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Highly connected monochromatic subgraphs

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Abstract

We conjecture that for $n > 4(k-1)$ every 2-coloring of the edges of the complete graph K_n contains a k-connected monochromatic subgraph with at least $n-2(k-1)$ vertices. This conjecture, if true, is best possible. Here we prove it for $k=2$, and show how to reduce it to the case $n < 7k-6$. We prove the following result as well: for $n > 16k$ every 2-colored K_n contains a k-connected monochromatic subgraph with at least $n-12k$ vertices.
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The following remark Paul Erdős made a long time ago is often used as a warm-up exercise (for example [5, Chapter VI, Ex. 1]): show that every 2-coloring of the edges of the complete graph K_n contains a connected monochromatic subgraph with n vertices, i.e., the graph contains a monochromatic spanning tree. Erdős's remark has been generalized in several directions. In particular, we may require a certain type of monochromatic spanning tree. For example, it is easy to find a monochromatic rooted tree of height at most two, with the root at any vertex (see [3]). Solving a conjecture of Bialostocki et al. [1], Burr [6] proved that every 2-coloring of the edges of K_n contains a monochromatic spanning *broom*, i.e., a path with a star at one end. Both results remain true if 2-colorings are replaced by colorings without multicolored triangles (conjectured in [2], proved in [12]).

For multicolorings, it is natural to ask the order of the largest monochromatic connected component. In fact, this question was posed in [10] and was rediscovered in [4]. For r-colorings of K_n, the answer is approximately $n/(r-1)$, proved independently in [8,11]. For most general results and further references see [9]. As another variant of the remark of Erdős, we may ask for a partition of the vertex set of an r-colored K_n into as few sets as possible, with each set spanning a monochromatic tree (see [7,14]).

This note explores a natural new variant: we ask for *large monochromatic subgraphs of high vertex connectivity* in 2-colored complete graphs.

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Example. Let B be the 2-colored complete graph on [6] with red edges 12, 23, 34, 25, 35 and with the other edges blue. (Both color classes form a "bull".) Assuming that $n > 4(k-1)$, $k \geqslant 2$, let $B(n,k)$ be a 2-colored complete graph with n vertices obtained by replacing vertices 1, 2, 3, 4 in B by arbitrary 2-colored complete graphs of $k-1$ vertices and replacing vertex 5 in B by a 2-colored complete graph of $n-4(k-1)$ vertices. All edges between the replaced parts retain their original colors from B. Note that $B(n,k)$ denotes a member of a rather large *family* of graphs. The definition of $B(n,k)$ is used in the case $n = 4(k-1)$ as well, but in this case vertices 1,4 (2,3) of B are replaced by red (blue) complete subgraphs (and vertex 5 is deleted). Thus in this case we have just one graph for each k, which we denote by $B(k)$. Observe that the color classes of $B(k)$ form isomorphic graphs and there is no monochromatic k-connected subgraph in $B(k)$.

It is easy to check that in $B(n,k)$ the maximal order of a k-connected monochromatic subgraph is $n-2(k-1)$. It is conceivable that each $B(n,k)$ is an optimal example for every k, i.e., the following assertion is true.

Conjecture 1. For $n > 4(k-1)$, every 2-colored K_n contains a k-connected monochromatic subgraph with at least $n-2(k-1)$ vertices.

The conjecture does hold for $k = 1$ and 2: for $k = 1$ it follows from the warm-up exercise and the case $k = 2$ is not much more difficult to prove.

Proposition 1. *For $n > 4$, every 2-coloring of the edges of K_n contains a 2-connected monochromatic subgraph with at least $n-2$ vertices.*

Proof. Every 2-coloring of K_5 contains a monochromatic cycle. Proceeding by induction, let (w.l.o.g.) H be a 2-connected red subgraph with $n-3$ vertices in a 2-coloring of K_n. If some vertex of $W = V(K_n)\backslash V(H)$ sends at least two red edges to H then we have a 2-connected red subgraph with $n-2$ vertices. Otherwise the blue edges between $V(H)$ and W determine a 2-connected blue subgraph of at least $n-2$ vertices (either a blue $K_{2,n-4}$ or a blue $K_{3,n-3}$ from which three pairwise disjoint edges are removed). □

The induction argument of Proposition 1 works for every fixed k and large n and implies that it is enough to prove Conjecture 1 for a relatively small range of vertices. (The case $k = 3$ can probably be settled by checking $n = 9, 10$.)

Proposition 2. *Conjecture 1 holds if the assertion holds for all n with $4(k-1) < n < 7k-5$.*

Proof. Assume that a 2-colored K_n is a minimal counterexample to Conjecture 1 with $n \geqslant 7k-5$. Without loss of generality, there is a k-connected red subgraph H with $n-2(k-1)-1$ vertices. Set $U = V(H)$, $W = V(K_n)\backslash U$ and observe that any vertex of W sends at most $k-1$ red edges to U, consequently each vertex of W sends at least $|U| - (k-1) = n-3k+2$ blue edges to U.

Let $B = [U, W]$ be the bipartite subgraph determined by the blue edges. Let U_1 denote the set of vertices in U with degree less than k in B. It follows easily that $|U_1| \leqslant (2k-1)(k-1)/k < 2k-1$. We claim that the subgraph $B_1 = [U\backslash U_1, W]$ of B is k-connected, contradicting the choice of K_n.

