Disjoint chorded cycles in graphs

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Abstract

We propose the following conjecture to generalize results of Pósa and Corrádi - Hajnal. Let r, s be nonnegative integers and let G be a graph with $|V(G)| \geq 3r + 4s$ and minimal degree $\delta(G) \geq 2r + 3s$. Then G contains a collection of $r + s$ vertex disjoint cycles, s of them with a chord. We prove the conjecture for $r = 0, s = 2$ and for $s = 1$. The corresponding extremal problem, to find the minimum number of edges in a graph on n vertices ensuring the existence of two vertex disjoint chorded cycles is also settled.

1 Introduction

Pósa proved (see in [6], problem 10.2) that any graph with minimum degree at least 3 contains a chorded cycle, i.e. a cycle with at least one chord and the same is true for any graph with $n \geq 4$ vertices and at least $2n - 3$ edges. Corrádi and Hajnal [3] proved that minimum degree at least $2r$ ensures that any graph with $n \geq 3r$ vertices contains r vertex disjoint cycles. For some related results see [1],[2],[5],[4].

We propose the following natural common generalization of the previous results.

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Conjecture 1. *Let r, s be nonnegative integers and let G be a graph with $|V(G)| \geq 3r + 4s$ and minimal degree $\delta(G) \geq 2r + 3s$. Then G contains a collection of r cycles and s chorded cycles, all vertex disjoint.*

Notice that $K_{2r+3s-1,n-2r-3s+1}$ shows shows that the the minimum degree can not be lowered if $n - 2r - 3s + 1 \geq 2r + 3s - 1$, i.e. if $n \geq 4r + 6s - 2$. In this paper we prove the conjecture for $r = 0, s = 2$ and for $s = 1$ and every non-negative r. Then we use these results to find the maximum number of edges in graphs that do not contain $r + s$ vertex disjoint cycles s of them chorded.

Theorem 1. *Let G be a graph with $|V(G)| \geq 8$ and minimum degree $\delta(G) \geq 6$. Then G contains two vertex disjoint chorded cycles.*

Proof. Let P be a maximal path in G. If $|V(P)| < 8$ then it is immediate that $|V(P)| = 7$ and that $|V(G)| > |V(P)| = 7$. So $G - P$ is nonempty. If $d(v; P) \geq 4$ for some $v \in V(G - P)$, then v either has two adjacent vertices of P as neighbors, or has an endpoint of P as a neighbor. Since both of these contradicts the maximality of P, we may assume $d(v; P) \leq 3$ for all $v \in V(G - P)$. Hence, both P and $G - P$ contain a chorded cycle.

Now assume that $|V(P)| \geq 8$, and $P = v_1 v_2 \dots v_k$. We treat separately the cases where $v_1 v_k$ is or is not an edge of G.

Case 1. $v_1 v_k$ is not an edge of G.

By hypothesis, $d(v_1; v_3 \dots v_{k-1}), d(v_k; v_2 \dots v_{k-2}) \geq 5$. Let $v_i \in V(P)$ be the vertex such that $d(v_1; v_3 \dots v_{i-1}) = 2$ and $d(v_1; v_3 \dots v_i) = 3$. Similarly, let v_j be such that $d(v_k; v_{k-2} \dots v_{j+1}) = 2$ and $d(v_k; v_{k-2} \dots v_j) = 3$. Observe that each of the vertex sets $\{v_1, \dots, v_{i-1}\}$, $\{v_1, v_i, \dots, v_{k-1}\}$, $\{v_k, v_{k-1}, \dots, v_{j+1}\}$, and $\{v_k, v_2, v_3, \dots, v_j\}$ induces a subgraph of G that contains a chorded cycle. If $i \leq j$, this allows us to take the disjoint subsets $\{v_1, \dots, v_{i-1}\}$ and $\{v_k, v_{k-1}, \dots, v_{j+1}\}$ of $V(P)$, which immediately gives two disjoint chorded cycles contained in the graph induced by $V(P)$. If $j < i$, then $\{v_1, v_i, \dots, v_{k-1}\}$ and $\{v_k, v_2, v_3, \dots, v_j\}$ are disjoint subset of $V(P)$, giving the same result.

