Highly connected monochromatic subgraphs

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Abstract

We conjecture that for \( n > 4(k - 1) \) every 2-coloring of the edges of the complete graph \( K_n \) contains a \( k \)-connected monochromatic subgraph with at least \( n - 2(k - 1) \) vertices. This conjecture, if true, is best possible. Here we prove it for \( k = 2 \), and show how to reduce it to the case \( n < 7k - 6 \). We prove the following result as well: for \( n > 16k \) every 2-colored \( K_n \) contains a \( k \)-connected monochromatic subgraph with at least \( n - 12k \) vertices.

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The following remark Paul Erdős made a long time ago is often used as a warm-up exercise (for example [5, Chapter VI, Ex. 1]): show that every 2-coloring of the edges of the complete graph \( K_n \) contains a connected monochromatic subgraph with \( n \) vertices, i.e., the graph contains a monochromatic spanning tree. Erdős’s remark has been generalized in several directions. In particular, we may require a certain type of monochromatic spanning tree. For example, it is easy to find a monochromatic rooted tree of height at most two, with the root at any vertex (see [3]). Solving a conjecture of Bialostocki et al. [1], Burr [6] proved that every 2-coloring of the edges of \( K_n \) contains a monochromatic spanning \textit{broom}, i.e., a path with a star at one end. Both results remain true if 2-colorings are replaced by colorings without multicolored triangles (conjectured in [2], proved in [12]).

For multicolorings, it is natural to ask the order of the largest monochromatic connected component. In fact, this question was posed in [10] and was rediscovered in [4]. For \( r \)-colorings of \( K_n \), the answer is approximately \( n/(r - 1) \), proved independently in [8,11]. For most general results and further references see [9]. As another variant of the remark of Erdős, we may ask for a partition of the vertex set of an \( r \)-colored \( K_n \) into as few sets as possible, with each set spanning a monochromatic tree (see [7,14]).

This note explores a natural new variant: we ask for \textit{large monochromatic subgraphs of high vertex connectivity} in 2-colored complete graphs.

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Example. Let $B$ be the 2-colored complete graph on $[6]$ with red edges 12, 23, 34, 25, 35 and with the other edges blue. (Both color classes form a “bull”.) Assuming that $n > 4(k - 1)$, $k \geq 2$, let $B(n, k)$ be a 2-colored complete graph with $n$ vertices obtained by replacing vertices 1, 2, 3, 4 in $B$ by arbitrary 2-colored complete graphs of $k - 1$ vertices and replacing vertex 5 in $B$ by a 2-colored complete graph of $n - 4(k - 1)$ vertices. All edges between the replaced parts retain their original colors from $B$. Note that $B(n, k)$ denotes a member of a rather large family of graphs. The definition of $B(n, k)$ is used in the case $n = 4(k - 1)$ as well, but in this case vertices 1, 4 (2, 3) of $B$ are replaced by red (blue) complete subgraphs (and vertex 5 is deleted). Thus in this case we have just one graph for each $k$, which we denote by $B(k)$. Observe that the color classes of $B(k)$ form isomorphic graphs and there is no monochromatic $k$-connected subgraph in $B(k)$.

It is easy to check that in $B(n, k)$ the maximal order of a $k$-connected monochromatic subgraph is $n - 2(k - 1)$. It is conceivable that each $B(n, k)$ is an optimal example for every $k$, i.e., the following assertion is true.

**Conjecture 1.** For $n > 4(k - 1)$, every 2-colored $K_n$ contains a $k$-connected monochromatic subgraph with at least $n - 2(k - 1)$ vertices.

The conjecture does hold for $k = 1$ and 2: for $k = 1$ it follows from the warm-up exercise and the case $k = 2$ is not much more difficult to prove.

**Proposition 1.** For $n > 4$, every 2-coloring of the edges of $K_n$ contains a 2-connected monochromatic subgraph with at least $n - 2$ vertices.

