Stability of the path-path Ramsey number

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July 9, 2012

*Research supported in part by OTKA Grant No. K68322.
†Research supported in part by the National Science Foundation under Grant No. DMS-0456401 and by OTKA Grant No. K68322.

Abstract

Here we prove a stability version of a Ramsey-type Theorem for paths. Thus in any 2-coloring of the edges of the complete graph K_n we can either find a monochromatic path substantially longer than $2n/3$, or the coloring is close to the extremal coloring.

1 Introduction

$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A + B$, and $E \subset A \times B$. For a graph G and a subset U of its vertices, $G|_U$ is the restriction of G to U. $N(v)$ is the set of neighbors of $v \in V$. Hence the size of $N(v)$ is $|N(v)| = deg(v) = deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum and $\Delta(G)$ for the maximum degree in G. When A, B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. In particular, we write $deg(v, U) = e(\{v\}, U)$ for the number of edges from v to U. A graph G_n on n vertices is γ-*dense* if it has at least $\gamma\binom{n}{2}$ edges. A bipartite graph $G(k, l)$ is γ-dense if it contains at least γkl edges.

For graphs $G_1, G_2, \ldots, G_r$, the *Ramsey number* $R(G_1, G_2, \ldots, G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes giving r graphs $H_1, H_2, \ldots, H_r$, then at least one H_i $(1 \le i \le r)$ has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's classical result [11]. The number $R(G_1, G_2, \ldots, G_r)$ is called the Ramsey number for the graphs $G_1, G_2, \ldots, G_r$. There is very little known about $R(G_1, G_2, \ldots, G_r)$ even for very special graphs (see eg. [5] or [10]). For $r = 2$ a theorem of Gerencsér and Gyárfás [4] states that

$$R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor. \tag{1}$$

In this paper we prove a stability version of this theorem. Since this is what we needed in a recent application [6], actually we prove the result in a slightly more general context; we work with 2-edge *multicolorings* (G_1, G_2) of a graph G. Here multicoloring means that the edges can receive more than one color, i.e. the graphs G_i are not necessarily edge disjoint. The subgraph colored with color i *only* is denoted by G_i^*, i.e.

$$G_1^* = G_1 \setminus G_2, G_2^* = G_2 \setminus G_1.$$

In order to state the theorem we need to define a relaxed version of the extremal coloring for (1).

Extremal Coloring (with parameter α): There exists a partition $V(G) = A \cup B$ such that

- $|A| \ge (2/3 - \alpha)|V(G)|$, $|B| \ge (1/3 - \alpha)|V(G)|$.

- The graph $G_1^*|_A$ is $(1-\alpha)$-dense and the bipartite graph $G_2^*|_{A\times B}$ is $(1-\alpha)$-dense. (Note that we have no restriction on the coloring inside the smaller set.)

Then the following stability version of the Gerencsér-Gyárfás Theorem claims that we can either find a monochromatic path substantially longer than $2n/3$, or the coloring is close to the extremal coloring.

Theorem 1.1. *For every $\alpha > 0$ there exist positive reals η, c_1 ($0 < \eta \ll \alpha \ll 1$ where $\ll$ means sufficiently smaller) and a positive integer n_0 such that for every $n \geq n_0$ the following holds: if the edges of the complete graph K_n are 2-multicolored then we have one of the following two cases.*

- *Case 1: K_n contains a monochromatic path P of length at least $(\frac{2}{3} + \eta)n$. Furthermore, in the process of finding P, for each vertex of the path P we have at least $c_1 \log n$ choices.*
- *Case 2: This is an Extremal Coloring (EC) with parameter α.*

Surprisingly, as far as we know, this natural question has not been studied, despite the fact that for some classical density results the corresponding stability versions are well-known (see [2]).

2 Tools

Theorem 1.1 can also be proved from the Regularity Lemma [12], however, here we use a more elementary approach using only the Kővári-Sós-Turán bound [7]. This is part of a new direction where we were able to "de-regularize" some of our proofs, namely to replace the Regularity Lemma with more elementary classical extremal graph theoretic results such as the Kővári-Sós-Turán bound (see e.g. [8]).

