Monochromatic Hamiltonian 3-Tight Berge Cycles in 2-Colored 4-Uniform Hypergraphs

András Gyárfás,[1] Gábor N. Sárközy,[1,2] and Endre Szemerédi[3,4]

[1] *COMPUTER AND AUTOMATION RESEARCH INSTITUTE HUNGARIAN ACADEMY OF SCIENCES P.O. BOX 63, BUDAPEST, H-1518, HUNGARY*
E-mail: gyarfas@sztaki.hu

[2] *COMPUTER SCIENCE DEPARTMENT WORCESTER POLYTECHNIC INSTITUTE WORCESTER, MASSACHUSETTS 01609*
E-mail: gsarkozy@cs.wpi.edu

[3] *COMPUTER SCIENCE DEPARTMENT, RUTGERS UNIVERSITY NEW BRUNSWICK, NEW JERSEY 08903*

[4] *INSTITUTE FOR ADVANCED STUDY PRINCETON, NEW JERSEY 08540*
E-mail: szemered@cs.rutgers.edu

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Abstract: Here improving on our earlier results, we prove that there exists an n_0 such that for $n \geq n_0$ in every 2-coloring of the edges of $K_n^{(4)}$ there is a monochromatic Hamiltonian 3-tight Berge cycle. This proves the $c=2$, $t=3$, $r=4$ special case of a conjecture from (P. Dorbec, S. Gravier, and G. N. Sárközy, J Graph Theory 59 (2008), 34–44). J Graph Theory 63: 288–299, 2010

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1. INTRODUCTION

$V(G)$ and $E(G)$ denote the vertex set and the edge set of the graph G. (A,B,E) denotes a bipartite graph $G=(V,E)$, where $V=A+B$ and $E \subset A \times B$. For a graph G and a subset U of its vertices, $G|_U$ is the restriction of G to U. $N(v)$ is the set of neighbors of $v \in V$. Hence the size of $N(v)$ is $|N(v)|=deg(v)=deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum and $\Delta(G)$ for the maximum degree in G. When A,B are subsets of $V(G)$, we denote by $e(A,B)$ the number of edges of G with one endpoint in A and the other in B. In particular, we write $deg(v,U)=e(\{v\},U)$ for the number of edges from v to U. A graph G_n on n vertices is γ-*dense* if it has at least $\gamma\binom{n}{2}$ edges. A bipartite graph $G(k,l)$ is γ-dense if it contains at least γkl edges.

Let $\mathcal{H}$ be an r-uniform hypergraph (a family of some r-element subsets of a set). The *shadow graph* of $\mathcal{H}$ is defined as the graph $\Gamma(\mathcal{H})$ on the same vertex set, where two vertices are adjacent if they are covered by at least one edge of $\mathcal{H}$. A coloring of the edges of an r-uniform hypergraph $\mathcal{H}$, $r \geq 2$, induces a multicoloring on the edges of the shadow graph $\Gamma(\mathcal{H})$ in a natural way; every edge e of $\Gamma(\mathcal{H})$ receives the color of all hyperedges containing e. We shall denote by $c(x,y)$ the color set of the edge xy in $\Gamma(\mathcal{H})$. A subgraph of $\Gamma(\mathcal{H})$ is *monochromatic* if the color sets of its edges have a nonempty intersection. Let $K_n^{(r)}$ denote the complete r-uniform hypergraph on n vertices.

In any r-uniform hypergraph $\mathcal{H}$ for $2 \leq t \leq r$ we define an r-uniform t-tight Berge-cycle of length ℓ, denoted by $C_\ell^{(r,t)}$, as a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$, such that for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of t consecutive vertices on the cycle, there is an edge e_i of $\mathcal{H}$ that contains these t vertices and the edges e_i are all distinct for i, $1 \leq i \leq \ell$ where addition is done modulo ℓ. This notion was introduced in [5] and for $t=2$ we get ordinary Berge-cycles [1] and for $t=r$ we get the tight cycle (see e.g. [14] or [19]). A Berge-cycle of length n in a hypergraph of n vertices is called a Hamiltonian Berge-cycle. It is important to keep in mind that, in contrast to the case $r=t=2$, for $r>t \geq 2$ a Berge-cycle $C_\ell^{(r,t)}$ is not determined uniquely; it is considered as an arbitrary choice from many possible cycles with the same triple of parameters.

