Three-color Ramsey numbers for paths *

András Gyárfás, Miklós Ruszinkó
Computer and Automation Research Institute
Hungarian Academy of Sciences
Budapest, P.O. Box 63
Budapest, Hungary, H-1518

Gábor N. Sárközy
Computer Science Department
Worcester Polytechnic Institute
Worcester, MA, USA 01609
gsarkozy@cs.wpi.edu
and
Computer and Automation Research Institute
Hungarian Academy of Sciences
Budapest, P.O. Box 63
Budapest, Hungary, H-1518

Endre Szemerédi
Computer Science Department
Rutgers University
New Brusnwick, NJ, USA 08903
szemered@cs.rutgers.edu

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Abstract

We prove - for sufficiently large n - the following conjecture of Faudree and Schelp :

$$R(P_n, P_n, P_n) = \begin{cases} 2n-1 & \text{for odd } n, \\ 2n-2 & \text{for even } n, \end{cases}$$

for the three-color Ramsey numbers of paths on n vertices.

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1 Introduction

1.1 Ramsey numbers for paths

For graphs $G_1, G_2, \ldots, G_r$, the Ramsey number $R(G_1, G_2, \ldots, G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes giving r graphs $H_1, H_2, \ldots, H_r$, then at least one H_i ($1 \le i \le r$) has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's classical result [19]. The number $R(G_1, G_2, \ldots, G_r)$ is called the Ramsey number for the graphs $G_1, G_2, \ldots, G_r$. There is very little known about $R(G_1, G_2, \ldots, G_r)$ for $r \ge 3$ even for very special graphs (see eg. [9] or [18]). In this paper we consider the case when each G_i is a path P_n on n vertices. For $r = 2$ a well-known theorem of Gerencsér and Gyárfás [8] states that

$$R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor.$$

For $r \ge 3$ the Ramsey numbers for P_n are not known. Set

$$r(n) = \begin{cases} 2n-1 & \text{for odd } n, \\ 2n-2 & \text{for even } n. \end{cases}$$

In [6] Faudree and Schelp determined the Ramsey numbers $R(P_{n_1}, P_{n_2}, P_{n_3})$ for the case when $n_1 \ge 6(n_2 + n_3)^2$ and wrote that they felt that $R(P_n, P_n, P_n) = r(n)$. In asymptotic form this was proved by Figaj and Łuczak in [5] as a corollary of more general results about the asymptotic of the Ramsey number for three long even cycles (in [10] the asymptotic of $R(P_n, P_n, P_n)$ was determined independently). In this paper we prove the conjecture in its original form for sufficiently large n.

Theorem 1. *There exists a positive integer n_0 such that for $n \ge n_0$ we have*

$$R(P_n, P_n, P_n) = r(n).$$

The 3-colorings of $K_{r(n)-1}$ without monochromatic P_n are not unique. One type comes from a "blow-up" of a factorization of K_4. More precisely, for odd n, partition the vertices of K_{2n-2} into four sets A, B, C, D of size $\frac{(n-1)}{2}$. Coloring the edges of $[A, B], [C, D]$ by color 1, the edges of $[A, D], [B, C]$ by color 2, the edges of $[A, C], [B, D]$ by color 3, no matter how the edges inside the sets are colored, there is no monochromatic P_n. Similarly, if n is even, three of the sets could be of size $(n-2)/2$ and one is of size $n/2$. In this case another type of coloring is obtained if all edges within $A \cup B, C, D$ are colored with color 1, the edges of $[A, C], [B, C]$ are colored with color 2 and the edges of $[A, D], [B, D]$ are colored with color 3. The edges of $[C, D]$ can be colored with either color 2 or 3.

We need a relaxation of these extremal colorings. A graph G_n on n vertices is γ-**dense** if it has at least $\gamma\binom{n}{2}$ edges. A bipartite graph $G(k,l)$ is γ-dense if it contains at least γkl edges. We work with 3-edge *multi-colorings* (G_1, G_2, G_3) of a $(1-\varepsilon)$-dense graph G. Here multi-coloring means that the edges can receive more than one color, i.e. the graphs G_i are not necessarily edge disjoint. The subgraph colored with color i only is denoted by G_i^*, i.e.

$$G_1^* = G_1 \setminus (G_2 \cup G_3), G_2^* = G_2 \setminus (G_1 \cup G_3), G_3^* = G_3 \setminus (G_1 \cup G_2).$$

Extremal Coloring 1 (with parameter α): There exists a partition $V(G) = A \cup B \cup C \cup D$ such that

- $|A|, |B|, |C|, |D| \geq (1-\alpha)\frac{|V(G)|}{4}$,
- The bipartite graphs $(A \times B) \cap G_1^*$, $(C \times D) \cap G_1^*$, $(A \times D) \cap G_2^*$, $(B \times C) \cap G_2^*$, $(A \times C) \cap G_3^*$ and $(B \times D) \cap G_3^*$ are all $(1-\alpha)$-dense.

