# Size of monochromatic components in local edge colorings

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**Abstract**

An edge coloring of a graph is a local $r$ coloring if the edges incident to any vertex are colored with at most $r$ distinct colors. We determine the size of the largest monochromatic component that must occur in any local $r$ coloring of a complete graph or a complete bipartite graph.

An easy exercise - in fact a note of Paul Erdős - is that in every 2-coloring of the edges of $K_n$ there is a monochromatic connected subgraph on $n$ vertices. For three colors the analogue problem was solved in [9],[1]. The problem was rediscovered in [2]. The generalization of this for $r$ colors is proved by the first author [10]: if the edges of $K_n$ are colored with $r$ colors then there is a monochromatic connected component

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with at least $\frac{n}{r-1}$ vertices. This result also follows from a more general result of Füredi [7]. The result is sharp if $r-1$ is a prime power and $r-1$ divides $n$. For sharp results when $r-1$ does not divide $n$ and $r$ is small, see [5]. Generalization of the problem for hypergraphs is treated in [8]. Recently some results are obtained for the case when connectivity is replaced by $k$-connectivity [6], [13].

We show how the answer changes if $r$-coloring is replaced by *local $r$-coloring* , where the number of colors can be larger than $r$, but the requirement is that edges incident to any vertex are colored with at most $r$ colors. Ramsey numbers in local $r$-colorings have been introduced in [11], [12].

Let $f(n,r)$ denote the largest $m$ such that in every local $r$-coloring of the edges of $K_n$ there is a monochromatic connected subgraph with $m$ vertices. This function has been also defined implicitly in [3], where mixed Ramsey numbers introduced. In particular, $RM(\mathcal{T}_n, G)$ was defined as the minimum $m$ such that in any edge coloring of $K_m$ there is either a monochromatic tree on $n$ vertices or a totally multicolored copy of $G$. The special case when $G$ is a forest on four edges were treated in [4]. Since the requirement of forbidding a multicolored $K_{1,4}$ is equivalent to local 3-colorings, $RM(\mathcal{T}_n, K_{1,4}) \leq \frac{7n}{3}$ follows from the case $r=3$ of our main result, settling a question raised in [4].

**Theorem 1.** *$f(n,r) \geq \frac{rn}{r^2-r+1}$ with equality if a finite plane of order $r-1$ exists and $r^2-r+1$ divides $n$.*

To show equality in the claimed case, consider the points of a finite plane of order $r-1$ as the vertices of a complete graph and color each pair of vertices by the line going through them. Then replace each vertex $i$ by a $k$-element set $A_i$ so that the $A_i$-s are pairwise disjoint. The coloring is extended naturally with the proviso that the edges within $A_i$-s are colored with some color among the colors that were incident to vertex $i$. The result is a locally $r$-colored $K_n$ where $n = k(r^2-r+1)$ and the largest monochromatic connected subgraph has $kr = \frac{nr}{r^2-r+1}$ vertices.

We give two proofs for Theorem 1. One is based on the following result, perhaps interesting in its own. A *double star* is a tree obtained from two vertex disjoint stars by connecting their centers.

**Theorem 2.** *Assume that the edges of a complete bipartite graph $G = [A, B]$ are colored so that the edges incident to any vertex of $A$ are colored with at most $p$ colors and the edges incident to any vertex of $B$ are colored with at most $q$ colors. Then there exists a monochromatic connected subgraph $H$ with at least $|A|/q + |B|/p$ vertices. In fact, $H$ can be selected as a double star.*

**Corollary 1.** *If the edges of a complete bipartite graph $G$ are locally $r$-colored, there exists a monochromatic connected subgraph (in fact a double star) with at least $|V(G)|/r$ vertices.*

The special case of Corollary 1, when local $r$-colorings are replaced by usual $r$-colorings was proved in [10] (without the remark about the double star). A considerably simpler proof (that gives the stronger result about the double star) was given by Liu, Morris and Prince [13]. We use their method to prove Theorem 2.

**Proof of Theorem 2.** Let $d_i(v)$ denote the degree of $v$ in color $i$. For any edge $ab$ of color $i$, $a \in A, b \in B$, set $c(a,b) = d_i(a) + d_i(b)$. Let $I(v)$ denote the set of colors on the edges incident to $v \in V(G)$. Then, by using the Cauchy-Swartz inequality and the local coloring conditions, we get

$$\sum_{ab\in E(G)} c(a,b) = \sum_{a\in A}\sum_{i\in I(a)} d_i^2(a) + \sum_{b\in B}\sum_{i\in I(b)} d_i^2(b) \geq |A|p\left(\frac{\sum_{a\in A}\sum_{i\in I(a)} d_i(a)}{|A|p}\right)^2 +$$

$$+|B|q\left(\frac{\sum_{b\in B}\sum_{i\in I(b)} d_i(b)}{|B|q}\right)^2 = |A||B|\left(\frac{|B|}{p}+\frac{|A|}{q}\right),$$

therefore for some $a \in A, b \in B$, $c(a,b) \geq |A|/q + |B|/p$. Since the edges incident to $a$ or $b$ in the color of $ab$ span a monochromatic connected subgraph with $c(a,b)$ vertices, Theorem 2 follows. □