In this paper, continuing investigations from [5, 8, 10, 11] and [12], we study long Berge-cycles in hypergraphs. In [5] (by generalizing an earlier conjecture from [8]) the following conjecture was formulated.

Conjecture 1.1. *For any fixed $2 \le c,t \le r$ satisfying $c+t \le r+1$ and sufficiently large n, if we color the edges of $K_n^{(r)}$ with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle.*

In [5] it was proved that if the conjecture is true it is best possible, since for any values of $2 \le c,t \le r$ satisfying $c+t > r+1$ the statement is not true. The conjecture can easily be proved for $c=t=2$ and $r=3$, see [8]. The asymptotic form of the conjecture was proved for $c=3$, $t=2$ and $r=4$ in [8] and for every r and $c=r-1$, $t=2$ in [11]—in both papers the Regularity Lemma [20] was used. In this paper we prove the conjecture in a *sharp* form for the first non-trivial special case: $c=2$, $t=3$ and $r=4$ and thus providing more evidence to the truth of the conjecture in general.

Theorem 1.2. *There exists an n_0 such that for $n \ge n_0$, in every 2-coloring of the edges of $K_n^{(4)}$ there is a monochromatic Hamiltonian 3-tight Berge-cycle.*

This improves a result of [12] where under the same assumptions we could only find a monochromatic 3-tight Berge-cycle of length at least $n-10$. It also improves a result from [5] where we did manage to find a Hamiltonian monochromatic 3-tight Berge-cycle but only in 2-colorings of the edges of the complete 5-uniform hypergraph. In the proof we combine the proof method of the weaker statement from [12] with stability arguments discussed in the next section.

2. A STABILITY VERSION OF THE GERENCSÉR–GYÁRFÁS THEOREM

For graphs $G_1, G_2, \ldots, G_r$, the Ramsey number $R(G_1, G_2, \ldots, G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes giving r graphs $H_1, H_2, \ldots, H_r$, then at least one H_i $(1 \le i \le r)$ has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's classical result [18]. The number $R(G_1, G_2, \ldots, G_r)$ is called the Ramsey number for the graphs $G_1, G_2, \ldots, G_r$. There is very little known about $R(G_1, G_2, \ldots, G_r)$ even for very special graphs (see e.g. [7] or [17]). For $r=2$ a theorem of Gerencsér and Gyárfás [6] states that

$$R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor.$$

To get the extremal example let $V(G) = A \cup B$ where $|A| = n-1$, $|B| = \lfloor (n-2)/2 \rfloor$, all the edges inside A are of one color (say red) and all the edges between A and B are of the other color (blue). (Note that we have no restriction on the coloring inside the smaller set.)

In the proof of Theorem 1.2 we will use a stability version of the Gerencsér–Gyárfás Theorem that we proved recently in [13]. For this purpose we need to define a relaxed version of the above extremal coloring. We work with 2-edge *multicolorings* (G_1, G_2) of a graph G. Here multicoloring means that the edges can receive more than one color, i.e. the graphs G_i are not necessarily edge disjoint. The subgraph colored with color i *only* is denoted by G_i^*, i.e.

$$G_1^* = G_1 \setminus G_2, \quad G_2^* = G_2 \setminus G_1.$$

Extremal Coloring 1 (*with parameter* α): There exists a partition $V(G) = A \cup B$ such that

- $|A| \geq (1-\alpha)2|V(G)|/3$, $|B| \geq (1-\alpha)|V(G)|/3$.
- The graph $G_1^*|_A$ is $(1-\alpha)$-dense and the bipartite graph $G_2^*|_{A\times B}$ is $(1-\alpha)$-dense, where say G_1 is red and G_2 is blue. (Note again that we have no restriction on the coloring inside the smaller set.)

Then the following stability version of the Gerencsér–Gyárfás Theorem from [13] claims that we can either find a monochromatic path substantially longer than $2n/3$ or the coloring is close to the extremal coloring.