Extremal Coloring 2 (with parameter α): There exists a partition $V(G) = A \cup B \cup C \cup D$ such that

- $|A|, |B|, |C|, |D| \geq (1-\alpha)\frac{|V(G)|}{4}$,
- The bipartite graphs $(A \times B) \cap G_1^*$, $((A \cup B) \times C) \cap G_2^*$, $((A \cup B) \times D) \cap G_3^*$ are all $(1-\alpha)$-dense.

The strategy to prove Theorem 1 is to apply the edge-colored version of the Regularity Lemma to a three-colored $K_{r(n)}$. Then the following lemma is applied to the so called **reduced graph,** the graph whose vertices are associated to the clusters and whose edges are associated to ε-regular pairs. The edges of the reduced graph will be multicolored with colors of density at least $\frac{\alpha}{4}$ between the clusters. A **connected matching** in a graph G is a matching M such that all edges of M are in the same connected component of G.

Lemma 1. *For every sufficiently small α there exist positive reals η, ε ($0 < \varepsilon \ll \eta \ll \alpha \ll 1$) and positive integer n_0 such that for every $n \geq n_0$ the following holds: if a $(1-\varepsilon)$-dense graph G_n is 3-multi-colored then we have one of the following cases.*

- *Case 1: G_n contains a monochromatic connected matching of size at least $(\frac{1}{4}+\eta)n$ vertices.*
- *Case 2: This is an Extremal Coloring 1 (EC1) with parameter $\alpha/2$.*
- *Case 3: This is an Extremal Coloring 2 (EC2) with parameter $\alpha/2$.*

In Case 1 we shall find a monochromatic P_n through the Regularity Lemma. In Case 2 or 3, the extremal colorings are lifted to the original graph and classical graph theoretical methods can be applied for finding a monochromatic P_n.

We notice that Lemma 1 is a strengthening of a lemma conjectured by Łuczak in [16]. (Similar ideas, namely finding a matching in the reduced graph and connecting the edges in the matching, have already appeared e.g. in [11].) The conjecture was proved by Figaj and Łuczak [5] (and independently by the authors [10]).

In Section 2 we prove Theorem 1 from Lemma 1, relying on the treatment of the two extremal colorings in Section 3. Section 4 gives some tools, old and new Ramsey-type results and their approximate versions to establish lemmas for the proof of Lemma 1 in Section 5.

1.2 Notation and definitions

For basic graph concepts see the monograph of Bollobás [3]. Disjoint union of sets will be sometimes denoted by $+$. $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. Usually G_n is a graph with n vertices. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A + B$, and $E \subset A \times B$. K_n is the complete graph on n vertices, $K(n_1, \ldots, n_k)$ is the complete k-partite graph with classes containing $n_1, \ldots, n_k$ vertices, P_n (C_n) is the path (cycle) with n vertices. $G(n_1, \ldots, n_k)$ is a k-partite graph with classes containing $n_1, \ldots, n_k$ vertices. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G. $\Gamma(v)$ is the set of neighbors of $v \in V$. Hence the size of $\Gamma(v)$ is $|\Gamma(v)| = deg(v) = deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G. For a vertex $v \in V$ and set $U \subset V - \{v\}$, we write $deg(v, U)$ for the number of edges from v to U. $G(k, k, k)$ is γ-dense if it contains at least $3\gamma k^2$ edges. When A, B are disjoint subsets of $V(G)$, we denote by $e_G(A, B)$ the number of edges of G with one endpoint in A and the other in B. For non-empty A and B,

$$d_G(A, B) = \frac{e_G(A, B)}{|A||B|}$$

is the **density** of the graph between A and B.

