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NOTE

A NOTE ON HYPERGRAPHS WITH THE HELLY-PROPERTY

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Let H be a hypergraph and t a natural number. The sets which can be written as the union of t different edges of H form a new hypergraph which is denoted by H^t. Let us suppose that H has the Helly property and we want to state something similar for H^t. We prove a conjecture of C. Berge and two negative results.

If $H=(V, \mathscr{E})$ is a finite simple hypergraph—the notions and notations of [1] are used throughout the paper—we define H^t as the hypergraph with vertex set V and edges $\bigcup_{k=1}^{t} E_{i_k}$ where $E_{i_k} \in \mathscr{E}$, $i_{k_1} \neq i_{k_2}$ for $k_1 \neq k_2$. ($H^1 = H$ is obvious). In the present note some properties of the transversal number $\mathscr{T}$ of H^t are investigated under the assumption that H has the Helly-property. (H has the Helly property if any pairwise intersecting set of edges has a non-empty intersection). The following theorem was a conjecture of Berge [2, p. 278]:

Theorem 1. *If H has the Helly-property and any $t+1$ edges of H^t have a non-empty intersection, then $\mathscr{T}(H^t) \leq t$.*

Theorem 1 is sharp in the following sense:

Theorem 2. *For every $t \geq 2$, $u \geq 1$ there is a hypergraph H with the Helly property so that any t edges of H^t have a non-empty intersection and $\mathscr{T}(H^t) \geq u$.*

It is natural to ask whether the partial hypergraphs of H^t have similar properties. The answer is negative:

Theorem 3. *For every $t \geq 2$, $u \geq 1$, $k \geq 1$ there is a hypergraph H with the Helly-property and a H' partial hypergraph of H^t so that any k edges of H' have a non-empty intersection and $\mathscr{T}(H') \geq u$.*

In the proofs we use the notion of the representative graph: a graph G is a representative graph (or line-graph) of a hypergraph H if $V(G)$ represents the edges of H and $x, y \in V(G)$ are connected by an edge if and only if the corresponding edges in H have a non-empty intersection. The representative graph of H is denoted by $L(H)$. We need the following simple proposition:

Proposition 1. *For every finite graph G there is a hypergraph H with the Helly-property so that $L(H)$ is isomorphic to G.*

Proof. The dual of the hypergraph of the maximal cliques of G satisfies the requirement.

Proof of Theorem 1. Suppose that H has the Helly-property. G will denote the complement of the graph $L(H)$. We consider the set

$$A=\{x: x \in V(G), d(x) \geq t\}$$

where $d(x)$ denotes the degree of vertex x. We prove that $|A|<t$. Suppose on the contrary that $|A| \geq t$—in that case we can choose different vertices $x_1, x_2, \ldots, x_t$ from $V(G)$ so that $d(x_i) \geq t$ for $i=1,2, \ldots, t$. The set $Y_i \subset V(G)$ is defined for $i=1,2, \ldots, t$ so that $|Y_i|=t$ and $(y, x_i) \in E(G)$ for $y \in Y_i$. We define $Y_0=\{x_1, x_2, \ldots, x_t\}$ and we consider the edges $E_i=\bigcup_{y \in Y_i} y$ for $i=0,1, \ldots, t$ of H^t. (Here y denotes the edge of H corresponding to the vertex y in $L(H)$). It is easy to check that $\bigcap_{i=0}^{t} E_i=\emptyset$ which is a contradiction. We proved therefore that $|A|<t$ which indicates $\chi(G) \leq t$ according to a theorem of Tomescu [1, p. 431]. $\chi(G) \leq t$ means that the vertices of $L(H)$ can be covered with at most t complete graphs and from that $\mathcal{T}(H) \leq t$ because H has the Helly-property. $\mathcal{T}(H) \leq t$ indicates $\mathcal{T}(H^t) \leq t$.

Proof of Theorem 2. Let G be a graph without cycles of length $\geq t^2$ and $\chi(G) \geq u+t$. The existence of such a graph follows from [3]. Let H be a hypergraph with the Helly-property so that $L(H)$ and the complement of G are isomorphic. H exists by Proposition 1. It is clear that $\mathcal{T}(H) \geq u+t$.

First we prove that $\mathcal{T}(H^t) \geq u$. If $\mathcal{T}(H^t)<u$, then there exists a transversal of at most $u-1$ elements in H^t which means that at most $t-1$ edges of H are disjoint from that transversal i.e. $\mathcal{T}(H)<u+t-1$—contradiction.

Now we show that any t edges of H^t have a non-empty intersection. Let

$$E_1'=\bigcup_{j=1}^{t} E_{1j},\ E_2'=\bigcup_{j=1}^{t} E_{2j}, \ldots, E_t'=\bigcup_{j=1}^{t} E_{1j}$$

be t edges of H^t ($E_{ij} \in \mathscr{E}(H)$ for $1 \leq i, j \leq t$). y_{ij} denotes the vertex in $L(H)$ which corresponds to E_{ij}. The complement of the subgraph in $L(H)$ spanned by $Y=\{y_{ij}: 1 \leq i, j \leq t\}$ contains no cycles because $|Y| \leq t^2$. Therefore we can order Y so that for any $y \in Y$ there is at most one $y' \in Y$ for which $y<y'$ and

$(y, y') \notin \mathscr{E}(L(H))$. We choose vertices from Y consecutively by the following algorithm:

Step 1: $Y' = \emptyset$.

Step 2: If there is a $y_{ij} \in Y - Y'$ so that y_{ij} is connected with every vertex of Y' and $y_{kl} \in Y'$ implies $i \neq k$, then the smallest y_{ij}—in the ordering defined above—is added to Y' and we repeat Step 2. If we can not choose y_{ij}, we stop.

We prove that Y' contains a vertex y_{ij_i} for every $1 \leq i \leq t$. Suppose, on the contrary that there is one index i_0 so that the (distinct) vertices $y_{i_0 1}, y_{i_0 2}, \ldots, y_{i_0 t}$ are not in Y'. For every k, $y_{i_0 k}$ there is a $y \in Y'$ so that $y < y_{i_0 k}$ and the edge $(y, y_{i_0 k})$ is not in $L(H)$ otherwise the algorithm would have added $y_{i_0 k}$ at some step to Y'. $|Y'| < t$ therefore there exists k_1, k_2 and $y \in Y'$ so that $(y, y_{i_0 k_1})$ and $(y, y_{i_0 k_2})$ are not edges of $L(H)$ and $y < y_{i_0 k_1}$, $y < y_{i_0 k_2}$ which contradicts the ordering of Y.

We conclude that Y' represents every edge E_i'—on the other hand Y' defines a complete graph which shows that the edges of H corresponding to Y' have a non-empty intersection. That means $\bigcap_{i=1}^{t} E_i' \neq \emptyset$.

Proof of Theorem 3. The graph G is defined as follows: Let G_1 be a graph for whcih $\chi(G_1) \geq 2u$ and without cycles of length $\leq k$. The vertices of G are placed in a matrix M which has t rows and $|V(G_1)|$ columns. The edges of G are defined by the first two rows of M: $\bar{G}_1$ is placed in the first and second row of M so that the vertices in the same column correspond to each other in an isomorphism. All edges between the first and second row of M are added to G. The hypergraph H is defined according to Proposition 1 so that $L(H)$ is isomorphic to G. $H' \subset H^t$ is defined as the unions of edges corresponding to the columns of G. It is easy to see that H' has the properties required in Theorem 3.

References

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