Lemma 2.1. *For every* $\alpha > 0$, *there exist positive reals* η, c_1 ($0 < \eta \ll \alpha \ll 1$ *where* $\ll$ *means sufficiently smaller*) *and a positive integer* n_0 *such that for every* $n \geq n_0$ *the following holds*: *if the edges of the complete graph* K_n *are* 2*-multicolored then we have one of the following two cases*:

- *Case 1*: *There exists a color* (*say red*) *and* $k \geq (2/3+\eta)n$ *such that* K_n *contains a red path* P *of length* k. *Furthermore*, *in the process of finding* P, *given a red initial subpath of* P, *for the next vertex of the path* P *we always have at least* $c_1 \log n$ *choices. More precisely*, *for each* $i = 1, 2, \ldots, k$, *given a red initial subpath* $p_1p_2\ldots p_{i-1}$, *there are at least* $c_1 \log n$ *choices of* $p_i \in V(K_n)$ *for which* $p_1p_2\ldots p_{i-1}p_i$ *is a red path.*
- *Case 2*: *This is an Extremal Coloring 1* (*EC1*) *with parameter* α.

Surprisingly, as far as we know, this natural question has not been studied, despite the fact that stability versions for some classical density (see [2]) and Ramsey-type results (see [9, 15]) are known.

Lemma 2.1 (and thus Theorem 1.2) can also be proved from the Regularity Lemma; however, in [13] we used a more elementary approach using only the Kővári–Sós–Turán bound [16] on the number of edges ensuring a balanced complete bipartite graph.

3. OUTLINE OF THE PROOF OF THEOREM 1.2

As in Lemma 2.1 we will use the following main parameters:

$$0 < \eta \ll \alpha \ll 1,$$

and we shall assume that n is sufficiently large.

We will follow the same rough outline as in [12]. Indeed, suppose that a 2-coloring c is given on the edges of $\mathcal{K} = K_n^{(4)}$. Let V be the vertex set of $\mathcal{K}$ and observe that c defines a 2-multicoloring on the complete 3-uniform hypergraph $\mathcal{T}$ with vertex set V by coloring a triple T with the colors of the edges of $\mathcal{K}$ containing T. We say that $T \in \mathcal{T}$ is *good in color* i if T is contained in at least two edges of $\mathcal{K}$ of color i ($i = 1, 2$). Let G be the shadow graph of $\mathcal{K}$. The following easy lemma is from [12].

Lemma 3.1. *Every edge $xy \in E(G)$ is in at least $n-4$ good triples of the same color.*

Proof. Consider an edge xy in G and let $W = V \setminus \{x,y\}$. Assume indirectly that the triples $\{x,y,a\}$, $\{x,y,b\}$ and $\{x,y,c\}$ are all bad in color 1 for some $a,b,c \in W$. Thus if the triple $\{x,y,z\}$ is bad in color 2 for some $z \in W$, then $z \notin \{a,b,c\}$ and without loss of generality we may assume that $c(xyza) = c(xyzb) = 1$. If also the triple $\{x,y,w\}$ is bad in color 2 for some $w \in W \setminus \{z\}$, then $w \notin \{a,b,c\}$ and without loss of generality we may assume that $c(xywa) = 1$, contradicting the fact that the triple $\{x,y,a\}$ is bad in color 1. We note that the lemma also follows from a result of Bollobás and Gyárfás [3]. ■

Using Lemma 3.1, we can define a 2-multicoloring c^* on the shadow graph $G = \Gamma(\mathcal{K})$ by coloring $xy \in E(G)$ with the color(s) of the (at least $n-4$) good triples containing xy. We apply the stability version of the Gerencsér–Gyárfás Theorem (Lemma 2.1) for this 2-multicoloring of G. Case 2, i.e. EC1 is handled in Section 6. Assuming that we have the non-extremal case, Case 1, we can find in G a monochromatic path P (say in red) of length $l \geq (2/3+\eta)n$. From now on in the non-extremal case we work in the color red. Label the edges of P by $e_j = \{p_j, p_{j+1}\}$, $j = 1,2,\ldots,l-1$. From Lemma 2.1 it also follows that we can guarantee that all the triples $\{p_j, p_{j+1}, p_{j+2}\}$, $j = 1,2,\ldots,l-2$, are good in red. Indeed, when we select p_{j+2} we select a vertex from the available $c_1 \log n$ choices that forms a good triple with $\{p_j, p_{j+1}\}$. Since only two vertices are forbidden we still have plenty to choose from.