Definition 1. *The bipartite graph $G = (A, B, E)$ is (ε, G)-***regular** *if*

$$X \subset A,\ Y \subset B,\ |X| > \varepsilon|A|,\ |Y| > \varepsilon|B| \quad \textit{imply} \quad |d_G(X, Y) - d_G(A, B)| < \varepsilon,$$

*otherwise it is (ε, G)-***irregular***. Furthermore, (A, B, E) is (ε, δ, G)-***super-regular** *if it is (ε, G)-regular and*

$$deg_G(a) > \delta|B| \ \forall\, a \in A, \qquad deg_G(b) > \delta|A| \ \forall\, b \in B.$$

1.3 The Regularity Lemma

In the proof a three-color version of the Regularity Lemma plays a central role.

Lemma 2 (Regularity Lemma [20]). *For every positive ε and positive integer m there are positive integers M and n_1 such that for $n \geq n_1$ the following holds. For all graphs G_1, G_2 and G_3 with $V(G_1) = V(G_2) = V(G_3) = V$, $|V| = n$, there is a partition of V into $l + 1$ classes (clusters)*

$$V = V_0 + V_1 + V_2 + ... + V_l$$

such that

- $m \leq l \leq M$
- $|V_1| = |V_2| = ... = |V_l|$
- $|V_0| < \varepsilon n$
- *apart from at most $\varepsilon\binom{l}{2}$ exceptional pairs, the pairs $\{V_i, V_j\}$ are (ε, G_s)-regular for $s = 1, 2, 3$.*

For an extensive survey on different variants of the Regularity Lemma see [14]. We will also use the following property of (ε, δ, G)-super-regular pairs.

Lemma 3. *For every $\delta > 0$ there exist an $\varepsilon > 0$ and m_0 such that the following holds. Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2| = m \geq m_0$, and let the pair (V_1, V_2) be (ε, δ, G)-super-regular. Then for every pair of vertices $v_1 \in V_1, v_2 \in V_2$, G contains a Hamiltonian path connecting v_1 and v_2.*

A lemma somewhat similar to Lemma 3 is used by Łuczak in [16]. Lemma 3 is a special case of the much stronger Blow-up Lemma (see [12] and [13]). Note that an easier approximate version of this lemma would suffice as well, but for simplicity we use this lemma.

2 Proof of Theorem 1

We shall assume that n is sufficiently large and use the following main parameters

$$0 < \varepsilon \ll \eta \ll \alpha \ll 1, \qquad (1)$$

where $a \ll b$ means that a is sufficiently small compared to b. In order to present the results transparently we do not compute the actual dependencies, although it could be done.

We need to show that each 3-edge coloring of $K_{r(n)}$ leads to a monochromatic P_n. Consider a 3-edge coloring (G_1, G_2, G_3) of $K_{r(n)}$. Apply the three-color version of the Regularity Lemma (Lemma 2), with ε as in (1) and get a partition of $V(K_{r(n)}) = V = \cup_{0\leq i\leq l} V_i$, where $|V_i| = m, 1 \leq i \leq l$. We define the following **reduced graph** G^r: The vertices of G^r are $p_1, \ldots, p_l$, and we have an edge between vertices p_i and p_j if the pair $\{V_i, V_j\}$ is (ε, G_s)-regular for $s = 1, 2, 3$. Thus we have a one-to-one correspondence $f : p_i \to V_i$ between the vertices of G^r and the clusters of the partition. Then,

$$|E(G^r)| \geq (1-\varepsilon)\binom{l}{2},$$

and thus G^r is a $(1-\varepsilon)$-dense graph on l vertices. Define a 3-edge multi-coloring (G_1^r, G_2^r, G_3^r) of G^r in the following way. The edge $p_i p_j \in G_s^r$ if $|E_{G_s}(V_i, V_j)| \geq \frac{\alpha}{4}|V_i||V_j|$.

Applying Lemma 1 to G^r we get three cases. Case 1 is that we can find in G^r a monochromatic connected matching $M = \{e_1, e_2, \ldots, e_{l_1}\}$ of size at least $(\frac{1}{4} + \eta)l$ in G^r. Assume that M is in G_1^r. Thus using (1) we have

$$\left|\cup_{i=1}^{l_1} \cup_{p\in e_i} f(p)\right| \geq (\frac{1}{2} + 2\eta)(1-\varepsilon)r(n) \geq 2(\frac{1}{2} + 2\eta)(1-\varepsilon)(n-1) \geq (1+3\eta)\, n. \quad (2)$$

Furthermore, define $f(e_i) = (V_1^i, V_2^i)$ for $1 \leq i \leq l_1$ where V_1^i, V_2^i are the clusters assigned to the end points of e_i.