Case 2. $v_1 v_k$ is an edge of G.

In this case, $V(P)$ induces a cycle of length k, so no vertex of P may have a neighbor outside $V(P)$, else the maximality of P is violated. So for all $v \in V(P)$ we have $d(v; P) \geq 6$. We can assume that $v_l v_{l+2}$ is not an edge of G for some l, else we immediately have two vertex disjoint chorded cycles contained in G. Relabel the vertices of P so that $v_l = v_k$, $v_{l+1} = v_1$, and $v_{l+2} = v_2$. Let $v_i \in V(P)$ be such that $d(v_2; v_4 \dots v_{i-1}) = 1$ and $d(v_2; v_4 \dots v_i) = 2$. Note that $d(v_2; v_i \dots v_{k-1}) \geq 3$. Similarly, let v_j be such that $d(v_k; v_3 \dots v_{j-1}) = 2$ and $d(v_k; v_3 \dots v_j) = 3$. It is easy to see that if $j < i$ or $i < j$ then we have two vertex disjoint chorded cycles contained in the graph induced by $V(P)$. Assume that $i = j$. Let $v_t \in V(P)$ be such that $d(v_1; v_3 \dots v_{t-1}) = 1$ and $d(v_1; v_3 \dots v_t) = 2$. If $t \geq i$ the sets $\{v_2, \dots, v_{i-1}\}$ and $\{v_1, v_k, \dots, v_i\}$ each induce a subgraph containing a chorded cycle. If $t < j$ then $\{v_k, \dots, v_j\}$ and $\{v_1, \dots, v_{j-1}\}$ each induce a subgraph containing a chorded cycle. Hence, there are two vertex disjoint chorded cycles in the subgraph induced by $V(P)$. $\square$

Theorem 2. *Let G be a graph with $|V(G)| \geq 3r + 4$ and minimum degree $\delta(G) \geq 2r + 3$. Then G contains $r + 1$ vertex disjoint cycles, one with a chord.*

Proof. Choose a vertex $v \in V(G)$, and let G' be the graph induced by $V(G) - v$. Then $\delta(G') \geq 2r + 2$ and $|V(G')| \geq 3r + 3$, so by the Corrádi - Hajnal result it contains $r + 1$ independent cycles, spanning a subgraph H in G. Let P be a maximal path in $G - H$. If P has one vertex only then it sends $2r + 3 > 2(r + 1)$ vertices to H thus at least three edges to some cycle of H and the proof is finished. If $d(w; P) \geq 3$ for an endpoint w of P then there is chorded cycle inside P and again the proof is finished. Otherwise each endpoint of P sends at least $2r + 1$ edges to H and we conclude as before that some cycle C in H receives at least three edges which easily gives a chorded cycle in $C \cup P$. $\square$

Now we proceed to Turán type problems for two vertex disjoint cycles when one or both are chorded. Let $f(n)$ ($g(n)$) be the smallest number of edges in a graph of n vertices that ensures two vertex disjoint cycles one of them (both of them) chorded. The inductive step of the following result is easy, the difficulty is to prove the anchoring cases which will be done in Theorem 4.

Theorem 3. *For $n \geq 10$, $f(n) = 4n - 15$ and for $n \geq 12$, $g(n) = 5n - 24$.*

Proof. Suppose the theorem were true for $n < N$, and take G with $|V(G)| = N$ and $|E(G)| = 4N - 15$. If G contains a vertex v with $d(v) \leq 4$ then the graph $G' = G - v$ contains two vertex disjoint cycles, one is chorded by the inductive hypothesis. If, on the other hand, $\delta(G) \geq 5$ then G contains two vertex disjoint cycles, one is chorded by Theorem 2.

Similarly, if G is a graph with $|V(G)| = N$ and $|E(G)| = 5N - 24$ and it contains a vertex v with $d(v) \leq 5$ then the graph $G' = G - v$ contains two vertex disjoint chorded cycles by the inductive hypothesis. If, on the other hand, $\delta(G) \geq 6$ then G contains two vertex disjoint chorded cycles by Theorem 1.

