Size of monochromatic double stars in edge colorings

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Abstract

We show that in every r-coloring of the edges of K_n there is a monochromatic double star with at least $\frac{n(r+1)+r-1}{r^2}$ vertices. This result is sharp in asymptotic for $r = 2$ and for $r \geq 3$ improves a bound of Mubayi for the largest monochromatic subgraph of diameter at most three. When r-colorings are replaced by local r-colorings, our bound is $\frac{n(r+1)+r-1}{r^2+1}$.

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1 Introduction

An easy exercise - in fact a note of Erdős and Rado - is that in every 2-coloring of the edges of K_n there is a monochromatic connected subgraph on n vertices. In other words, there is a monochromatic spanning tree in every 2-coloring of the edges of a complete graph. One can also require further properties, spanning trees of radius two and spanning trees having only at most one vertex of degree at least three can be found monochromatically in every 2-coloring of K_n as shown in [3]. Also, Burr proved ([2]) that there is a monochromatic spanning broom (a path with a star at one end) in every 2-coloring of K_n (see also [11]). Largest monochromatic subgraphs with further properties have been also investigated, such as given diameter, [6], [15], given connectivity, [4], [14]. For three colors the order of the largest monochromatic subtree was determined in [9],[1]. The generalization for r colors have been proved by the first author [13]:

Theorem 1. *If the edges of K_n are colored with r colors then there is a monochromatic subtree with at least $\frac{n}{r-1}$ vertices.*

Theorem 1 is sharp if $r-1$ is a prime power and $(r-1)^2$ divides n. An important generalization has been obtained by Füredi [7]. The proof of Theorem 1 in [13] was based on the following lemma.

Lemma 1. *In every r-coloring of a complete bipartite graph on n vertices there is a monochromatic subtree with at least $\frac{n}{r}$ vertices.*

A *double star* is the tree obtained from two vertex disjoint stars by connecting their centers. The next lemma of Mubayi [15], found also independently by Liu, Morris and Prince [14], generalizes Lemma 1.

Lemma 2. *In every r-coloring of a complete bipartite graph on n vertices there is a monochromatic double star with at least $\frac{n}{r}$ vertices.*

A corollary of Lemma 2 is that in any r-coloring of K_n either all color classes have just one component or there is a monochromatic double star with at least $\frac{n}{r-1}$ vertices. This naturally raises the question to find $f(n,r)$, the maximum m such that there is a monochromatic double star with m vertices in any r-coloring of K_n. We shall use the proof method (averaging) of Lemma 2 to get our main result, Theorem 2. Note that finding $f(n,r)$ is different from finding the Ramsey number of a *fixed* double star, determined in [8].

Theorem 2. *For $r \geq 2$ there is a monochromatic double star with at least $\frac{n(r+1)+r-1}{r^2}$ vertices in any r-coloring of the edges of K_n.*

The bound in Theorem 2 is close to best possible for $r = 2$: 2-colorings of K_n where the size of the largest monochromatic double star is asymptotic to $\frac{3n}{4}$ can be obtained from random graphs or from Paley graphs. In [5] (Theorem 2) the existence of such a 2-coloring is proved by the random method. (In fact, the proof of Theorem 2 for $r = 2$ is also implicitly in [5].) However, for $r \geq 3$ the random method seems to fail to provide good bounds for $f(n,r)$ and it is conceivable that $f(n,r) = \frac{n}{r-1}$, a good test case would be $r = 3$.

Observing that a double star has diameter at most three, the bound in Theorem 2 provides an improvement (for $r \geq 3$) of a result of Mubayi [15], who proved that there is a monochromatic subgraph of diameter at most three with at least $\frac{n}{r-1+1/r}$ vertices in every r-coloring of K_n. (For $r = 2$ one can find a monochromatic subgraph of diameter at most three spanning all the n vertices, see in [15].)

The size of the largest monochromatic connected subgraph for *local* r-colorings (where the number of colors is arbitrary but the edge set incident to any vertex is r-colored) is determined in [12]: it is $\frac{rn}{r^2-r+1}$ and this bound is sharp if a finite projective plane of order $r-1$ exists and $r^2 - r + 1$ divides n. Our second result, Theorem 3, is the local variant of Theorem 2.