Since M is a connected matching in G_1^r we can find a connecting path P_i^r in G_1^r from $f^{-1}(V_2^i)$ to $f^{-1}(V_1^{i+1})$ for every $1 \leq i \leq l_1 - 1$. Note that these paths in G_1^r may not be internally vertex disjoint. From these paths P_i^r in G_1^r we can construct vertex disjoint connecting paths P_i in G_1 connecting a typical vertex v_2^i of V_2^i to a typical vertex v_1^{i+1} of V_1^{i+1}. More precisely we construct P_1 with the following simple greedy strategy. Denote $P_1^r = (p_1, \ldots, p_t), 2 \leq t \leq l$, where according to the definition $f(p_1) = V_2^1$ and $f(p_t) = V_1^2$. Let the first vertex u_1 $(= v_2^1)$ of P_1 be a vertex $u_1 \in V_2^1$ for which $deg_{G_1}(u_1, f(p_2)) \geq \frac{\alpha m}{5}$ and $deg_{G_1}(u_1, V_1^1) \geq \frac{\alpha m}{5}$. By (ε, G_1)-regularity most of the vertices satisfy this in V_2^1. The second vertex u_2 of P_1 is a vertex $u_2 \in (f(p_2) \cap N_{G_1}(u_1))$ for which $deg_{G_1}(u_2, f(p_3)) \geq \frac{\alpha m}{5}$. Again by (ε, G_1)-regularity most vertices satisfy this in $f(p_2) \cap N_{G_1}(u_1)$. The third vertex u_3 of P_1 is a vertex $u_3 \in (f(p_3) \cap N_{G_1}(u_2))$ for which $deg_{G_1}(u_3, f(p_4)) \geq \frac{\alpha m}{5}$. We continue in this fashion, finally the last vertex u_t $(= v_1^2)$ of P_1 is a vertex $u_t \in (f(p_t) \cap N_{G_1}(u_{t-1}))$ for which $deg_{G_1}(u_t, V_2^2) \geq \frac{\alpha m}{5}$.

Then we move on to the next connecting path P_2. Here we follow the same greedy procedure, we pick the next vertex from the next cluster in P_2^r. However, if the cluster has occurred already on the path P_1^r, then we just have to make sure that we pick a vertex that has not been used on P_1.

We continue in this fashion and construct the vertex disjoint connecting paths P_i in G_1, $1 \leq i \leq l_1 - 1$. These will be parts of the final path in G_1. We remove the internal vertices of these paths from G_1. Furthermore, we remove some more vertices from each $(V_1^i, V_2^i), 1 \leq i \leq l_1$ to achieve super-regularity in all of these pairs. From V_1^i we remove all exceptional vertices v_1 for which

$$deg_{G_1}(v_1, V_2^i) < \frac{\alpha m}{5},$$

and from V_2^i all exceptional vertices v_2 for which

$$deg_{G_1}(v_2, V_1^i) < \frac{\alpha m}{5}.$$

(ε, G)-regularity guarantees that at most $\varepsilon|V_j^i|$ vertices are removed from each cluster V_j^i. By doing this we may create some discrepancies in the cardinalities of the clusters of this connected matching. We remove an additional at most $2\varepsilon|V_j^i|$ vertices from each cluster V_j^i of the matching to assure that now we have the same number of vertices left in each cluster of the matching. Then by applying Lemma 3 for $1 \leq i \leq l_1$, we get a path in $G_1|_{f(e_i)}$ connecting v_1^i and v_2^i that contains all of the remaining vertices of $f(e_i)$ (in case of $i = 1$ we just select a long path of $f(e_1)$ starting from v_2^1 and in case of $i = l_1$, we select a long path of $f(e_{l_1})$ starting from $v_1^{l_1}$). Finally using (1) and (2) we get a path in G_1 that contains at least

$$(1 + 3\eta - 3\varepsilon)\, n \geq n$$

vertices, finishing Case 1 in the application of Lemma 1.