To see that $f(n) > 4n - 16, g(n) > 5n - 25$ consider $K_{4,n-4}$ and $K_{5,n-5}$ respectively. $\square$

Theorem 4. $f(7) = 17, f(8) = 19, f(9) = 22, f(10) = 25; g(8) = 23, g(9) = 25, g(10) = 28, g(11) = 32, g(12) = 36.$

Proof. The values of f, g in the theorem can not be decreased. For f and $8 \leq n \leq 10$ consider $K_{3,n-3}$ with a triangle inside the 3-element partite class. Then $f(7) > 16$ and $g(8) > 22$ are demonstrated by K_6 and K_7 with a pendant edge. To see that $g(9) > 24$, consider $K_{4,5}$ with a $K_{1,4}$ inside the 5-element partite class. Finally, $g(n) > 4n - 13$ is shown by $K_{4,n-4}$ with a $K_{1,3}$ inside the 4-element partite class.

To see that the stated values are giving an upper bound, we proceed by induction. The starting cases, $f(7) = 17$ and $g(8) = 23$, are easy since only very few edges (4 or 5) are missing from the complete graphs K_7, K_8.

Assume G is a graph with the given number of edges that does not contain two vertex disjoint cycles, one of them chorded (in case of f) or both chorded (in case of g) - we simply refer to them as forbidden configurations. If $\delta(G)$ is small, induction applies. Otherwise, when f is considered we have

- $\delta(G) \geq 3$ if $n = 8$,
- $\delta(G) \geq 4$ if $n = 9, 10$.

When g is considered, we have

- $\delta(G) \geq 3$ if $n = 9$,
- $\delta(G) \geq 4$ if $n = 10$,
- $\delta(G) \geq 5$ if $n = 11, 12$.

Using these conditions, one can easily see that in each case we may assume that our graph G is connected. Select a path $P = \{v_1, \ldots, v_k\}$ of maximum length in G. Let $A = \{2 = a_1 < \ldots < a_p\}$ denote the set of indices i for which v_1 is adjacent to v_i. Similarly, let $B = \{b_1 < \ldots < b_q = k - 1\}$ denote the set of indices j for which v_k is adjacent to v_j. The maximality of P and the connectivity assumption on G implies in each case that

- If $v \notin P, a \in A, b \in B$ then $vv_{a-1}, vv_{b+1} \notin E(G)$,
- If $v \notin P$ then $v_1 v_k \notin E(G)$.

Moreover, from the assumption that G has no forbidden configuration, we get

- $b_{q-1} \leq a_3, b_{q-2} \leq a_2$ in case of f,
- $b_{q-2} \leq a_3$ in case of g.

It is not difficult to check that the conditions listed above imply that in each case $k = n$ and G has a Hamiltonian cycle $C = \{0, \ldots, n-1\}$. That brings in enough symmetry to handle the cases. However, the arguments are still not easy, we show them for $f(n)$, $n = 8, 9, 10$ and for $g(12)$. In fact, it would be of some interest to find proofs with less case analysis.

Case 1. $f(8) = 19$. Define X_0 a the set of pairs $17, 27, 35, 36$ and for $i = 1, 2, 3$ let X_i be $X_0 + i$ in mod8 arithmetic. The condition on G implies that G has at most two edges from each $X_i, i = 0, 1, 2, 3$. Therefore G has at least three edges not on C and not on any X_i - i.e. has at least three of the four long diagonals of C. This implies that we may assume that G contains two edges from both X_0 and X_2 and two long diagonals $04, 26$ (otherwise we have two edges from both X_1, X_3 and the two long diagonals $15, 37$). If the two edges of X_0, X_2 are $27, 36$ and $41, 50$ respectively in G then we have two vertex disjoint chorded C_4-s. Thus we may assume that from X_0 we have either $17, 27$ or $35, 36$ in G, by symmetry $17, 27$. This gives two choices: either $13, 14$ or $50, 57$ are in G. In both cases we have two disjoint cycles, the first chorded: $(1, 3, 4, 0), (2, 6, 7)$ (chord 14) or $(1, 2, 6, 7), (0, 4, 5)$ (chord 27).

