Corrigendum to “Three-color Ramsey numbers for paths” [Combinatorica 27 (1) (2007), pp. 35-69.]

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In February 2007 Fabricio Benevides [1] reported an easily correctable error in the proof of our main result. We wrote ([2], p. 2, lines $18-22$) that one type of extremal colorings comes form an equal part blow up of a faktorization of K_4. In fact, this blow up must not be necessarily equal part. A similar coloring with $|A|, |B|, |C| \geq (1-\alpha_1)\frac{|V(G)|}{4}, |D| \geq \alpha_2|V(G)|, |A|+|D| \geq (1-\alpha_1)\frac{|V(G)|}{2}$ and coloring all edges in A with color 2 gives an extremal coloring, too. The remedy is to relax the condition in Extremal Coloring 1 (EC1) as follows, to allow one unbalanced pair (A, D).

Extremal Coloring 1 (with parameters α_1, α_2, where $\alpha_1 \ll \alpha_2$): There exists a partition $V(G) = A \cup B \cup C \cup D$ such that

- $|A|, |B|, |C| \geq (1-\alpha_1)\frac{|V(G)|}{4}, |D| \geq \alpha_2|V(G)|, |A|+|D| \geq (1-\alpha_1)\frac{|V(G)|}{2}$,
- The bipartite graphs $(A \times B) \cap G_1^*$, $(C \times D) \cap G_1^*$, $(A \times D) \cap G_2^*$, $(B \times C) \cap G_2^*$, $(A \times C) \cap G_3^*$ and $(B \times D) \cap G_3^*$ are all $(1-\alpha_1)$-dense.

The proof of the fact that we can find the desired monochromatic path of length n in case we have this relaxed Extremal Coloring 1 is similar to the proof for the original EC1. For the sake of completeness we restate and prove Lemma 5 from our paper here .

Lemma 1. *For every $0 < \alpha_1 \ll \alpha_2 \ll 1$ there exists a positive integer $n_0 = n_0(\alpha_1, \alpha_2)$ such that the following is true for $n \geq n_0$. If a 3-edge coloring (G_1, G_2, G_3) of $K_{r(n)}$ is an Extremal Coloring 1 (EC1) with parameters α_1, α_2 then there is a monochromatic path of length n.*

Proof: First we will remove certain exceptional vertices (denote their set by E) from the four sets A, B, C, D in EC1. A vertex $v \in A$ is **exceptional** if one of the following is true:

$$deg_{G_1}(v, B) < (1 - \sqrt{\alpha_1})|B|, deg_{G_2}(v, D) < (1 - \sqrt{\alpha_1})|D|,$$

$$\text{or } deg_{G_3}(v, C) < (1 - \sqrt{\alpha_1})|C|.$$

From the density conditions in EC1 it follows that the number of these exceptional vertices is at most $3\sqrt{\alpha_1}|A|$. We remove these vertices from A and add them to E. Similarly, for the other three sets we define exceptional vertices and add them to E. Thus altogether (since we have at most $2n$ vertices)

$$|E| \leq 24\sqrt{\alpha_1}n. \tag{1}$$

Next we redistribute these vertices among the 4 sets in such a way that we are not creating new exceptional vertices. Let us take the first exceptional vertex v from E, the procedure will be similar for the other vertices. Consider the G_1-neighbors of v. We may assume that these neighbors are either all in $A \cup B$, or in $C \cup D$ (say they are in $A \cup B$). Indeed, otherwise we can connect $A \cup B$ with $C \cup D$ in color G_1 through v and this would give a monochromatic path in G_1 of length more than n (applying Lemma 4 from [2] inside the bipartite graphs $A \times B$ and $C \times D$ and using $\alpha_1 \ll \alpha_2$). Hence, all the edges between $C \cup D$ and v are in colors G_2 and G_3. By a similar reasoning, we may assume that v does not have G_2 neighbors in both $A \cup D$ and $B \cup C$, and it does not have G_3 neighbors in both $A \cup C$ and $B \cup D$. Thus either all the edges in $C \times \{v\}$ are in G_2, and all the edges in $D \times \{v\}$ are in G_3, or the other way around. Say we have the first case. Then all the edges in $A \times \{v\}$ are in G_1 and we may safely add v to B.

We repeat this procedure for all the exceptional vertices in E. Let us consider the largest set (say A) of the four sets A, B, C and D.