Lemma 2.1 (Theorem 3.1 on page 328 in [2])**.** *There is an absolute constant $\beta > 0$ such that if $0 < \epsilon < 1/r$ and we have a graph G with*

$$|E(G)| \geq \left(1 - \frac{1}{r} + \epsilon\right) \frac{n^2}{2}$$

then G contains a $K_{r+1}(t)$, where

$$t = \lfloor \frac{\beta \log n}{r \log 1/\epsilon} \rfloor.$$

For $r = 1$ this is essentially the Kővári-Sós-Turán bound [7] and for general r this was proved by Bollobás, Erdős and Simonovits [3]. Here we will use the result only for $r = 1$.

3 Outline of the proof

We will need the following definition. Given a graph G and a positive integer k, we say that a subset W of the vertex set $V(G)$ is *k-well-connected* if for any two vertices $u, v \in W$ there are at least k internally vertex disjoint paths of length at most *three* connecting u and v in G (note that these paths might leave W). We will use this definition with $k = \eta n$, in this case we just say shortly that W is well-connected.

We will follow a similar outline as in applications of the Regularity Lemma. However, a regular pair will be replaced with a complete balanced bipartite graph $K(t,t)$ with $t \geq c \log n$ for some constant c (thus the size of the pair is somewhat smaller but this is still good enough for our purposes). Then a monochromatic connected matching in the reduced graph (the usual tool in these types of proofs using the Regularity Lemma) will be replaced with a monochromatic cover consisting of vertex disjoint complete balanced bipartite graphs $K_i(t_i, t_i)$, $1 \leq i \leq s$ such that these monochromatic complete balanced bipartite graphs are all contained in a set W that is well-connected in this color and $t_i \geq c \log n$ for every $1 \leq i \leq s$ for some constant c. Let us call a cover like this a monochromatic well-connected complete balanced bipartite graph cover. The size of this cover is the total number of vertices in the union of these complete bipartite graphs.

Then Theorem 1.1 will follow from the following lemma.

Lemma 3.1. *For every $\alpha > 0$ there exist a positive real η ($0 < \eta \ll \alpha \ll 1$ where $\ll$ means sufficiently smaller) and a positive integer n_0 such that for every $n \geq n_0$ the following holds: if the edges of the complete graph K_n are 2-multicolored then we have one of the following two cases.*

- *Case 1: K_n contains a monochromatic well-connected complete balanced bipartite graph cover of size at least $(\frac{2}{3} + 2\eta)n$.*
- *Case 2: This is an Extremal Coloring (EC) with parameter α.*

Indeed, let us assume that we have Case 1 in this Lemma. Denote the two color classes of $K_i(t_i, t_i)$ by V_1^i and V_2^i for $1 \leq i \leq s$. Since this cover is inside the same set W that is well-connected in this color (say red), we can find red vertex disjoint connecting paths P_i of length at most 3 from a vertex of V_2^i to a vertex of V_1^{i+1} for every $1 \leq i \leq s-1$. Furthermore, we can also guarantee that in this connecting process from any V_j^i, $1 \leq j \leq 2$, $1 \leq i \leq s$ we never use up more than $\eta |V_j^i|$ vertices. Indeed, then during the whole process the total number of forbidden vertices is at most

$$\frac{4s}{\eta} \leq \frac{4n}{2c\eta \log n} \ll \eta n,$$

(if n is sufficiently large) and thus we can always select the next connecting path that is vertex disjoint from the ones constructed so far.

We remove the internal vertices of these connecting paths P_i from the complete balanced bipartite graphs $K_i(t_i, t_i)$. By doing this we may create some discrepancies in the cardinalities of the two color classes. We remove some additional vertices to assure that now we have the same number t_i' ($\geq (1-\eta)t_i$) of vertices left in both color classes. Now we can connect the endpoint of P_{i-1} in V_1^i and the endpoint of P_i in V_2^i by a red Hamiltonian path Q_i in the remainder of $K_i(t_i, t_i)$ (using the fact that this is a balanced complete bipartite graph). Putting together the connecting paths P_i and these Hamiltonian paths Q_i we get a red path P of length at least

$$(\frac{2}{3} + 2\eta - \frac{2}{3}\eta)n \geq (\frac{2}{3} + \eta)n.$$

Furthermore, clearly for each vertex of P we have at least $(c/2)\log n$ choices, as desired in Theorem 1.1. $\square$

4 Monochromatic well-connected components

In this section we show that in any 2-multicoloring of K_n there is a large set W and a color such that W is k-well-connected in this color.