Case 2 implies that the 3-edge multi-coloring (G_1^r, G_2^r, G_3^r) is an Extremal Coloring 1 (EC1) of G^r with parameter $\alpha/2$. We will show that this implies that (G_1, G_2, G_3) is an EC1 coloring of $K_{r(n)}$ as well with parameter α. Consider this EC1 partition of $V(G^r) = A \cup B \cup C \cup D$ and consider $f(A), f(B), f(C)$ and $f(D)$ in V. Let us add the remaining exceptional vertices of V_0 to $f(A)$ and for simplicity let us preserve the notation. Now we have a partition $V = f(A) \cup f(B) \cup f(C) \cup f(D)$ such that

$$|f(A)|, |f(B)|, |f(C)|, |f(D)| \geq \left(1 - \frac{\alpha}{2}\right) \frac{(1-\varepsilon)r(n)}{4} \geq (1-\alpha)\frac{r(n)}{4},$$

giving the first condition in the definition of EC1 with parameter α. Next we will show that the second condition in EC1 is true as well for this partition and for the (G_1, G_2, G_3) coloring of $K_{r(n)}$. To make calculations easier, we disregard here the exceptional vertices of V_0 added to A. Then, for the number of edges between $f(A)$ and $f(B)$ not in color G_1 we get the following upper bound.

$$|(f(A) \times f(B)) \cap (G_2 \cup G_3)| \leq \frac{\alpha}{2}|f(A)||f(B)| + 2\left(\frac{\alpha}{4}|f(A)||f(B)|\right) =$$

$$= \alpha |f(A)||f(B)|. \tag{3}$$

Here the first term comes from edges that are between $f(p_i)$ and $f(p_j)$, where $p_i \in A$, $p_j \in B$ and $(p_i, p_j) \notin G_1^{r*}$. The second term comes from edges that are between $f(p_i)$ and $f(p_j)$, where $p_i \in A$, $p_j \in B$ and $(p_i, p_j) \in G_1^{r*}$. Indeed, since $(p_i, p_j) \in G_1^{r*}$, we have $(p_i, p_j) \notin G_2^r$ and $(p_i, p_j) \notin G_3^r$, and thus

$$|E_{G_2}(f(p_i), f(p_j))|, |E_{G_3}(f(p_i), f(p_j))| < \frac{\alpha}{4}|f(p_i)||f(p_j)|.$$

Then (3) implies that $(f(A) \times f(B)) \cap G_1^*$ is $(1-\alpha)$-dense. Similarly we get the other density conditions in the definition of EC1, and thus indeed (G_1, G_2, G_3) is an EC1 coloring of $K_{r(n)}$ with parameter α. However, in this case Lemma 5 (see in next section) finds a monochromatic P_n in this coloring.

Case 3 implies that the 3-edge multi-coloring (G_1^r, G_2^r, G_3^r) is an Extremal Coloring 2 (EC2) of G^r with parameter $\alpha/2$. Similarly as above in Case 2 we can show that this implies that (G_1, G_2, G_3) is an EC2 coloring of $K_{r(n)}$ as well with parameter α. In this case Lemma 6 (see in next section) finds a monochromatic P_n in this coloring. This completes the proof of Theorem 1. □

3 Extremal colorings

In this section we handle Extremal Colorings 1,2. Note that here we will deal with complete graphs with usual 3-edge colorings, multi-colorings are not allowed. To construct long paths, we shall use a variant of Pósa's result [17] on Hamiltonian graphs.

Lemma 4. *([2] Ch.10, Theorem 15.) Let $G = (A, B)$ be a bipartite graph with $|A| = |B| = n \geq 2, \delta(G) \geq 2$ such that for each j, $2 \leq j \leq \frac{n+1}{2}$, in both color classes A, B, the set of vertices of degree at most j is smaller than $j - 1$. Then G is Hamilton-connected i.e. each pair of vertices can be connected by a Hamiltonian path.*

3.1 Extremal Coloring 1.

First we will prove that we can find the desired monochromatic path of length n in case we have Extremal Coloring 1.

