Gallai colorings of non-complete graphs *

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October 29, 2009

*2000 Mathematics Subject Classification: 05C15, 05C35, 05C55.

†Research supported in part by OTKA Grant No. K68322.

‡Research supported in part by OTKA Grant No. K68322 and by a János Bolyai Research Scholarship.

Abstract

Gallai-colorings of complete graphs - edge colorings such that no triangle is colored with three distinct colors - occur in various contexts such as the theory of partially ordered sets (in Gallai's original paper), information theory and the theory of perfect graphs. We extend here Gallai-colorings to non-complete graphs and study the analogue of a basic result - any Gallai-colored complete graph has a monochromatic spanning tree - in this more general setting.

We show that edge colorings of a graph H without multicolored triangles contain monochromatic connected subgraphs with at least $(\alpha(H)^2 + \alpha(H) - 1)^{-1}|V(H)|$ vertices, where $\alpha(H)$ is the independence number of H. In general, we show that if the edges of an r-uniform hypergraph $\mathcal{H}$ are colored so that there is no multicolored copy of a fixed F then there is a monochromatic connected subhypergraph $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $|V(\mathcal{H}_1)| \geq c|V(\mathcal{H})|$ where c depends only on $\mathcal{F}$, r, and $\alpha(\mathcal{H})$.

1 Gallai-colorings of non-complete graphs

Edge colorings of complete graphs in which no triangle is colored with three distinct colors were called Gallai-partitions in [10], Gallai-colorings in [5], [6]. More than just the term, the concept occurs again and again in relation to deep structural properties of fundamental objects. A main result in Gallai's original paper [4] – translated to English and endowed by comments in [11] – can be reformulated in terms of Gallai-colorings. Further occurrences are related to generalizations of the perfect graph theorem [2], or applications in information theory [9].

In this paper we start investigating whether Gallai-colorings can be fruitfully extended from complete graphs to arbitrary graphs, i.e. we say that *an edge coloring of a graph G is a Gallai-coloring - or G-coloring - if no triangle of G is colored with three distinct colors.* In particular, every edge coloring of a triangle-free graph is a G-coloring. A less obvious example can be obtained by considering a labeling of the vertices of a graph of order n by $1, 2, \ldots, n$ and for all $1 \leq i < j \leq n$ color the edge ij by color i.

A basic remark of Erdős and Rado states that in any coloring of the edges of a complete graph with two colors there is a monochromatic spanning tree. This remains true for G-colorings of complete graphs as proved in [1], see also [6]. Our starting point is another generalization of the above remark, we state it as Theorem 1. Let $\alpha(H)$ be the *independence number* of H, that is, the maximum size of an *independent set*, set of vertices not containing both endpoints of any edge.

Theorem 1. *If the edges of an arbitrary graph H are colored with two colors, there exists a monochromatic subtree $T \subset H$ with at least $\alpha(H)^{-1}|V(H)|$ vertices.*

We derive Theorem 1 from König's theorem and note that Theorem 1 can be extended for r-colorings as well, with $(r-1)\alpha(H)$ in the role of $\alpha(H)$, this more general form can be obtained from Füredi's result [3] on fractional transversals (see [7]). Notice that Theorem 1 is sharp if $\alpha(H)$ is a divisor of $|V(H)|$, simply consider $\alpha(H)$ disjoint monochromatic complete graphs of equal order. Our main result is an analogue of Theorem 1 for G-colorings.