Lemma 4.1. *For every integer k and for every 2-multicolored K_n there exist $W \subset V(K_n)$ and a color (say color 1) such that $|W| \geq n - 28k$ and W is k-well-connected in the color 1 subgraph of K_n.*

Proof. Assume that a 2-multicoloring is given on K_n - in fact it is enough to prove the lemma for a coloring obtained by ignoring one of the colors of every 2-colored edge. A pair $u, v \in V(K_n)$ is bad for color 1 if there are no k internally vertex disjoint paths of length at most three from u to v all monochromatic in color 1. Let m be the maximum number of vertex disjoint bad pairs for color 1. If $m < 2k$ then deleting $2m < 4k$ vertices of a maximum matching of bad pairs for color 1, we have a set W of more than $n - 4k$ vertices that is k-well-connected for color 1 and the proof is finished (with $24k$ to spare). Otherwise select a matching $\{u_1v_1, \ldots, u_{2k}v_{2k}\}$ of $2k$ bad pairs for color 1. For $t \in [2k], i, j \in [1, 2]$ define $A_{t,i,j}$ as the set of vertices adjacent to u_t in color i and to v_t in color j. From the definition of bad pairs $|A(t, 1, 1)| < k$. For the same reason - using König's theorem - all edges of color 1 in the bipartite graph $[A(t, 1, 2), A(t, 2, 1)]$ can be met by a set S_t of less than k vertices. Set $B_t = A(t, 1, 2) \setminus S_t, C_t = A(t, 2, 1) \setminus S_t$. Observe that between B_t and C_t we have a complete bipartite graph in color 2 (if both are non-empty). A set B_t (C_t) is small if it has less than $2k$ elements. Define H_t as the union of $A(t, 1, 1) \cup S_t$ and the small sets B_t, C_t. Observe that $|H_t| < 6k$. Consider the hypergraph $\mathcal{H}$ with vertex set $X = V(K_n) \setminus (\cup_{t\in[2k]}\{u_t, v_t\}$ and edge set $X \cap H_t$ for $t \in [2k]$. Let M be the set of vertices of $\mathcal{H}$ with degree at least $k/2$. Then

$$12k^2 = 2k \times 6k \geq \sum_{t=1}^{2k} |H_t| = \sum_{x\in X} d(x) \geq \sum_{x\in M} d(x) \geq |M|(k/2)$$

implying that $24k \geq |M|$. Therefore $|X \setminus M| \geq n - 24k - 4k = n - 28k$. We claim that $W = X \setminus M$ is k-well-connected in color 2 and that will finish the proof.

To prove the claim, consider a pair of vertices $x, y \in W$. From the definition of W, x, y are both in less than $k/2$ edges of $\mathcal{H}$. Thus x, y are both covered by at least k sets in the form $Y_t = (\cup_{i,j\in[1,2]} A(t,i,j)) \setminus H_t$ where $t \in [2k]$. Therefore in every Y_t the pair x, y can be connected in color 2 either by a path of length two through u_t (if $x, y \in A(t,2,2) \cup A(t,2,1)$) or through v_t (if $x, y \in A(t,2,2) \cup A(t,1,2)$) or through a path of length three (if one of x, y is in $A(t,2,1)$ and the other is in $A(t,1,2)$. It is easy to see - using that there are at least k choices for t, $|B_t| = |A(t,1,2) \setminus S_t| \geq 2k, |C_t| = |A(t,2,1) \setminus S_t| \geq 2k$ and that we have a complete bipartite graph between B_t and C_t in color 2 - that there are at least k internally edge disjoint paths of length at most three in color 2 connecting x, y. $\Box$

5 Proof of Lemma 3.1

Assume that we have an arbitrary 2-multicoloring (red/blue) of K_n. We shall assume that n is sufficiently large and use the following main parameters

$$0 \ll \eta \ll \alpha \ll 1, \tag{2}$$

where $a \ll b$ means that a is sufficiently small compared to b. In order to present the results transparently we do not compute the actual dependencies, although it could be done. We will use the constant

$$c = \left(\frac{\eta}{2}\right)^{\frac{8}{\eta^3}} \frac{\beta}{2 \log \frac{1}{\eta}},$$

where β is from Lemma 2.1.