Claim 1. If $|B| \geq \lfloor \frac{n}{2} \rfloor$, then there is a monochromatic path of length n in color G_1 in the bipartite graph $G_1|_{A \times B}$.

Proof of Claim 1: If n is even, then take arbitrary subsets $A' \subseteq A$, $B' \subseteq B$ with $|A'| = |B'| = \frac{n}{2}$. Applying Lemma 4 for $G_1|_{A' \times B'}$ (the conditions of the lemma are satisfied with much room to spare) we get a monochromatic path of length n in color G_1.

If n is odd, then we must have $|A| \geq \frac{n+1}{2}$, since we have $2n-1$ vertices. Then take arbitrary subsets $A' \subseteq A$, $B' \subseteq B$ with $|A'| = \frac{n+1}{2}$, $|B'| = \frac{n-1}{2}$. Again applying Lemma 4 we can find a Hamiltonian path in $G_1|_{A' \times B'}$ beginning and ending in A'. This gives the desired monochromatic path of length n in color G_1 and proves Claim 1.

Thus we may assume that

$$|B|, |C|, |D| < \lfloor \frac{n}{2} \rfloor. \tag{2}$$

At this point we consider the colors of the edges inside A. If for the density of the G_1-edges inside A we have $d(G_1|_A) \geq \sqrt[3]{\alpha_1}$, then using $\alpha_1 \ll 1$ we can clearly find a path P_1 in $G_1|_A$ that has length

$$p = \min(|A| - |B|, 2(\lceil n/2 \rceil - |B|)).$$

Remove this path from A except for one of the endpoints u. In case we have $p < |A| - |B|$ we remove some more vertices from A until we have exactly $|B|$ vertices left. Denote the resulting set in A by A'. Then in both cases $|A'| = |B|$. Again applying Lemma 4 we can find a Hamiltonian path P_2 in $G_1|_{A' \times B}$ starting with u. P_1 together with P_2 gives us the desired path P in $G_1|_{A \cup B}$. Indeed, in case $p = 2(\lceil n/2 \rceil - |B|)$, P trivially has length at least n. In case $p = |A| - |B|$, P is a Hamiltonian path in $G_1|_{A \cup B}$. By (2), in case n is even we get

$$|C| + |D| = 2n - 2 - |P| \leq 2\left(\frac{n}{2} - 1\right) = n - 2,$$

and in case n is odd we get

$$|C| + |D| = 2n - 1 - |P| \leq 2\frac{n-1}{2} = n - 1.$$

Thus in both cases

$$|P| \geq n,$$

and thus P is a monochromatic path of length at least n.

Thus we may assume $d(G_1|_A) < \sqrt[3]{\alpha_1}$. Similarly we may assume $d(G_3|_A) < \sqrt[3]{\alpha_1}$, otherwise we can find a path of length at least n in $G_3|_{A \cup C}$. This implies that $d(G_2|_A) > (1 - 2\sqrt[3]{\alpha_1})$. From this and $\alpha_1 \ll \alpha_2$ it easily follows that the monochromatic subgraph $G_2|_{A \cup D}$ satisfies the Pósa-condition (for nondecreasing degree sequence $d_k \geq k+1$ for all $k < \frac{|A \cup B|}{2}$ see [3]) and thus has a Hamiltonian path.

Similarly as above, from (2) it follows that this path has length at least n, completing the proof of the lemma. □

The reason to relax EC1 is that in Subcase 1.2 the statement about the size of the matching N_i (at least $m_i - \sqrt{\epsilon}n$) is valid only if the condition $m_i \leq |X_4| - 2|M_i|$ holds. If this condition is not true for some i, say for $i = 1$ then we easily get $|X_4| + m_1 < (\frac{1}{2} + 4\eta)n$ which implies $m_2 + m_3 > \frac{n}{2} - 4\eta n$. Since we know that $m_2, m_3 \leq \frac{n}{4} + 2\eta n$, this implies that we have the relaxed EC1. The authors thank F. Benevides the careful reading of their manuscript.

References

[1] F. Benevides, private communication.

[2] A. Gyárfás, M. Ruszinkó, G. N. Sárközy and E. Szemerédi, Three-color Ramsey number for paths, Combinatorica 27 (1) (2007), pp. 35-69.

[3] L. Pósa, A theorem concerning Hamilton lines, *Publ. Math. Inst. Hung. Acad. Sci.* 7 (1962) 225-226.