Theorem 2. *If the edges of an arbitrary graph H are G-colored, there exists a monochromatic subtree with at least $(\alpha^2(H) + \alpha(H) - 1)^{-1}|V(H)|$ vertices and a monochromatic double star with at least $(\alpha^2(H) + \alpha(H) - \frac{2}{3})^{-1}|V(H)|$ vertices.*

In fact, the coefficient of $|V(H)|$ in Theorem 2 is not very far from the truth, showing that the bound of Theorem 1 does not extend to G-colorings. Indeed, consider a triangle-free graph $H = H(\alpha)$ such that $\alpha(H) = \alpha$ and has as many vertices as possible - in other words, H is a so called Ramsey-graph with $R(3, \alpha+1)-1$ vertices. It is well-known that $|V(H)|$ is "almost" quadratic in α, its order of magnitude is $\frac{\alpha^2}{\log \alpha}$ (see [8]). One can get a trivial G-coloring on H by coloring each edge of H with a different color. Then substituting into each vertex of H a G-colored complete graph K_p, we get a G-colored graph H^* on $p|V(H)|$ vertices, with $\alpha(H^*) = \alpha$ and with largest monochromatic connected subgraph at most $2p$ - order of magnitude $\frac{log \alpha}{\alpha^2}|V(H^*)|$ - vertices.

The problem of determining $f(\alpha)$, the largest value such that every G-colored graph H has a monochromatic connected subgraph with at least $f(\alpha(H))|V(H)|$ vertices remains open even for $\alpha = 2$. From Theorem 2 and from the construction above we get

$$\frac{1}{(\alpha^2 + \alpha - 1)} \le f(\alpha) \le \frac{c \log \alpha}{\alpha^2}$$

where c is a constant, coming from Kim's [8] estimate of $R(3, \alpha + 1)$.

For $\alpha = 2$ we have the example above from coloring the edges of a C_5 with five distinct colors, substituting arbitrarily G-colored complete graphs of equal sizes to the vertices. However, this coloring is not the best for $\alpha = 2$. We may take another Ramsey graph, H_8, the complement of the Wagner graph, its missing edges form an eight cycle $1, 2, \ldots 8$ plus its main diagonals. (The graph H_8 is a smallest graph with $\alpha = 2, \omega = 3$, see for example [13].) The edges of H_8 can be G-colored without monochromatic connected subgraph on four vertices, by using color i on the edges $(i, i+2), (i, i+5)$ for $i = 1, 2, \ldots, 8$ (modulo 8 arithmetic). Using substitutions into this coloring of H_8 and applying Theorem 2 we have

Corollary 1. $\frac{1}{5} \le f(2) \le \frac{3}{8}$.

Theorem 2 shows that colorings of G without multicolored K_3 have monochromatic connected subgraphs with order proportional to $V(G)$. This property remains true in a much more general setting. Extending the definition from graphs, let $\alpha(\mathcal{H})$ denote the maximum m such that $\mathcal{H}$ contains m vertices that does not contain any edge of $\mathcal{H}$. A hypergraph is *connected* if both parts of every nontrivial 2-partition of its vertex set have nonempty intersection with some edge of the hypergraph.

Theorem 3. *Suppose that the edges of an r-uniform hypergraph $\mathcal{H}$ are colored so that $\mathcal{H}$ does not contain multicolored copies of an r-uniform hypergraph F. Then there is a monochromatic connected subhypergraph $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $|V(\mathcal{H}_1)| \geq c|V(\mathcal{H})|$ where c depends only on F, r, and $\alpha(\mathcal{H})$ (thus does not depend on $\mathcal{H}$).*

2 Proofs

Proof of Theorem 1.

Consider a coloring of the edges of H with two colors, say red and blue. Let $\mathcal{H}$ be the hypergraph on vertex set $V(H)$ whose hyperedges are the vertex sets of the connected components (in both colors). Since each vertex of $\mathcal{H}$ is in one red component and in one blue component, the dual of $\mathcal{H}$ is a bipartite graph B. Observe that two vertex-disjoint edges $e, f \in E(B)$ correspond to two vertices $v_e, v_f \in V(H)$ that are not covered by any hyperedge (component) in $\mathcal{H}$, in particular $(v_e, v_f) \notin E(H)$. Therefore the maximum number of pairwise disjoint edges in B, $\nu(B)$, satisfies $\nu(B) \leq \alpha(H)$. By König's theorem, the edges of B has a transversal of $\nu(B)$ vertices, i.e. there is $T \subseteq V(B)$ such that $|T| = \nu(B)$ and T intersects all edges of B in at least one vertex. From this it follows that some $t \in T$ is incident to at least $k = \frac{|E(B)|}{\nu(B)}$ edges of B. Thus the hyperedge (component) in $\mathcal{H}$ corresponding to t contains at least

$$k = \frac{|E(B)|}{\nu(B)} = \frac{|V(H)|}{\nu(B)} \geq \frac{|V(H)|}{\alpha(H)}$$

vertices, finishing the proof. □

Proof of Theorem 2.