We plan to splice in the remaining vertices in $V(G) \setminus V(P)$ into (most of) the edges $e_{2j} = \{p_{2j}, p_{2j+1}\}$. For this purpose we make sure that if we plan to splice in the vertex $v \in V(G) \setminus V(P)$ into the edge e_{2j}, then all 3 triples $\{p_{2j-1}, p_{2j}, v\}$, $\{p_{2j}, v, p_{2j+1}\}$ and $\{v, p_{2j+1}, p_{2j+2}\}$ are good in red. This will guarantee, through Hall's condition, that we will be able to make this into a 3-tight Berge-cycle later. Note that we need the technical condition for the triples to be *good* in red rather than just red to make sure that we can select through Hall's condition *distinct* 4-edges containing the consecutive triples along the cycle. This is a necessary condition for being a 3-tight Berge-cycle.

However, as in [12], there could be a small (constant) number of exceptional vertices in $V(G) \setminus V(P)$ that simply cannot be spliced in into *any* of the edges e_{2j}. In order to avoid this technical difficulty we do the following. First we build an initial red path P' that has length 26. This determines a small number of exceptional vertices in $V(G) \setminus V(P')$ that cannot be spliced in into P'. For each such exceptional vertex v we make sure artificially that we will be able to splice it in; we will build a *v-absorbing bridge*.

Definition 3.2. We define a *v-absorbing bridge* $\{p_1, p_2, p_3, p_4\}$ (in red) in the following way. The edges $\{p_1, p_2\}$, $\{p_2, p_3\}$ and $\{p_3, p_4\}$ are all red (under c^*) in G and we have one of the following two cases:

- All 3 triples $\{p_1, p_2, v\}$, $\{p_2, v, p_3\}$ and $\{v, p_3, p_4\}$ are good in red (type 1 bridge),
- Otherwise there exists a vertex $w \notin \{v, p_1, p_2, p_3, p_4\}$ such that the 4-edges $\{p_1, p_2, v, w\}$, $\{p_2, v, p_3, w\}$ and $\{v, p_3, p_4, w\}$ are all red edges of $\mathcal{K}$ (type 2 bridge).

Note that in the second case the triples $\{p_1,p_2,v\}$, $\{p_2,v,p_3\}$ and $\{v,p_3,p_4\}$ might not be good in red, as w might be the only vertex that can be added to them. However, this definition will imply that in both cases v can be spliced in into the edge $\{p_2,p_3\}$.

Now we are ready to define our second extremal coloring.

Extremal Coloring 2 (*with parameter* α): We will call the coloring an Extremal Coloring 2 (EC2) if the following statement is *not* true: For *both* colors and for *every* vertex $v\in V(G)$ there are at least $\sqrt{\alpha}n^4$ v-absorbing bridges (with the same w if they are type 2).

EC2 is handled later in Section 5. If the condition in the definition of EC2 is violated for red, then we call it an EC2-red; if the condition is violated for blue, then we call it an EC2-blue (thus potentially we might have a coloring that is both in EC2-red and in EC2-blue).

Assuming that EC2 does not hold we connect P' and these red absorbing bridges for the exceptional vertices into a path P'' that still has a constant length. Then we extend this to a red cycle C' that has length at least $(2/3+\eta)n$ and that contains P'' as a subpath. Now we are able to splice in all the remaining vertices into the cycle C' and thus resulting in a red Hamiltonian 3-tight Berge-cycle.