Let us apply Lemma 4.1 with $k = \eta n$ to find a $W \subset V(K_n)$ and a color (say red) such that $|W| \geq (1 - 28\eta)n$ and W is ηn-well-connected (or shortly well-connected) in the red subgraph of K_n. Put $R = K_n \setminus W$, then we have $|R| \leq 28\eta n$. From now on we will work inside W.

We may assume that inside W the red density is at least η, since otherwise we can switch colors as the blue-only subgraph is almost complete. Thus we can apply Lemma 2.1 with $r = 1$ and $\epsilon = \eta$ to the red subgraph inside W to find a red complete balanced bipartite subgraph $K_1(t_1, t_1)$ in W with

$$t_1 = \frac{\beta}{2 \log \frac{1}{\eta}} \log n$$

(for simplicity we assume that this is an integer). We remove this $K_1(t_1, t_1)$ from W and in the remainder of W iteratively we find red complete balanced bipartite graphs $K(t_1, t_1)$ until we can. Suppose that we found this way the red well-connected complete balanced bipartite graph cover

$$M_1 = (K_1(t_1, t_1), K_2(t_1, t_1), \ldots, K_{s_1}(t_1, t_1))$$

for some positive integer s_1. If this red cover M_1 has size $|M_1| \geq (2/3 + 2\eta)n$, then we are done, we have Case 1 in Lemma 3.1. Otherwise we will show that we can either increase the size of this red cover by an $\eta^2/2$-fraction, or we can find directly a monochromatic well-connected complete bipartite graph cover of size at least $(2/3 + 2\eta)n$ unless we are in the Extremal Coloring (Case 2), as desired.

We know that at least we have

$$|M_1| \geq \frac{\eta}{4}n, \tag{3}$$

since otherwise in the remainder of W the red density is still at least $\eta/2$, and we can still apply Lemma 2.1 in the remainder to find a red $K(t_1, t_1)$. Let $K_i(t_1, t_1) = (V_1^i, V_2^i), 1 \leq i \leq s_1$. Denote

$$V_1 = \cup_{i=1}^{s_1} V_1^i,\ V_2 = \cup_{i=1}^{s_1} V_2^i \text{ and } V_3 = W \setminus (V_1 \cup V_2).$$

From

$$|V_1| + |V_2| < \left(\frac{2}{3} + 2\eta\right) n,$$

we have

$$|V_3| > \left(\frac{1}{3} - 30\eta\right) n. \tag{4}$$

Furthermore, since in V_3 we cannot pick another red complete balanced bipartite subgraph $K(t_1, t_1)$, by Lemma 2.1 V_3 is $(1-\eta)$-dense in the blue-only subgraph.

Next let us look at the bipartite graphs (V_1, V_3) and (V_2, V_3). We will show that either one of them is $(1-2\eta)$-dense in blue-only or we can increase our red cover M_1. Indeed, assume first the following: (i) There is a subcover of M_1

$$M_1' = (K_{i_1}(t_1, t_1), K_{i_2}(t_1, t_1), \ldots, K_{i_{s_1'}}(t_1, t_1))$$

with $1 \leq s_1' \leq s_1$ such that if we denote $V_1' = \cup_{j=1}^{s_1'} V_1^{i_j}$, $V_2' = \cup_{j=1}^{s_1'} V_2^{i_j}$ we have

- $|V_1'| = |V_2'| \geq \eta |V_1| = \eta |V_2|$, and
- the bipartite graphs $(V_1^{i_j}, V_3)$ and $(V_2^{i_j}, V_3)$ are both η-dense in red for every $1 \leq j \leq s_1'$.