Theorem 3. *For $r \geq 2$ there is a monochromatic double star with at least $\frac{n(r+1)+r-1}{r^2+1}$ vertices in any local r-coloring of the edges of K_n.*

As in Theorem 2, we could not close the gap between the upper bound $\frac{rn}{r^2-r+1}$ and the lower bound of Theorem 3, except for $r = 2$ when the upper bound gives the right answer and it can be proved easily.

Theorem 4 *In every local 2-coloring of K_n there is a monochromatic double star with at least $\lceil \frac{2n}{3} \rceil$ vertices. This bound is sharp for every n.*

2 Proofs

Proof of Theorem 2. We show the existence of a monochromatic double star with $M = \frac{n(r+1)+r-1}{r^2}$ vertices in an r-colored K_n. Let p be a vertex of K_n and let A_i denote the set of vertices adjacent to p in color i ($i \in [r]$). We may assume that any vertex $a \in A_i$ sends less than $M - |A_i| - 1$ edges of color i to $\cup_{j \neq i} A_j$ otherwise we have a monochromatic double star in color i with M vertices. Consider the r-partite graph G with partite classes A_i obtained by the removal of edges of color i going out of A_i (for all $i \in [r]$). From the previous remark and from the Cauchy-Schwartz inequality we get

$$2|E(G)| > \sum_{i=1}^{r} |A_i| \left(n - 1 - |A_i|\right) - 2\sum_{i=1}^{r} |A_i| \left(M - 1 - |A_i|\right) =$$

$$= \sum_{i=1}^{r} |A_i|(n + |A_i| + 1 - 2M) = \sum_{i=1}^{r} |A_i|^2 + (n-1)(n+1-2M) \geq$$

$$\geq \frac{(n-1)^2}{r} + (n-1)(n+1-2M) \tag{1}$$

Notice that for $r = 2$ the last line of (1) is zero. This is a contradiction (since $|E(G)| = 0$ for $r = 2$), proving the theorem for $r = 2$. Thus assume $r \geq 3$.

Define $G(i,j)$ as the bipartite subgraph of G spanned by $[A_i, A_j]$. Let $d_k(v, H)$ denote the degree of v in color k in the graph H. For any edge $e = xy$ of color k, $x \in A_i, y \in A_j$, we define

$$s_{ijk}(x,y) = d_k(x,G) + d_k(y, G(i,j)), t_{ijk}(x,y) = d_k(x, G(i,j)) + d_k(y,G).$$

Notice that this definition ensures that there are two double stars, $S_{ijk}(x,y)$, resp. $T_{ijk}(x,y)$ of color k in G with $s_{ijk}(x,y)$, resp. $t_{ijk}(x,y)$ vertices. We estimate the sum of $s_{ijk}(x,y) + t_{ijk}(x,y)$ over all edges of G. Using the Cauchy-Schwartz inequality and (1) we get :

$$\sum_{1 \leq i < j \leq r} \sum_{k \neq i,j} \sum_{xy \in E(G(i,j))} s_{ijk}(x,y) + t_{ijk}(x,y) =$$

$$= \sum_{i=1}^{r} \sum_{x \in A_i} \sum_{k \neq i} d_k^2(x,G) + \sum_{i=1}^{r} \sum_{j \neq i} \sum_{k \neq i, k \neq j} \sum_{x \in A_i} d_k^2(x, G(i,j)) \geq$$

$$\geq \frac{\left(\sum_{i=1}^{r} \sum_{x \in A_i} \sum_{k \neq i} d_k(x,G)\right)^2}{(r-1) \sum_{i=1}^{r} |A_i|} + \frac{\left(\sum_{i=1}^{r} \sum_{j \neq i} \sum_{k \neq i, k \neq j} \sum_{x \in A_i} d_k(x, G(i,j))\right)^2}{(r-1)(r-2) \sum_{i=1}^{r} |A_i|} =$$

$$= \frac{(2|E(G)|)^2 (r-2) + (2|E(G)|)^2}{(r-1)(r-2)(n-1)} = \frac{(2|E(G)|)^2}{(r-2)(n-1)} >$$