**Proof.** Every 2-coloring of $K_5$ contains a monochromatic cycle. Proceeding by induction, let (w.l.o.g.) $H$ be a 2-connected red subgraph with $n - 3$ vertices in a 2-coloring of $K_n$. If some vertex of $W = V(K_n) \setminus V(H)$ sends at least two red edges to $H$ then we have a 2-connected red subgraph with $n - 2$ vertices. Otherwise the blue edges between $V(H)$ and $W$ determine a 2-connected blue subgraph of at least $n - 2$ vertices (either a blue $K_{2,n-4}$ or a blue $K_{3,n-3}$ from which three pairwise disjoint edges are removed). □

The induction argument of Proposition 1 works for every fixed $k$ and large $n$ and implies that it is enough to prove Conjecture 1 for a relatively small range of vertices. (The case $k = 3$ can probably be settled by checking $n = 9, 10$.)

**Proposition 2.** Conjecture 1 holds if the assertion holds for all $n$ with $4(k - 1) < n < 7k - 5$.

**Proof.** Assume that a 2-colored $K_n$ is a minimal counterexample to Conjecture 1 with $n \geq 7k - 5$. Without loss of generality, there is a $k$-connected red subgraph $H$ with $n - 2(k - 1) - 1$ vertices. Set $U = V(H)$, $W = V(K_n) \setminus U$ and observe that any vertex of $W$ sends at most $k - 1$ red edges to $U$, consequently each vertex of $W$ sends at least $|U| - (k - 1) = n - 3k + 2$ blue edges to $U$.

Let $B = [U, W]$ be the bipartite subgraph determined by the blue edges. Let $U_1$ denote the set of vertices in $U$ with degree less than $k$ in $B$. It follows easily that $|U_1| \leq (2k - 1)(k - 1)/k < 2k - 1$. We claim that the subgraph $B_1 = [U \setminus U_1, W]$ of $B$ is $k$-connected, contradicting the choice of $K_n$.

Indeed, assume that $B_1$ has a disconnecting set $S$ with at most $k - 1$ vertices and write $C_1, \ldots, C_p$ for the components of $B_1 \setminus S$, $p > 1$. Each component $C_i$ intersects $W$, since otherwise a vertex of $C_i$ could send at most $|S \cap W| < k - 1$ blue edges to $W$. Furthermore, each component $C_i$ intersects $U \setminus U_1$ as well, since otherwise a vertex of $C_i$ could send at most $2k - 2 + (k - 1) < n - 3k + 2$ blue edges to $U$. As

\[
|U \setminus (U_1 \cup (S \cap U))| \geq n - 2k + 1 - (2k - 2) - (k - 1)
= n - 5k + 4 \geq (7k - 5) - 5k + 4 = 2k - 1,
\]

selecting an index $j$ such that $|C_j \cap U| (1 \leq j \leq p)$ is smallest, we have that $|U \setminus (S \cup C_j)| \geq k$. But this implies that any vertex of $C_j \cap W$ is nonadjacent to at least $k$ vertices of $U$ in $B$, which is a contradiction. □

For general $k$ we prove only a simple result with constants weaker than in Conjecture 1.
Theorem 1. For \( n \geq 16k - 22, k \geq 2 \), every 2-colored \( K_n \) contains a \( k \)-connected monochromatic subgraph with at least \( n - 6(2k - 3) \) vertices.

Corollary 1. Every 2-colored \( K_n \) contains a \( \lceil n/16 \rceil \)-connected subgraph with at least \( n/4 \) vertices.

It is natural to define
\[
f(n) = \max\{k : \text{every 2-coloring of } K_n \text{ contains a } k\text{-connected monochromatic subgraph}\}.
\]

Corollary 1 and the graphs \( B(k) \) show that
\[
\left\lfloor \frac{n}{16} \right\rfloor \leq f(n) \leq \left\lfloor \frac{n}{4} \right\rfloor.
\]

An affirmative answer to Conjecture 1 would show that the second inequality in (1) is, in fact, an equality.

Conjecture 2. Every 2-colored \( K_n \) contains an \( \lceil n/4 \rceil \)-connected monochromatic subgraph.