Lemma 5. *For every sufficiently small α there exists a positive integer $n_1 = n_1(\alpha)$ such that the following is true for $n \geq n_1$. If a 3-edge coloring (G_1, G_2, G_3) of $K_{r(n)}$ is an Extremal Coloring 1 (EC1) with parameter α then there is a monochromatic path of length n.*

Proof: First we will remove certain exceptional vertices (denote their set by E) from the four sets A, B, C, D in EC1. A vertex $v \in A$ is **exceptional** if one of the following is true:

$$deg_{G_1}(v, B) < (1 - \sqrt{\alpha})|B|, deg_{G_2}(v, D) < (1 - \sqrt{\alpha})|D|,$$
$$\text{or } deg_{G_3}(v, C) < (1 - \sqrt{\alpha})|C|.$$

From the density conditions in EC1 it follows that the number of these exceptional vertices is at most $3\sqrt{\alpha}|A|$. We remove these vertices from A and add them to E. Similarly, for the other three sets we define exceptional vertices and add them to E. Thus altogether (since we have $2n$ vertices)

$$|E| \leq 24\sqrt{\alpha}n. \tag{4}$$

Next we redistribute these vertices among the 4 sets in such a way that we are not creating new "very" exceptional vertices. Let us take the first exceptional vertex v from E, the procedure will be similar for the other vertices. Consider the color (say G_1) that contains the most out of the edges incident to v, and consider these G_1-neighbors of v. We may assume that these neighbors are either all in $A \cup B$, or in $C \cup D$ (say they are in $A \cup B$). Indeed, otherwise we can connect $A \cup B$ with $C \cup D$ in color G_1 through v and this would give a monochromatic path in G_1 of length close to $2n$ (certainly much more than the desired n). Thus we have

$$deg_{G_1}(v, A \cup B) \geq \frac{2n - 3}{3},$$

which implies that

$$deg_{G_1}(v, A), deg_{G_1}(v, B) \geq \frac{n}{7},$$

if α is sufficiently small.

Furthermore, all the edges between $C \cup D$ and v are in colors G_2 and G_3. By a similar reasoning as above, we may assume that v does not have G_2 neighbors in both C and D, and it does not have G_3 neighbors in both C and D. Thus either all the edges in $C \times \{v\}$ are in G_2, and all the edges in $D \times \{v\}$ are in G_3, or the other way around. Say we have the first case. Then we add v to B, certainly we will have

$$deg_{G_1}(v, A), deg_{G_2}(v, C), deg_{G_3}(v, D) \geq \frac{n}{7}. \tag{5}$$

We repeat this procedure for all the exceptional vertices in E. Let us consider the largest set (say A) of the fours sets A, B, C and D.

Claim 1. If $|B| \geq \lfloor \frac{n}{2} \rfloor$, then there is a monochromatic path of length n in color G_1 in the bipartite graph $G_1|_{A \times B}$.

Proof of Claim 1: If n is even, then take arbitrary subsets $A' \subseteq A$, $B' \subseteq B$ with $|A'| = |B'| = \frac{n}{2}$. Applying Lemma 4 for $G_1|_{A' \times B'}$ (the conditions of the lemma are satisfied with much room to spare because of (4) and (5)) we get a monochromatic path of length n in color G_1.

If n is odd, then we must have $|A| \geq \frac{n+1}{2}$, since we have $2n-1$ vertices. Then take arbitrary subsets $A' \subseteq A$, $B' \subseteq B$ with $|A'| = \frac{n+1}{2}$, $|B'| = \frac{n-1}{2}$. Again applying Lemma 4 we can find a Hamiltonian path in $G_1|_{A' \times B'}$ beginning and ending in A'. This gives the desired monochromatic path of length n in color G_1 and proves Claim 1.

Thus we may assume that

$$|B|, |C|, |D| < \lfloor \frac{n}{2} \rfloor. \tag{6}$$

Consider the color (say G_1) that contains the most edges inside A.

Claim 2. There is a Hamiltonian path P in $G_1|_{A \cup B}$.

Proof of Claim 2: Since $G_1|_A$ contains a subgraph of minimum degree $|A| - |B|$, we can find a path P_1 in $G_1|_A$ that has length $|A| - |B|$. Remove this path from A except for one of the endpoints u. Denote the resulting set in A by A'. Then $|A'| = |B|$. Again applying Lemma 4 we can find a Hamiltonian path P_2 in $G_1|_{A' \times B}$ starting with u. P_1 together with P_2 gives us the desired Hamiltonian path P in $G_1|_{A \cup B}$, and this proves the claim.

By (6), in case n is even we get

$$|C| + |D| \leq 2\left(\frac{n}{2} - 1\right) = n - 2,$$

and in case n is odd we get

$$|C| + |D| \leq 2\frac{n-1}{2} = n - 1.$$

Thus in both cases

$$|A| + |B| \geq n,$$

and thus P is a monochromatic path of length at least n.□

3.2 Extremal Coloring 2.

Next we will prove that we can find the desired monochromatic path of length n in case we have Extremal Coloring 2.

