|C|=n-k$ edges go from x_1 to a subset B of $V(P)$, each colored with a color not in C. Let B^- denote the set of predecessors of the vertices in B along P. Observe that the color of $x_i x_{i+1}$ cannot be used on any $x_k y\in E$, for any $x_i\in B^-$; otherwise, the path

$$y x_k x_{k-1} \dots x_{i+1} x_1 x_2 \dots x_i$$

obtained by edge-switching would give a longer R-path. We conclude that $n-k=|A|\le |C|-(n-k)=(k-1)-(n-k)$, giving the desired result. □

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