Case 2. $f(9) = 22$. Define X_0 as the path with edges $81, 17, 72, 26, 63, 35$, and for $i = 1, 2, 3$ let X_i be $X_0 + i$ in $\bmod 9$ arithmetic. The condition on G implies that G has at most three edges on any X_i therefore at least one of the edges of $Y = \{07, 16, 25\}$. By symmetry this is true for all $Y + i$. Moreover no $Y + i$ contains three edges of G thus some (in fact five) of them contains precisely one edge. Thus w.l.o.g. Y contains precisely one edge of G and all the X_i-s contain three. This is possible only if G intersects X_i in a path P_4. Assume $07 \in E(G)$. To avoid the forbidden configuration in $X_0 \cup C \cup 07$, $81, 17, 72 \in E(G)$ follows. Similarly we get $05, 06, 68 \in E(G)$ but now $(0, 5, 6, 8)$ with chord 06 and the triangle $7, 1, 2$ gives contradiction. Assume $16 \in E(G)$. Now it is easy to check that all (16) choices of P_4-s from X_0 and X_2 generate two disjoint cycles, one chorded. Finally, if $25 \in E(G)$ then $81, 17, 72 \in E(G)$ or $17, 72, 26 \in E(G)$ generate the forbidden configuration and this happens also for all (4) combinations of $72, 26, 63$ and $26, 63, 35$ with a P_4 of X_2.

Case 3. $f(10) = 25$. Define X_0 as the set of edges $91, 18, 27, 36, 64$. Then $G - C$ is partitioned into $X_i = X_0 + i$ for $i = 0, 1, 2, 3, 4$ and to the set Y of diagonals $i, i + 4$ (in mod10 arithmetic). It is clear that G can contain at most two edges from each X_i thus must contain at least five edges from Y. We first look at the case when G contains two parallel edges of Y, say $04, 59$.

Clearly none of the five-cycles $C_1 = 0, 1, 2, 3, 4$, $C_2 = 5, 5, 7, 8, 9$ can contain diagonals so G has 13 edges from the bipartite graph B defined by removing $09, 45$ from the complete bipartite graph between the vertex sets of C_1, C_2. Partition B into five subgraphs B_i as follows. $B_1 = \{07, 71, 62, 25, 39, 94, 48\}$, $B_2 = \{47, 73, 82, 29, 15, 50, 06\}$, $B_3 = \{91, 18, 27, 36, 64\}$, $B_4 = \{08, 16, 35\}$, $B_5 = \{38\}$. The "geometry" of this partition shows that G can contain at most $4, 4, 2, 2, 1$ edges from these sets, implying that equality must hold since the sum of these numbers is 13. In particular, G must contain the edge 38 and by symmetry the edge 16 as well. But then, since none of the cycles $0, 4, 5, 9$ and $1, 2, 3, 8, 7, 6$ can be chorded, G must contain nine edges from the union of the following two $K_{2,3}$-s: $[\{0, 4\}, \{6, 7, 8\}]$ and $[\{5, 9\}, \{1, 2, 3\}]$. It follows easily that it is impossible without generating the forbidden configuration.

We conclude that G does not contain two parallel edges from Y. Since Y is the union of five pairs of parallel edges, it follows that G has precisely five edges from Y, one from each parallel pair. It also follows that G has precisely two edges from each X_i. One of the two crossing pentagons of Y must contain at least three edges of G. Two of these must form a path, w.l.o.g $04, 48$. However, it is easy to check that C, $04, 48$ and any two edges of X_2 gives a forbidden configuration. □

Case 4. $g(12) = 36$.

During the proof we count edges of G that are not on C. Define X_0 as the path $1, 11, 2, 10, 3, 9, 4, 8, 5, 7$. Then $G - C$ is partitioned into $X_i = X_0 + i$ for $i = 0, \ldots, 5$. It is obvious that from each path X_i G has at most four edges, thus G can have at most $6 \times 4 + 12 = 36$ edges. Thus G contains precisely four edges from each X_i. It is immediate that to avoid the forbidden configuration, some three of these four edges must form a path Y_i on X_i. The pair completing Y_i to a four-cycle is denoted by z_i.