4. THE NON-EXTREMAL CASE

Assume in this case that we do not have Extremal Colorings 1 or 2. Following the outline above first we build an initial red path P' in G that has length 26. Label the edges of P' by $e_j=\{p_j,p_{j+1}\}$, $j=1,2,\dots,25$. P' determines a small number of exceptional vertices in $V(G)\setminus V(P')$ in the following way. As indicated above for a vertex $v\in V(G)\setminus V(P')$ and for an edge $e_{2j}=\{p_{2j},p_{2j+1}\}$, $j=1,2,\dots,12$, of P' we say that v can be spliced in into e_{2j} if all 3 triples $\{p_{2j-1},p_{2j},v\}$, $\{p_{2j},v,p_{2j+1}\}$ and $\{v,p_{2j+1},p_{2j+2}\}$ are good in red. A vertex $v\in V(G)\setminus V(P')$ is *exceptional* if it can be spliced in into at most 6 edges e_{2j} of P'. We claim that the number of these exceptional vertices in $V(G)\setminus V(P')$ is at most 12. Indeed, for each fixed edge e_{2j} of P', $1\le j\le 12$, there could be only at most 6 vertices of $V(G)\setminus V(P')$ that cannot be spliced in into e_{2j} since for each of the pairs $\{p_{2j-1},p_{2j}\}$, $\{p_{2j},p_{2j+1}\}$ and $\{p_{2j+1},p_{2j+2}\}$ there could be at most 2 exceptional vertices. Then, as usual, we define an auxiliary bipartite graph G_b between the edges e_{2j} and the vertices $v\in V(G)\setminus V(P')$ where we put an edge between e_{2j} and v, if v cannot be spliced in into e_{2j}. By the above G_b has at most $6\cdot 12=72$ edges. Then indeed the number of exceptional vertices is at most 12, since otherwise the number of edges of this bipartite graph would be more than $12\cdot 6=72$, a contradiction. Note that the degree of all non-exceptional vertices of $V(G)\setminus V(P')$ in $\overline{G_b}$ is at least 6, i.e. each non-exceptional vertex can be spliced in into at least 6 edges e_{2j} of P'; a fact that will be important later.

For the at most 12 exceptional vertices we will find vertex disjoint absorbing bridges in red where they will be spliced in. The fact that we are not in EC2 makes this possible. Indeed, we do the following for the exceptional vertices. Denote the exceptional vertices with $v_1,v_2,\dots,v_{12}$ (we may assume that there are exactly 12 such vertices by taking arbitrary vertices from $V(G)\setminus V(P')$.). We find vertex disjoint v_i-absorbing bridges $P_i=\{p_1^i,p_2^i,p_3^i,p_4^i\}$ for $1\le i\le 12$ such that the following are true (to make sure that the paths can be connected and that this new path can be a part of a 3-tight Berge-cycle).

- The triples $\{p_{c_2-1},p_{c_2},p_1^1\}$ and $\{p_{c_2},p_1^1,p_2^1\}$ are good in red. This allows us to connect P' and the bridge P_1.
- The triples $\{p_3^i,p_4^i,p_1^{i+1}\}$ and $\{p_4^i,p_1^{i+1},p_2^{i+1}\}$ are good in red for $1\le i\le 11$. This allows us to connect the bridges P_i and P_{i+1}.
- If P_i is a type 2 bridge with the 4th vertex w_i, then the vertices p_4^{i-1} (or p_{c_2} if $i=1$) and p_1^{i+1} are not equal to w_i.

Indeed, from the fact that we have at least $\sqrt{\alpha}n^4$ v_i-absorbing bridges for each $1\le i\le 12$ (since we are not in EC2) we can find vertex disjoint $\{p_2^i,p_3^i\}$ in such a way that we have at least $\sqrt{\alpha}n/4$ available choices for both p_1^i and p_4^i. Then clearly we can pick p_1^i and p_4^i such that the above properties hold and the resulting v_i-absorbing bridges P_i are vertex disjoint.

Thus indeed we can connect $P',P_1,P_2,\dots,P_{12}$ into one path. Splice in the vertices $v_1,v_2,\dots,v_{12}$ into their bridges between p_2^i and p_3^i. Denote the resulting path by P''. For technical reasons let us "leave open" the endpoints of this path. This P'' has the following properties. Any triple of consecutive three vertices on P'' is good in red if it does not contain any of the vertices $v_i, 1\le i\le 12$, or if it does contain a vertex v_i with a type 1 bridge. For the consecutive triples T that contain a vertex v_i with a type 2 bridge with the 4th vertex w_i, the corresponding 4-edge of $\mathcal{K}$ containing T will be $T\cup\{w_i\}$. The above construction guarantees that there will not be any repetitions of these 4-edges and thus indeed P'' can be a part of a 3-tight Berge-cycle. Note that the length of P'' is still a constant $(26+60=86)$.