Consider the bipartite graph $(V_1^{i_j}, V_3)$ for some $1 \leq j \leq s_1'$. Since this is η-dense in red, there must be at least $\eta|V_3|/2$ vertices in V_3 for which the red degree in $V_1^{i_j}$ is at least $\eta|V_1^{i_j}|/2 = \eta t_1/2$. Indeed, otherwise the total number of red edges would be less than

$$\frac{\eta}{2}|V_3||V_1^{i_j}| + \frac{\eta}{2}|V_3||V_1^{i_j}| = \eta|V_3||V_1^{i_j}|,$$

a contradiction with the fact that $(V_1^{i_j}, V_3)$ is η-dense in red. Consider all the red neighborhoods of these vertices in $V_1^{i_j}$. Since there can be at most

$$2^{t_1} = n^{\frac{\beta}{2 \log \frac{1}{\eta}}}$$

such neighborhoods, by averaging and using (2) and (4) there must be a red neighborhood that appears for at least

$$\frac{\frac{\eta}{2}|V_3|}{n^{\frac{\beta}{2\log\frac{1}{\eta}}}} \geq \frac{\eta}{8} n^{1-\frac{\beta}{2\log\frac{1}{\eta}}} \gg \frac{\eta}{2} t_1$$

such vertices of V_3. This means that we can find a red complete balanced bipartite graph $K(t_2, t_2)$ in the red bipartite graph $(V_1^{i_j}, V_3)$, where $t_2 = \eta t_1/2$. We can proceed similarly for the red bipartite graph $(V_2^{i_j}, V_3)$. Thus the red complete balanced bipartite graph $K_{i_j}(t_1, t_1)$ can be replaced with 3 red complete balanced bipartite graphs. We can proceed similarly for all $K_{i_j}(t_1, t_1), 1 \leq j \leq s_1'$. This way we obtain a new red well-connected complete bipartite graph cover

$$M_2 = (K_1(t_1^2, t_1^2), K_2(t_2^2, t_2^2), \ldots, K_{s_2}(t_{s_2}^2, t_{s_2}^2))$$

such that

- $|M_2| \geq \left(1 + \frac{\eta^2}{2}\right) |M_1|$, and
- $t_1 \geq t_i^2 \geq t_2 = \eta t_1/2$ for every $1 \leq i \leq s_2$.

Iterating this process $(j-1)$ times we get a new cover

$$M_j = (K_1(t_1^j, t_1^j), K_2(t_2^j, t_2^j), \ldots, K_{s_j}(t_{s_j}^j, t_{s_j}^j))$$

such that

- $|M_j| \geq \left(1 + \frac{\eta^2}{2}\right)^{j-1} |M_1|$, and
- $t_1 \geq t_i^j \geq t_j = (\eta/2)^{j-1} t_1$ for every $1 \leq i \leq s_j$.

This and (3) imply that if we could iterate this process $8/\eta^3$ times, then we would get a red complete balanced bipartite graph cover of size at least $(2/3 + 2\eta)n$, where for each $K_i(t_i, t_i)$ in the cover we would still have

$$t_i \geq (\frac{\eta}{2})^{\frac{8}{\eta^3}} t_1 = (\frac{\eta}{2})^{\frac{8}{\eta^3}} \frac{\beta}{2\log\frac{1}{\eta}} \log n = c \log n$$

and thus we would have Case 1 in Lemma 3.1.

We may assume that this is not the case and after $(j-1)$ iterations for some $j < 8/\eta^3$ we get a cover M_j as above for which (i) does not hold. In this case we will show that we can find directly a monochromatic well-connected complete bipartite graph cover of size at least $(2/3 + 2\eta)n$ unless we are in the Extremal Coloring (Case 2). For simplicity we still use the notation V_1, V_2, V_3 for M_j just as for M_1. Since (i) does not hold for M_j, for most of the complete bipartite graphs $K_i(t_i^j, t_i^j) = (V_1^i, V_2^i)$ (namely for complete bipartite graphs covering at least an $(1-\eta)$-fraction of the total size of M_j) the red density in one of the bipartite graphs (V_1^i, V_3) and (V_2^i, V_3) is less than η. By renaming we may assume that this is always the bipartite graph (V_2^i, V_3). This means that indeed (V_2, V_3) is $(1-2\eta)$-dense in blue-only, as we wanted. However, this implies that