Lemma 1. *If at least five edges connect two vertex disjoint paths of a graph then there is a chorded cycle in the graph.*

Proof. Label the vertices of the two paths with increasing numbers. Define the graph H with vertices representing the edges between the two paths as follows. Two vertices i, j and k, l are adjacent if i, k and j, l are ordered in the same way or $i = k$ or $j = l$. Orient the edge from i, j to k, l if $i \leq k$ and $j \leq l$. The orientation is obviously transitive so H is a perfect graph. Five vertices in a perfect graph either have a clique or independent set of size three and both represents a chorded cycle. □

First we eliminate the case when some Y_i, say Y_0 is the middle of X_0, i.e. $Y_0 = \{10, 3, 9, 4\}$ and $z_0 = \{4, 10\}$. Partition $V(G)$ into four sets, $A = \{11, 0, 1, 2\}$, $B = \{5, 6, 7, 8\}$, $U_1 = \{9, 10\}$, $U_2 = \{3, 4\}$. The lemma implies that at most 4 edges of G are in $[A, B]$ (since $U_1 \cup U_2$ is a chorded cycle).

We claim that $[U_1, A] \cup [U_2, B] \cup [U_2, A] \cup [U_1, B] \cup z_0$ contains at most 14 edges of $G \setminus C$.

First we prove that $[U_1, A] \cup [U_2, B] \cup z_0$ has at most seven edges. Notice that $[10, A] \cup [4, B]$ have at most one edge. Indeed, if both has at least one then $4, 5, \ldots, 9$ - $3, 10, 11, 0, 1, 2$ is a forbidden pair and if one of them, say $[10, A]$ has at least two then $10, 11, \ldots, 2$ contains a chorded cycle disjoint from $3, 4, \ldots, 9$. Also, both $[9, A]$ and $[3, B]$ can not contain at least two edges. If one of them, say $[9, A]$ contains at least three then $[3, B] \cup [4, B] \cup z_0$ contains at most three giving the statement.

Next we show that $[U_2, A]$ has at most five edges. Indeed, if $[3, A]$ has three edges (without the cycle edge 23) then we can have at most two edges from $[4, A]$ to avoid the forbidden configuration: in case of $4, 11 \in E(G)$ we have the chorded cycles $4, 5, \ldots, 9, 10, 11$ - $0, 1, 2, 3$; in case of $04, 14, 24 \in E(G)$ we have $0, 1, 2, 4$ - $4, 9, 10, 11$. If $[3, A]$ has two edges then $[4, A]$ can not have four: since $3, 11$ is not an edge ($3, 11, 10, 9$ - $4, 0, 1, 2$) we have $30, 31 \in E(G)$ and $0, 1, 2, 3$ - $4, 5, \ldots, 11$ give contradiction.

Thus $[U_2, A]$ and (by symmetry) $[U_1, B]$ has at most five edges. But notice that if $[U_2, A]$ contains at least three edges, there is a chorded cycle in $11, 0, 1, 2, 3, 4$. The same is true for $[U_1, B]$ so one of them has at most two edges and the other has at most five, proving the claim.

Since $G \setminus C$ has at most one edge within A (same is true for B) , and has three edges in Y_0, moreover C has 12 edges, we get with the previous estimates that G has at most $1 + 1 + 3 + 12 + 4 + 14 = 35$ edges - contradiction.

Similar - but slightly more complicated - argument works for the case when Y_0 is "next to the middle", $Y_0 = \{2, 10, 3, 9\}$. The partition of $V(G)$ into four sets is similar, $A = \{11, 0, 1\}$, $B = \{4, 5, 6, 7, 8\}$, $U_1 = \{9, 10\}$, $U_2 = \{2, 3\}$. First we show that $A, B, [A, B]$ altogether contain at most 5 edges. Notice that B can contain at most two edges with equality in two ways: $47, 58$ or $46, 68$. If A has one edge then the triangle $0, 1, 11$ sends at most two edges to B. If A has no edge and B has one, we are done by the lemma. If B has two edges then we have one of the two graphs on B described before. With one exception, any two distinct vertices x, y of B is connected by a chorded path in B implying that x and y cannot send an edge to A at the same time. The exceptional case is when $x = 5, y = 7$ and $47, 58$ are edges. It is easy to check that $\{x, y\}$ sends at most three edges to A.