Using the fact that P'' has length 85, we can still apply Lemma 2.1 to find in G a red path $Q=\{q_1,q_2,\dots,q_l\}$, $f_i=\{q_i,q_{i+1}\}$, $l\ge(2/3+\eta)n$ that is vertex disjoint from P''. Indeed, we mark the vertices in P'' as forbidden vertices, and by Lemma 2.1 we still have at least $c_1\log n-86\ge c_1\log n/2$ available choices for each vertex of Q (using that n is sufficiently large). Furthermore, as in P', we can also guarantee that any triple of consecutive three vertices on Q is good in red and that we can connect the endpoints of P'' and Q similarly as above. Thus we get a cycle $C'=P''\cup Q$. Consider the bipartite graph $\overline{G_b}$ between the remaining vertices in $V(G)\setminus V(C')$ and the set of edges

$$E=\{e_{2j}|2\le 2j\le 24\}\cup\{f_{2i}|2\le 2i\le l-2\},$$

where we put en edge between a vertex $v\in V(G)\setminus V(C')$ and an edge e_{2j} or f_{2i} if the vertex can be spliced in into the edge.

Claim 1. *There is a perfect matching M in $\overline{G_b}$ from $V(G)\setminus V(C')$.*

Indeed, we have to check Hall's condition, i.e. for every $S\subset V(G)\setminus V(C')$ we need $|N_{\overline{G_b}}(S)|\ge|S|$. For $|S|\le 6$, this is true as

$$|N_{\overline{G_b}}(S)|\ge deg(v)\ge 6\ge|S|,$$

for an arbitrary $v \in S$. However, for $|S| \geq 7$ we have

$$|N_{\overline{G_b}}(S)| = |E| \geq (1/3 + \eta/2)n \geq |S|, \tag{1}$$

as desired (since for each $e \in E$ we can have at most six exceptional vertices that cannot be spliced in into e).

We splice in the vertices of $V(G) \setminus V(C')$ into the edges where they are matched under M. Now we finish the proof of the non-extremal case by claiming that the Hamiltonian cycle C that we get after splicing in the vertices of $V(G) \setminus V(C')$ is indeed a red 3-tight Berge-cycle. Indeed, every triple of three consecutive vertices on C that does not contain a vertex v_i with a type 2 bridge is good in red. For the triples containing a vertex v_i with a type 2 bridge we already found the distinct red 4-edges of $\mathcal{K}$ containing them (by adding the corresponding w_i to the triple). For the other triples, since they are good in red, there are at least two red 4-edges of $\mathcal{K}$ available to cover them. However, no edge of $\mathcal{K}$ can cover more than two of these triples of C. Thus, by Hall's theorem again, there is a matching from these triples of C to the set of red edges of $\mathcal{K}$ containing them, and thus resulting in a red Hamiltonian 3-tight Berge-cycle finishing the proof in the non-extremal case.

5. EXTREMAL COLORING 2

For technical reasons we treat first EC2. In fact, this can be reduced to the non-extremal case. Let us assume that we have an EC2, say an EC2-red. By the definition there must exist a vertex v_r, such that we cannot find at least $\sqrt{\alpha}n^4$ v_r-absorbing bridges in red. In this case we will show that either we can find a Hamiltonian 3-tight Berge-cycle in blue or we can find sufficiently many v_r-absorbing bridges in red after all with a somewhat weaker definition of a bridge, which is just as good.

We will show first that we may assume that the blue and *weak blue* (to be defined later) edges form a $(1-\alpha^{1/10})$-dense subgraph in G. Indeed, if the density of the red edges is at most $\alpha^{1/10}$, then this is immediate. Otherwise, consider the set of red edges and mark those red edges e for which v_r is not among the at most two exceptional vertices, i.e. for which (e, v_r) forms a good triple in red. If the density of the marked red edges in G is still at least $\alpha^{1/10}$, then we could clearly find at least $\sqrt{\alpha}n^4$ paths of length 3 consisting of marked red edges. However, these paths are v_r-absorbing bridges in red, a contradiction with our assumption. Indeed, one may take a subgraph of the marked red edges where the minimum degree is at least half of the original average degree (see e.g. Proposition 1.2.2 in [4]), and then use a greedy procedure and the fact that $\alpha \ll 1$.