Suppose $f(\alpha)$ is the largest value such that every G-colored graph H has a monochromatic connected subgraph with at least $f(\alpha(H))|V(H)|$ vertices. We estimate $f(\alpha)$ by induction. Clearly $f(1) = 1$ because G-colored complete graphs have monochromatic spanning trees see [1], [5], [7]. Suppose $\alpha(H) \geq 2$ and consider a G-coloring on H. Let v be an arbitrary vertex, X is the set of vertices nonadjacent to v, A_i is the set of vertices adjacent to v in color i. By induction, the graph $H[X]$ has a monochromatic component C_1 with

$$|V(C_1)| \geq f(\alpha - 1)|X|. \tag{1}$$

Set $G = H[\cup_i A_i \cup \{v\}]$ and define the graph G' on the same vertex set as G as follows. For every i, A_i is replaced by a complete graph with each edge colored with color i (all other edges retain their colors). It is enough to find a large monochromatic component in G' because of the following claim.

Claim 1. *Suppose C' is a connected component of G' in color i. Then there is a connected component C of G in color i such that $V(C') = V(C)$.*

Proof. Let C be the subgraph of C' obtained by removing edges of C' in color i that were added when A_i was replaced by a complete graph with each edge colored with color i. Observe that C is connected in color i because for each removed edge xy, the edges xv, vy have color i in G. □

Notice that every edge of G' between A_i and A_j must be colored with either color i or color j because we had a G-coloring on H. Based on this, we can orient the edges of G' so that the edge sets going out from any vertex are monochromatic. To achieve that, for any $1 \leq i < j \leq n$, orient edges of color i between A_i and A_j from A_i to A_j; similarly, orient edges of color j between A_i and A_j from A_j to A_i. The edges within the A_i's (they are colored with color i) can be oriented arbitrarily. All edges incident to v are oriented into v.

Applying the complementary form of a well-known consequence of Turán's theorem (see for example in [12]) to G'

$$|E(G')| \geq \frac{|V(G')|}{2}\left(\frac{|V(G')|}{\alpha(G')} - 1\right) = M.$$

Thus - by looking at the average outdegree of G' - there exists a vertex $w \in V(G')$ (in fact $w \neq v$ because v has outdegree zero) with outdegree at least

$$\frac{M}{|V(G')|} = \frac{|V(G')|}{2\alpha(G')} - \frac{1}{2}.$$

Since all edges going out of w have the same color, we have a monochromatic component C_2 such that

$$|V(C_2)| \geq \frac{|V(G')|}{2\alpha(G')} \geq \frac{|V(G)|}{2\alpha(G)} \geq \frac{|V(H)| - |X|}{2\alpha(H)}.$$

Combining this estimate with (1), we get a monochromatic component of H with at least

$$\max\left(f(\alpha - 1)|X|, \frac{|V(H)| - |X|}{2\alpha(H)}\right) \tag{2}$$

vertices. Suppose $|X| = \nu|V(H)|$, then $|V(H)| - |X| = (1-\nu)|V(H)|$ and the two terms in (2) are equal when

$$\nu = \frac{1}{1 + 2\alpha f(\alpha - 1)}$$

and their common value gives the recursion

$$f(\alpha) = \frac{f(\alpha - 1)}{1 + 2\alpha f(\alpha - 1)}$$

and that is equivalent to

$$\frac{1}{f(\alpha)} = \frac{1}{f(\alpha - 1)} + 2\alpha$$

and that easily yields

$$\frac{1}{f(\alpha)} = \alpha^2 + \alpha - 1$$

as desired.