Lemma 6. *For every sufficiently small α there exists a positive integer $n_2 = n_2(\alpha)$ such that the following is true for $n \geq n_2$. If a 3-edge coloring (G_1, G_2, G_3) of $K_{r(n)}$ is an Extremal Coloring 2 (EC2) with parameter α then there is a monochromatic path of length n.*

Proof: Like in Lemma 5, we remove certain exceptional vertices (denote their set by E) from the four sets A, B, C, D in EC2. A vertex $v_1 \in A$ is exceptional if one of the following is true:

$$deg_{G_1}(v_1, B) < (1-\sqrt{\alpha})|B|, deg_{G_2}(v_1, C) < (1-\sqrt{\alpha})|C|,$$

$$\text{or } deg_{G_3}(v_1, D) < (1-\sqrt{\alpha})|D|.$$

We remove these vertices from A and add them to E. Similarly, a vertex $v_2 \in B$ is exceptional if one of the following is true:

$$deg_{G_1}(v_2, A) < (1-\sqrt{\alpha})|A|, deg_{G_2}(v_2, C) < (1-\sqrt{\alpha})|C|,$$

$$\text{or } deg_{G_3}(v_2, D) < (1-\sqrt{\alpha})|D|.$$

A vertex $v_3 \in C$ is exceptional if

$$deg_{G_2}(v_3, A \cup B) < (1-\sqrt{\alpha})(|A| + |B|),$$

and finally a vertex $v_4 \in D$ is exceptional if

$$deg_{G_3}(v_4, A \cup B) < (1-\sqrt{\alpha})(|A| + |B|).$$

We remove these exceptional vertices and add them to E. From the density conditions in EC2 it follows that the number of these exceptional vertices is at most a small constant times $\sqrt{\alpha}|A|$, say

$$|E| \leq 100\sqrt{\alpha}n. \tag{7}$$

Next we redistribute these vertices among the 4 sets in such a way that we are not creating new "very" exceptional vertices. Let us take the first exceptional vertex v from E, the procedure will be similar for the other vertices. Consider the color that contains the most out of the edges in $\{v\} \times (A \cup B)$. If this is G_1, then we add v to B in case it has more G_1-neighbors in A than in B, and to A otherwise. If it is G_2, we add v to C and finally if it is G_3, then we add it to D.

Claim 1. If $|C| \geq \lfloor \frac{n}{2} \rfloor$, then there is a monochromatic path of length n in color G_2 in the bipartite graph $G_2|_{(A \cup B) \times C}$. If $|D| \geq \lfloor \frac{n}{2} \rfloor$, then there is a monochromatic path of length n in color G_3 in the bipartite graph $G_3|_{(A \cup B) \times D}$.

Proof of Claim 1: Assume first that $|C| \geq \lfloor \frac{n}{2} \rfloor$, the second case is symmetrical. If n is even, then select subsets $A' \subseteq A \cup B$, $C' \subseteq C$ with $|A'| = |C'| = \frac{n}{2}$ such that A' is a random subset of $A \cup B$ with cardinality $\frac{n}{2}$ and C' is an arbitrary subset of C with cardinality $\frac{n}{2}$. Applying Lemma 4 for $G_2|_{A' \times C'}$ (with high probability the conditions of the lemma are satisfied with much room to spare) we get a monochromatic path of length n in color G_2.

If n is odd, then we select similarly subsets $A' \subseteq A \cup B$, $C' \subseteq C$ with $|A'| = \frac{n+1}{2}$, $|C'| = \frac{n-1}{2}$. Again applying Lemma 4 we can find a Hamiltonian path in $G_2|_{A' \times C'}$ beginning and ending in A'. This gives the desired monochromatic path of length n in color G_2 and proves Claim 1.