Thus we may assume that this is not the case, the density of the marked red edges is less than $\alpha^{1/10}$. Next we will show that we may assume that all unmarked red edges are blue as well in this 2-multicoloring. Let us take an unmarked red edge f. By definition, the triple (f, v_r) is not a good triple in red, so apart from at most one edge all 4-edges of $\mathcal{K}$ containing the triple are blue. In other words f is contained in at least $(n-4)$ blue triples. It seems as this is a slightly weaker condition than being blue in G, as these $(n-4)$ blue triples might not be good in blue. On the other hand, it is always the same vertex (namely v_r) that we have to add to each of these triples to get a blue 4-edge of $\mathcal{K}$, and this property is just as good for building a 3-tight Berge-cycle and that is our ultimate goal. Let us call these edges *weak blue* edges, since they are almost as good as blue edges. Then every unmarked red edge of G is weak blue and thus the density of the blue edges (blue or weak blue) is at least $(1-\alpha^{1/10})$, as claimed.

Thus, certainly in this case in blue (or weak blue) we can find a monochromatic path much longer than $(2/3+\eta)n$. Next we will show that we may assume that in blue we have sufficiently many absorbing bridges for every vertex, and thus we are in EC2 only because of the red color, i.e. this is an EC2-red but not an EC2-blue. Then we can proceed similarly in blue, as in the non-extremal case in red. Indeed, having weak blue edges instead of blue edges is not going to create any difficulties since we can always choose v_r as the 4th vertex of the blue 4-edge containing a triple of three consecutive vertices with a weak blue edge. This finishes the proof in this case.

Thus to finish let us assume that we are in EC2-blue as well. Thus we do not have sufficiently many absorbing bridges for every vertex in blue, i.e. there exists a vertex v_b such that we cannot find at least $\sqrt{\alpha}n^4$ v-absorbing bridges in blue. Similarly as above (with the colors playing the opposite roles), we may assume that the density of red edges (red or weak red, where here for the weak red edges we always have to add v_b as the 4th vertex) is at least $(1-\alpha^{1/10})$. Thus at least $(1-2\alpha^{1/10})$-portion of all the edges are both red and blue. Consider all those 4-edges of $\mathcal{K}$ that we get when we add $\{v_r, v_b\}$ (if $v_r = v_b$, we add an arbitrary other vertex) to these edges and the majority color induced by these edges. If this color is red, then we can find many (certainly much more than $\sqrt{\alpha}n^4$) v_r-absorbing type 2 bridges in red where the vertex w in the definition of the type 2 bridge can be chosen as v_b. We might have to use weak red edges on these red bridges instead of just red edges, but they are just as good for building bridges. We just have to make sure that v_b is never used on these bridges. Thus we have sufficiently many v_r-absorbing bridges in red after all. If the majority color is blue then we have sufficiently many v_r-absorbing type 2 bridges in blue, as desired.

We can repeat the same argument for blue as well if blue also violates the condition of having sufficiently many bridges. Thus in summary we can claim that either we can find a monochromatic Hamiltonian 3-tight Berge-cycle or we can assume that we have sufficiently many bridges for every vertex in both colors.

6. EXTREMAL COLORING 1

Assume finally that we have an EC1. Thus there exists a partition $V(G)=A\cup B$ such that

- $|A| \geq (1-\alpha)2|V(G)|/3$, $|B| \geq (1-\alpha)|V(G)|/3$.
- The graph $G_1^*|_A$ is $(1-\alpha)$-dense and the bipartite graph $G_2^*|_{A\times B}$ is $(1-\alpha)$-dense, where say G_1 is red and G_2 is blue.

The main idea is the same as in the non-extremal case; either in red or in blue we have to find a long enough monochromatic cycle in G and then we splice in the remaining vertices into roughly every other edge on the cycle. In light of the previous section we may assume that we have sufficiently many bridges for every vertex in both colors, so this is not going to be a problem. However, here we might not be able to find a long enough monochromatic cycle since we are in EC1.