**Proof of Theorem 1.** If any monochromatic, say red component $C$ satisfies $|C| \geq \frac{rn}{r^2-r+1}$, we have nothing to prove. Otherwise apply Theorem 2 for the complete bipartite graph $[A,B] = [V(C), V(G)\setminus V(C)]$. The edges incident to any $v \in A$ are colored with at most $p = r-1$ colors and the edges incident to any $v \in B$ are colored with at most $q = r$ colors. Thus, by Theorem 2, there is a monochromatic component of size at least

$$|A|/q + |B|/p = \frac{|A|}{r} + \frac{n-|A|}{r-1} = \frac{n}{r-1} - |A|\left(\frac{1}{r-1} - \frac{1}{r}\right) \geq$$

$$\geq n\left(\frac{1}{r-1} - \frac{r}{r^2-r+1}\left(\frac{1}{r(r-1)}\right)\right) = \frac{rn}{r^2-r+1}.$$

□

Our second proof for Theorem 1 applies a result of Füredi [7]. Assume that the edges of $K_n$ are locally $r$-colored. Consider the hypergraph $H$ whose vertices are the vertices of $K_n$ and whose edges are the vertex sets of the connected monochromatic components. In the dual of $H$, $H^*$, every edge has at most $r$ vertices and each pair of edges has a nonempty intersection. Füredi proved ([7]) that in such hypergraphs the fractional transversal number, $\tau^*(H^*) \leq r - 1 + \frac{1}{r}$. Using well-known elementary facts,

$$\frac{|E(H^*)|}{D(H^*)} \leq \nu^*(H^*) = \tau^*(H^*) \leq r - 1 + \frac{1}{r}$$

where $D$ is the maximum degree of $H^*$. Thus we have $\frac{r|E(H^*)|}{r^2-r+1} \leq D(H^*)$. Noting that $|E(H^*)| = n$ and $D(H^*)$ equals to the maximum size of an edge in $H$, i.e. the maximum size of a connected component in the local $r$-coloring, the inequality of Theorem 1 follows.

## References


[1] B. Andrásfai, Remarks on a paper of Gerencsér and Gyárfás, Ann. Univ. Sci. Eötvös, Budapest **13** (1970) 103-107.

[2] J. Bierbrauer, A. Brandis, On generalized Ramsey numbers for trees, *Combinatorica* **5** (1985) 95-107.

[3] A. Bialostocki, P. Dierker, W. Voxman, Either a graph or its complement is connected: a continuing saga, manuscript (2001)

[4] A. Bialostocki, W. Voxman, Monochromatic trees and a Rainbow forest on four edges, preprint.

[5] J. Bierbrauer, A. Gyárfás, On (n,k)-colorings of complete graphs, *Congressus Num.* **58** (1987) 123-139.

[6] B. Bollobás, A. Gyárfás, Highly connected monochromatic subgraphs, submitted.

[7] Z. Füredi, Maximum degree and fractional matchings in uniform hypergraphs, *Combinatorica* **1** (1981) 155-162.

[8] Z. Füredi, A. Gyárfás, Covering $t$-element sets by partitions, *Europ. J. Combinatorics* **12** (1991) 483-489.

[9] L. Gerencsér, A. Gyárfás, On Ramsey type problems, *Ann. Univ. Sci. Eötvös, Budapest* **10** (1967) 167-170.

[10] A. Gyárfás, Partition coverings and blocking sets in hypergraphs (in Hungarian) *Communications of the Computer and Automation Institute of the Hungarian Academy of Sciences* **71** (1977) 62 pp.

[11] A. Gyárfás, J. Lehel, R. H. Schelp, Zs. Tuza, Ramsey numbers for local colorings, *Graphs and Combinatorics* **3** (1987) 267-277.

[12] A. Gyárfás, J. Lehel, J. Nesetril, V. Rödl, R. H. Schelp, Zs. Tuza, Local $k$-colorings of graphs and hypergraphs, *J. of Comb. Theory B* **43** (1987) 127-139.

[13] H. Liu, R. Morris, N. Prince, Highly connected monochromatic subgraphs of multicoloured graphs, submitted.