The second statement of the theorem follows easily. For $\alpha = 1$ we have to show that every G-coloring of a complete graph K_n has a monochromatic double star with at least $\frac{3n}{4}$ vertices - this was proved in [6]. This changes only the initial value in the proof above and yields a directed star S of the required size in color i with center $w \in A_i$. Deleting the leaves of S from A_i and adding all edges from v to A_i, S is transformed into the required monochromatic double star. □

Proof of Theorem 3. It is enough to prove Theorem 3 for $F = K_m^r$. Assume that the edges of an r-uniform hypergraph $\mathcal{H}$ on n vertices are colored without multicolored F. Set $\alpha = \alpha(\mathcal{H})$ and $t = R(m, \alpha + 1)$, i.e. t is the smallest integer for which every r-uniform hypergraph on t vertices contains either F or $\alpha + 1$ independent vertices. This definition ensures that every t vertices of $\mathcal{H}$ contains a copy of F. From this - by an obvious counting argument - we observe that $\mathcal{H}$ contains at least $c_1 n^m$ copies of F where c_1 - as any other c_i later - depends only on m, r, α. Indeed, since any t vertices of $\mathcal{H}$ contains a copy of F, there are at least $\frac{\binom{n}{t}}{\binom{n-m}{t-m}} \geq c_1 n^m$ copies of F.

Furthermore - by counting again - there exists $e \in E(\mathcal{H})$ such that e is in at least $c_2 n^{m-r}$ copies of F and the color of e - say red - is repeated in all of these copies. Indeed, assigning two edges of $\mathcal{H}$ with the same color to each copy of F, at least $2c_1 n^m$ edges are assigned, thus some edge e is assigned to at least

$$\frac{2c_1 n^m}{\binom{n}{r}} \geq c_2 n^{m-r}$$

copies of F. We select a second red edge in all of these copies and partition these copies into two parts, $\mathcal{F}_1, \mathcal{F}_2$, the first part contains those copies where the other red edge intersects e, the second part contains those copies where the other red edge does not intersect e. Set $c_3 = \frac{c_2}{2}$, then $\mathcal{F}_i$ must contain at least $c_3 n^{m-r}$ copies of F for $i \in \{1,2\}$. The union of the second red edges of $\mathcal{F}_i$ forms an r-uniform hypergraph $\mathcal{R}$, its edges cover a (nonempty) set S in $V \setminus e$.

If $i = 1$ then $\mathcal{R} \cup e$ is part of a connected red component and $|S| n^{m-r-1} \geq |\mathcal{F}_1| \geq c_3 n^{m-r}$ thus $|S| \geq c_3 n$. If $i = 2$ then S can be the union of several components of $\mathcal{R}$, say it is the union of components $C_1, C_2, \dots, C_k$. Let C_1 be the largest component. Then - using that $\sum_{i=1}^{k} \binom{|C_i|}{r}$ is largest (k may vary, $\sum |C_i|$ is fixed and $|C_1| \geq |C_i|$) if all but at most one $|C_i|$ are equal to $|C_1|$ -

$$c_3 n^{m-r} \leq |\mathcal{F}_2| \leq \sum_{i=1}^{k} \binom{|C_i|}{r} \binom{n-r}{m-2r} \leq$$

$$\leq c_4 n^{m-2r} \sum_{i=1}^{k} \binom{|C_i|}{r} \leq c_4 n^{m-2r} \left\lceil \frac{n}{|C_1|} \right\rceil \binom{|C_1|}{r} \leq c_4 n^{m-2r+1} |C_1|^{r-1}.$$

Thus we conclude $|C_1| \geq c_5 n$.

In both cases we have a red component in $\mathcal{H}$ with at least $c|V(\mathcal{H})|$ vertices. □

Acknowledgement. Thanks to Miklós Ruszinkó for fruitful conversations on the subject.

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