$$|V_3| < \left(\frac{1}{3} + \alpha^3\right),$$

since otherwise using (2) we can easily find a blue well-connected complete balanced bipartite graph cover of size at least $(2/3+2\eta)n$ by iteratively applying Lemma 2.1 in blue, first in the bipartite graph (V_2, V_3) until we can, and then continuing inside V_3. Similarly if the blue density in the bipartite graph (V_1, V_2) is at least α^3, then again we can find a blue well-connected complete balanced bipartite graph cover of size at least $(2/3+2\eta)n$ by iteratively applying Lemma 2.1 in blue, first in the bipartite graph (V_1, V_2), then in the bipartite graph (V_2, V_3), and finally inside V_3. Thus we may assume that the bipartite graph (V_1, V_2) is $(1-\alpha^3)$-dense in red-only.

Making progress towards the Extremal Coloring, next let us look at the red density inside V_2. Assume first that this density is at least α^2. Similarly as above this implies that the bipartite graph (V_1, V_3) is $(1-\alpha^2)$-dense in blue-only, since otherwise we can find again a red cover of size at least $(2/3+2\eta)n$. Thus the bipartite graph $(V_1 \cup V_2, V_3)$ is $(1-\alpha^2)$-dense in blue-only. This in turn implies that $V_1 \cup V_2$ is $(1-\alpha)$-dense in red-only since otherwise we can find again a blue cover of size at least $(2/3+2\eta)n$. This gives us the Extremal Coloring where $A = V_1 \cup V_2$, $B = V_3$, G_1 is red and G_2 is blue (actually here in a somewhat stronger form since inside B most of the edges are blue-only as well).

Hence we may assume that V_2 is $(1-\alpha^2)$-dense in blue-only. This implies again that (V_1, V_3) is $(1-\alpha)$-dense in red-only and this gives us the Extremal Coloring again where $A = V_2 \cup V_3$, $B = V_1$, G_1 is blue and G_2 is red. This finishes the proof of Lemma 3.1. $\square$

References

[1] C. Berge, Graphs and Hypergraphs, North Holland, Amsterdam and London, 1973.

[2] B. Bollobás, *Extremal Graph Theory*, Academic Press, London (1978).

[3] B. Bollobás, P. Erdős, M. Simonovits, On the structure of edge graphs II, *J. London Math. Soc.*, 12 (2), 1976, pp. 219-224.

[4] L. Gerencsér, A. Gyárfás, On Ramsey-type problems, *Ann. Univ. Sci. Budapest Eötvös Sect. Math.* 10 (1967), pp. 167-170.

[5] R.L. Graham, B.L. Rothschild, J.H. Spencer, *Ramsey Theory*, John Wiley & Sons, New York, 1990.

[6] A. Gyárfás, G.N. Sárközy, E. Szemerédi, Monochromatic Hamiltonian 3-tight Berge cycles in 2-colored 4-uniform hypergraphs, submitted for publication.

[7] P. Kővári, V.T. Sós, P. Turán, On a problem of Zarankiewicz, *Colloq. Math.*, 3 (1954), pp. 50-57.

[8] I. Levitt, G.N. Sárközy, E. Szemerédi, How to avoid using the Regularity Lemma; Pósa's Conjecture revisited, submitted for publication.

[9] L. Lovász, Combinatorial Problems and Exercises, 2nd edition, North-Holland, Amsterdam, 1979.

[10] S.P. Radziszowski, Small Ramsey numbers, *Electronic Journal of Combinatorics*, (2002) DS1.

[11] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc., 2nd Ser.*, 30 (1930), pp. 264-286.

[12] E. Szemerédi, Regular partitions of graphs, Colloques Internationaux C.N.R.S. N$^{\underline{o}}$ 260 - Problèmes Combinatoires et Théorie des Graphes, Orsay (1976), pp. 399-401.