$$\frac{2|E(G)|(\frac{(n-1)^2}{r} + (n-1)(n+1-2M))}{(r-2)(n-1)} = 2|E(G)|M.$$

Since altogether we summed the cardinalities of the vertex sets of $2|E(G)|$ monochromatic double stars ($S_{ijk}(x,y)$ and $T_{ijk}(x,y)$), for some $k \in [r]$, $x \in A_i, y \in A_j$, either $|V(S_{ijk}(x,y))|$ or $|V(T_{ijk}(x,y))|$ is at least M, proving the theorem. $\square$

Proof of Theorem 3. The proof follows the proof of Theorem 2 with obvious modifications, now $M = \frac{n(r+1)+r-1}{r^2+1}$. We use the same notation. Inequality (1) remains the same. Using $I(x)$ for the set of colors appearing on the edges incident to vertex x, the argument of the proof of Theorem 2 is followed. A difference worth noting is that in a local r-coloring an edge $xy, x \in A_i, y \in A_j$ of color k with $k \in I(x) \setminus \{i,j\}$

implies that k can have $r-1$ distinct values (in contrast to the ordinary r-coloring, where k can have only $r-2$ values). Now the "local" variant of the argument is as follows.

$$\sum_{1\le i<j\le r}\sum_{k\neq i,j}\sum_{xy\in E(G(i,j))} s_{ijk}(x,y)+t_{ijk}(x,y)=$$

$$=\sum_{i=1}^{r}\sum_{x\in A_i}\sum_{k\in I(x)\setminus\{i\}} d_k^2(x,G)+\sum_{i=1}^{r}\sum_{j\neq i}\sum_{k\in I(x)\setminus\{i,j\}}\sum_{x\in A_i} d_k^2(x,G(i,j))\ge$$

$$\ge\frac{\left(\sum_{i=1}^{r}\sum_{x\in A_i}\sum_{k\in\{I(x)\setminus i\}} d_k(x,G)\right)^2}{(r-1)\sum_{i=1}^{r}|A_i|}+\frac{\left(\sum_{i=1}^{r}\sum_{j\neq i}\sum_{k\in\{I(x)\setminus\{i,j\}\}}\sum_{x\in A_i} d_k(x,G(i,j))\right)^2}{(r-1)^2\sum_{i=1}^{r}|A_i|}=$$

$$=\frac{(r-1)(2|E(G)|)^2+(2|E(G)|)^2}{(r-1)^2(n-1)}=\frac{r(2|E(G)|)^2}{(r-1)^2(n-1)}>$$

$$>\frac{2|E(G)|r\left(\frac{(n-1)^2}{r}+(n-1)(n+1-2M)\right)}{(r-1)^2(n-1)}=2|E(G)|M.$$

As before, we summed the cardinalities of the vertex sets of $2|E(G)|$ monochromatic double stars, ($S_{ijk}(x,y)$ and $T_{ijk}(x,y)$), thus for some $k\in[r]$, $x\in A_i, y\in A_j$, either $|V(S_{ijk}(x,y))|$ or $|V(T_{ijk}(x,y))|$ is at least M. $\square$

Proof of Theorem 4. It is easy to see ([10]) that a local 2-coloring of K_n is one of three types: case A. a 2-coloring; case B. the vertices of K_n are partitioned into $m\ge 3$ parts $A_{12},\dots,A_{1m}$ all edges within A_{1i} are colored with color 1 or color i, edges between A_{1i}, A_{1j} are colored with color 1; case C. the vertices of K_n are partitioned into three parts A_{12}, A_{13}, A_{23}, edges within A_{ij} are colored with i or j and cross edges are colored with the color of the intersection of their index pairs. In case A. there is monochromatic double star with at least $\frac{3n}{4}\ge\lceil\frac{2n}{3}\rceil$ vertices. In case B. there is a monochromatic double star in color 1 spanning all vertices. In case C. the two largest sets, say A_{12}, A_{13} span a double star of the required size in color 1. This proves the lower bound, and any local 2-coloring according to case C. with evenly distributed sets shows that equality is possible. $\square$

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