Lemma 1. Assume that \( G \) is a graph with \( n \) vertices and has minimal degree at least \( 2(k - 1) \). Then either \( G \) is \( k \)-connected or \( \overline{G} \) has a \( k \)-connected subgraph with at least \( n - k + 1 \) vertices.

Proof. If \( G \) is not \( k \)-connected then for some set \( W \) of \( k - 1 \) vertices the graph \( F = G \setminus W \) is disconnected. Since each component of \( F \) has at least \( k \) vertices, \( F \subseteq \overline{G} \) is \( k \)-connected. \( \square \)

Lemma 1 (which will be also used in the proof of Theorem 1) combined with a result of Tuza and the second author [13] improves the first inequality in (1).

Theorem 2. For all \( n \geq 1 \), we have \( f(n) \geq n/(4 + 2\sqrt{2}) \).

Proof. An easy calculation shows that any 2-colored \( K_n \) has a monochromatic subgraph with minimal degree at least \( n/(2 + \sqrt{2}) \) (see [13], where it is also mentioned that this estimate is sharp up to a constant error term). Now the theorem follows from Lemma 1. \( \square \)

Proof of Theorem 1. Select sets of vertices \( X = \{x_1, \ldots, x_p\} \) and \( Y = \{y_1, \ldots, y_q\} \) so that (i) for every \( i = 1, \ldots, p \) the degree of \( x_i \) in the red subgraph spanned by \( V \setminus \{x_1, \ldots, x_{i-1}\} \) is less than \( 2(k - 1) \); (ii) for every \( j = 1, \ldots, q \) the degree of \( y_j \) in the blue subgraph spanned by \( V \setminus \{y_1, \ldots, y_{j-1}\} \) is less than \( 2(k - 1) \); (iii) \( p + q \) is maximal with respect to the first two conditions. The existence of the required sets \( X, Y \) follows immediately from an application of the obvious greedy algorithm: build \( X, Y \) by adding new vertices until the first two conditions hold. To start the algorithm, notice that the red (blue) subgraph has a vertex \( x_i \) (\( y_j \)) of degree less than \( 2(k - 1) \) otherwise Theorem 1 follows from Lemma 1. Thus \( X = \{x_i\}, Y = \{y_j\} \) is a good initial choice.

We claim that \( \min\{p, q\} \leq 6(2k - 3) \). Set \( |X \cap Y| = r, |X \setminus Y| = u, |Y \setminus X| = v \). Observe that (i) and (ii) imply the following edge-count inequalities:
\[
uv \leq (u + v)(2k - 3), \quad \binom{r}{2} + ur \leq (r + u)(4k - 6) \quad \text{and} \quad \binom{r}{2} + vr \leq (r + v)(4k - 6).
\]

Assume that \( u \leq v \), so that \( \min\{p, q\} = u + r \). The first inequality gives \( u \leq 2(2k - 3) \) and the second yields
\[
r \leq \frac{r + u}{(r - 1)/2 + u} \leq 6(2k - 3),
\]

proving the claim.

Without loss of generality, assume that \( |X| \leq 6(2k - 3) \). By conditions (i) and (iii), the red subgraph \( G_R \) on \( V \setminus X \) has minimal degree \( 2(k - 1) \). By Lemma 1, either \( G_R \) is \( k \)-connected or has a \( k \)-connected blue subgraph \( G_B \) with at least \( |G_R| - k + 1 \) vertices. In the first case \( |G_R| \geq n - p \geq n - 6(2k - 3) \), so \( G_R \) satisfies our requirements.
In the second case every vertex of $X$ sends to $G_B$ at least $|G_B| - (2k - 3) \geq n - 6(2k - 3) - (k - 1) - (2k - 3) \geq k$ blue edges, with the last inequality coming from the assumption $n \geq 16k - 22$. Therefore the subgraph $G_B \cup X$ is $k$-connected in blue and has at least $n - k + 1$ vertices, so $G_B \cup X$ has the required properties. □

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