First we will redistribute certain exceptional vertices from A and B. A vertex $u \in A$ is *exceptional* if its red-only degree in A is significantly less than $|A|$, i.e. we have

$$deg_{G_1^*}(u,A) < (1-\alpha)|A|, \tag{2}$$

From the density condition in $G_1^*|_A$, it follows that the number of these exceptional vertices in A is at most $\alpha|A|$. If in (2) we have the stronger inequality

$$deg_{G_1^*}(u,A) < \sqrt{\alpha}|A|,$$

then we move u from A to B, since indeed now we have

$$deg_{G_2}(u,A) > (1-\sqrt{\alpha})|A|.$$

Similarly, a vertex $v \in B$ is *exceptional* if its blue-only degree in A is significantly less than $|A|$, i.e. we have

$$deg_{G_2^*}(v,A) < (1-\alpha)|A|. \tag{3}$$

From the density condition in $G_2^*|_{A\times B}$, it follows again that the number of these exceptional vertices in B is at most $\alpha|B|$. If in (3) we have the stronger inequality

$$deg_{G_2^*}(v,A) < \sqrt{\alpha}|A|,$$

then we move v from B to A, since now we have

$$deg_{G_1}(v,A) > (1-\sqrt{\alpha})|A|.$$

For simplicity let us denote the resulting sets still by A and B. It is easy to see that in these new sets A,B we still have the following degree conditions. Apart from at most $2\alpha|A|$ exceptional vertices in $G_1|_A$, all the degrees are at least $(1-2\sqrt{\alpha})|A|$, and the degrees of the exceptional vertices are at least $\sqrt{\alpha}|A|/2$. Similarly, in the bipartite graph $G_2|_{A\times B}$ apart from at most $2\alpha|B|$ exceptional vertices in B all the degrees from B to A are at least $(1-2\sqrt{\alpha})|A|$, and the degrees of the exceptional vertices are at least $\sqrt{\alpha}|A|/2$.

We distinguish two cases.

Case 1. $|B| \le \lfloor n/3 \rceil$. In this case we will find a red Hamiltonian 3-tight Berge-cycle. We proceed exactly as in the non-extremal case, but we have to be slightly more careful because of the sharp size conditions. We will build P'' consisting of P' and the absorbing bridges for the exceptional vertices as in the non-extremal case. However, here we also make sure that the connecting edges between the subpaths are also red edges (it is not hard to see from the degree conditions that this is possible). Furthermore, we can also see from the degree conditions (using $\alpha \ll 1$ and a Pósa-type condition on Hamilton-connectedness, see [1]) that $C' = P'' \cup Q$ may cover all vertices in A. Indeed, let us remove from A the vertices of P'' other than the endpoints, then from the above degree conditions and $\alpha \ll 1$ it follows that the remainder of $G_1|_A$ still satisfies a

Pósa-type condition on Hamilton-connectedness, and thus we can connect the endpoints of P'' by a Hamiltonian path Q, resulting in a red Hamiltonian cycle C' in G.

The set of edges E where we can splice in the remaining vertices includes now literally every second edge on C', so it has size $|E| \geq \lfloor |C'|/2 \rfloor \geq \lfloor n/3 \rfloor$. Then corresponding to (1) we still have

$$|N_{\overline{G_b}}(S)| = |E| \geq \left\lfloor \frac{n}{3} \right\rfloor \geq |S|, \tag{4}$$

and thus we can still splice in every remaining vertex of $V(G) \setminus V(C')$ resulting in a red Hamiltonian 3-tight Berge-cycle.

Case 2. $|B| > \lfloor n/3 \rceil$. In this case we will find a blue Hamiltonian 3-tight Berge-cycle. Now we build $C' = P'' \cup Q$ in the blue almost-complete bipartite graph between A and B in such a way that we cover all vertices of B with C' (again using $\alpha \ll 1$ and a bipartite Pósa-type condition on Hamilton-connectedness, see [1]). Then (4) is true again, and we can splice in every remaining vertex of $V(G) \setminus V(C')$ resulting in a blue Hamiltonian 3-tight Berge-cycle. This finishes the proof of Theorem 1.2.

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