First we prove that $[U_2, A] \cup [U_1, B] \cup z_0$ has at most 7 edges. Like before, $[9, B] \cup [2, A]$ has at most one edge. Also, both $[10, B]$ and $[3, A]$ can not contain at least two edges. If $[3, A]$ contains three edges then $[10, B]$ contains at most two and the statement follows immediately. If $[10, B]$ contains at least three then $[3, A] \cup z_0$ contains at most one edge. Indeed, if $z_0, 3i$ are in G then $9, 2, \ldots, i, 3$ is a chorded cycle, if there are two edges of G in $[3, A]$ then these edges with the path $11, \ldots, 2, 3$ define a chorded cycle. In both cases the three edges of $[10, B]$ span a disjoint chorded cycle and the inequalities imply the statement.

Next we claim that $[U_1, A] \cup [U_2, B]$ has at most 8 edges. Indeed, if $[U_1, A]$ has at least three edges then $U_1 \cup A$ has a chorded cycle, thus $[U_2, B]$ has at most two edges. The claim follows since $[U_1, A]$ contains at most 5 edges. If $[U_1, A]$ has two edges and $U_1 \cup A$ has a chorded cycle, the claim follows as before, otherwise $1, 10$ and $0, 9$ are edges. Now $0, 1, 2, 10, 11$ is chorded implying that $[3, B]$ has no edges (otherwise $3, 4, \ldots, 9$ is chorded). The same argument works also if $[10, A]$ has an edge. Thus $[U_2, B]$ has at most 5 edges and the claim follows. If $[U_1, A]$ has one edge,

$9i$, then the cycle $3, 10, 11, \dots, i, 9$ is chorded so $[2, B]$ contains at most two edges. Finally, if $[U_1, A]$ has no edge and $[U_2, B]$ has 9 then we get two disjoint chorded cycles: $2, 8, 9, \dots, 1$ and $3, 4, 5, 6, 7$.

Putting together the previous estimates, G has at most $3 + 5 + 7 + 8 + 12 = 35$ edges - contradiction.

Thus we may assume that no Y_i is in middle or near middle position. Thus all of them must be selected at "peripheral" position (the shifts of $1, 11, 2, 10$ and $11, 2, 10, 3$).

We eliminate the case when some Y_i, say Y_0 contains two P_4-s. Apart from symmetry this can happen only if Y_0 is the path $1, 11, 2, 10, 3$.

Observe that $0, 1, 2, 11$ spans a chorded cycle so the cycle $3, 4, \dots, 10$ has no diagonal. Since G has minimum degree at least five, 3 is adjacent to at least two vertices in $\{11, 0, 1\}$ and 10 is adjacent to at least one vertex in $\{0, 1\}$. This ensures that $A = \{0, 1, 2, 3, 10, 11\}$ spans a subgraph $G[A]$ in G such that the deletion of any vertex of A leaves a chorded cycle in $G[A]$. This implies that $v \in \{0, 1, 2, 11\}$ sends at most two edges to $\{4, 5, 6, 7, 8, 9\}$, altogether at most eight edges go between those sets. There are 12 edges on C and at most ten further edges within A. Thus G has at most $8 + 12 + 10$ edges - contradiction.

We conclude that G must contain precisely one peripheral Y_i from each X_i. There are 6×4 choices for these positions. Define a graph H with vertices as positions (four for each X_i) and with edges if two positions are excluded because they define two vertex disjoint chorded cycles (with the edges of C). The graph obtained has two 12-cycles, an outer ring A and an inner ring B. The outer ring contains diagonals $(i, i + 2)$, $(i, i + 3)$ and $(i, i + 6)$. The inner ring has diagonals $(i, i + 4)$. Vertex i on the inner ring is adjacent to vertices $i - 1, i, i + 1, i + 4$ on the outer ring. A moments reflection shows that there is no independent set of size six in H and this concludes the proof. □

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