Indeed, assume that B_1 has a disconnecting set S with at most $k-1$ vertices and write $C_1, \ldots, C_p$ for the components of $B_1\backslash S$, $p > 1$. Each component C_i intersects W, since otherwise a vertex of C_i could send at most $|S \cap W| < k-1$ blue edges to W. Furthermore, each component C_i intersects $U\backslash U_1$ as well, since otherwise a vertex of C_i could send at most $2k-2+(k-1) < n-3k+2$ blue edges to U. As

$$\begin{aligned}|U\backslash(U_1 \cup (S \cap U))| &\geqslant n-2k+1-(2k-2)-(k-1)\\ &= n-5k+4 \geqslant (7k-5)-5k+4 = 2k-1,\end{aligned}$$

selecting an index j such that $|C_j \cap U|$ $(1 \leqslant j \leqslant p)$ is smallest, we have that $|U\backslash(S \cup C_j)| \geqslant k$. But this implies that any vertex of $C_j \cap W$ is nonadjacent to at least k vertices of U in B, which is a contradiction. □

For general k we prove only a simple result with constants weaker than in Conjecture 1.

Theorem 1. *For $n \geqslant 16k - 22$, $k \geqslant 2$, every 2-colored K_n contains a k-connected monochromatic subgraph with at least $n - 6(2k - 3)$ vertices.*

Corollary 1. *Every 2-colored K_n contains a $\lceil n/16 \rceil$-connected subgraph with at least $n/4$ vertices.*

It is natural to define

$$f(n) = \max\{k : \text{every 2-coloring of } K_n \text{ contains a } k\text{-connected monochromatic subgraph}\}.$$

Corollary 1 and the graphs $B(k)$ show that

$$\left\lceil \frac{n}{16} \right\rceil \leqslant f(n) \leqslant \left\lceil \frac{n}{4} \right\rceil. \tag{1}$$

An affirmative answer to Conjecture 1 would show that the second inequality in (1) is, in fact, an equality.

Conjecture 2. Every 2-colored K_n contains an $\lceil n/4 \rceil$-connected monochromatic subgraph.

Lemma 1. *Assume that G is a graph with n vertices and has minimal degree at least $2(k - 1)$. Then either G is k-connected or $\overline{G}$ has a k-connected subgraph with at least $n - k + 1$ vertices.*

Proof. If G is not k-connected then for some set W of $k - 1$ vertices the graph $F = G \backslash W$ is disconnected. Since each component of F has at least k vertices, $\overline{F} \subseteq \overline{G}$ is k-connected. □

Lemma 1 (which will be also used in the proof of Theorem 1) combined with a result of Tuza and the second author [13] improves the first inequality in (1).

Theorem 2. *For all $n \geqslant 1$, we have $f(n) \geqslant n/(4 + 2\sqrt{2})$.*

Proof. An easy calculation shows that any 2-colored K_n has a monochromatic subgraph with minimal degree at least $n/(2 + \sqrt{2})$ (see [13], where it is also mentioned that this estimate is sharp up to a constant error term). Now the theorem follows from Lemma 1. □

Proof of Theorem 1. Select sets of vertices $X = \{x_1, \ldots, x_p\}$ and $Y = \{y_1, \ldots, y_q\}$ so that (i) for every $i = 1, \ldots, p$ the degree of x_i in the red subgraph spanned by $V \backslash \{x_1, \ldots, x_{i-1}\}$ is less than $2(k - 1)$; (ii) for every $j = 1, \ldots, q$ the degree of y_j in the blue subgraph spanned by $V \backslash \{y_1, \ldots, y_{j-1}\}$ is less than $2(k - 1)$; (iii) $p + q$ is maximal with respect to the first two conditions. The existence of the required sets X, Y follows immediately from an application of the obvious greedy algorithm: build X, Y by adding new vertices until the first two conditions hold. To start the algorithm, notice that the red (blue) subgraph has a vertex x_1 (y_1) of degree less than $2(k - 1)$ otherwise Theorem 1 follows from Lemma 1. Thus $X = \{x_1\}$, $Y = \{y_1\}$ is a good initial choice.

We claim that $\min\{p, q\} \leqslant 6(2k - 3)$. Set $|X \cap Y| = r$, $|X \backslash Y| = u$, $|Y \backslash X| = v$. Observe that (i) and (ii) imply the following edge-count inequalities:

$$uv \leqslant (u + v)(2k - 3), \quad \binom{r}{2} + ur \leqslant (r + u)(4k - 6) \quad \text{and}$$
$$\binom{r}{2} + vr \leqslant (r + v)(4k - 6).$$

Assume that $u \leqslant v$, so that $\min\{p, q\} = u + r$. The first inequality gives $u \leqslant 2(2k - 3)$ and the second yields

$$r \leqslant \frac{r + u}{(r - 1)/2 + u}(4k - 6) \leqslant 2(4k - 6),$$

proving the claim.

Without loss of generality, assume that $|X| \leqslant 6(2k - 3)$. By conditions (i) and (iii), the red subgraph G_R on $V \backslash X$ has minimal degree $2(k - 1)$. By Lemma 1, either G_R is k-connected or has a k-connected blue subgraph G_B with at least $|G_R| - k + 1$ vertices. In the first case $|G_R| \geqslant n - p \geqslant n - 6(2k - 3)$, so G_R satisfies our requirements.

In the second case every vertex of X sends to G_B at least $|G_B| - (2k-3) \geqslant n - 6(2k-3) - (k-1) - (2k-3) \geqslant k$ blue edges, with the last inequality coming from the assumption $n \geqslant 16k - 22$. Therefore the subgraph $G_B \cup X$ is k-connected in blue and has at least $n - k + 1$ vertices, so $G_B \cup X$ has the required properties. □

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