Thus we may assume that

$$|C|, |D| < \lfloor \frac{n}{2} \rfloor. \tag{8}$$

Consider $A \cup B$ and denote $a = |A \cup B|$. Assume first that $G_2|_{A \cup B}$ is $\sqrt[3]{\alpha}$-dense, so

$$|E\left(G_2|_{A \cup B}\right)| \geq \sqrt[3]{\alpha} \binom{a}{2}. \tag{9}$$

As every graph of average degree d has a subgraph of minimum degree $d/2$, we can clearly find $\geq \frac{\sqrt[3]{\alpha}}{2} a$ vertices $v \in A \cup B$ with

$$deg_{G_2}(v, A \cup B) \geq \frac{\sqrt[3]{\alpha}}{2}(a-1). \tag{10}$$

We can move some of these vertices v satisfying (10) from $A \cup B$ to C to achieve that now $|C| \geq \lfloor \frac{n}{2} \rfloor$ holds. Now similarly, as above in Claim 1, we can find a monochromatic path of length n in $G_2|_{(A \cup B) \times C}$. Thus we may assume that (9) does not hold and similarly for G_3. Hence

$$|E\left(G_1|_{A \cup B}\right)| > \left(1 - 2\sqrt[3]{\alpha}\right) \binom{a}{2}. \tag{11}$$

Equation (11) with the minimum degree condition in $G_1|_{A \cup B}$ clearly implies Pósa-condition [17] for ordinary graphs, i.e. that $d_k \geq k+1$ for $k < n/2$ for the nondecreasing degree sequence d_i. Thus $G_1|_{A \cup B}$ has a Hamiltonian path.

As in $EC1$, (8)implies for even n that

$$|C| + |D| \leq 2\left(\frac{n}{2} - 1\right) = n - 2,$$

and for odd n we get

$$|C| + |D| \leq 2\frac{n-1}{2} = n - 1.$$

Thus in both cases

$$|A| + |B| \geq n,$$

and thus P is a monochromatic path of length at least n.□

4 Tools, Ramsey-type results and their approximate versions

A set M of pairwise disjoint edges of a graph G is called a matching. The size $|M|$ of a maximum matching is the matching number, $\nu(G)$. A key notion in our approach is the notion of a connected matching. A matching M is *connected* in G if all edges of M are in the same component of G. The following result is often referred to as the Tutte - Berge formula (see for example in [15] Theorem 3.1.14). We shall use $c(G)$ and $c_o(G)$ for the number of components and odd components of a graph G and $\mathrm{def}(G)$, the deficiency of G, is defined as $|V(G)| - 2\nu(G)$.

Lemma 7. *For any graph G, def(G) $= \max\{c_o(G \setminus S) - |S|\}$ where the maximum is taken over all $S \subseteq V(G)$.*

We also need the following obvious property of maximum matchings.

Lemma 8. *Suppose $M = \{e_1, \ldots, e_k\}$ is a maximum matching in a graph G. Then $V(G) \setminus V(M)$ spans an independent set and one can select one end point x_i of each e_i - we call it* **strong point** *- so that for each i, $1 \leq i \leq k$, there is at most one edge in G from x_i to $V(G) \setminus V(M)$.*

The next lemmas collect some simple properties of graphs of high density.

Lemma 9. *Assume that G_n is $(1 - \varepsilon)$-dense. Then G_n has a subgraph H with at least $(1 - \sqrt{\varepsilon})n$ vertices such that: A. $\Delta(\overline{H}) < \sqrt{\varepsilon}n$; B. $\delta(H) \geq (1 - 2\sqrt{\varepsilon})n$; C. H is $(1 - 2\sqrt{\varepsilon})$-dense.*

Proof: If G_n has p vertices with degree at least $\sqrt{\varepsilon}n$ in $\overline{G}$ then $\overline{G}$ has at least $\frac{p\sqrt{\varepsilon}n}{2}$ edges. Therefore

$$\frac{p\sqrt{\varepsilon}n}{2} \leq \varepsilon\binom{n}{2}$$

implying $p < \sqrt{\varepsilon}n$. Removing these p vertices, the remaining (at least $(1 - \sqrt{\varepsilon})n$) vertices induce the subgraph H. Properties A. and B. are obvious, C. follows from

$$|E(H)| \geq \frac{|V(H)|\delta(H)}{2} \geq \frac{|V(H)|(1 - 2\sqrt{\varepsilon})n}{2} \geq (1 - 2\sqrt{\varepsilon})\frac{|V(H)|^2}{2} >$$

$$> (1 - 2\sqrt{\varepsilon})\binom{|V(H)|}{2}. \qquad \square$$

Lemma 10. *Assume $\Delta(\overline{G_n}) < \sqrt{\varepsilon}n$ and $H = [A, B]$ is a bipartite subgraph of G_n with $2\sqrt{\varepsilon}n < |A| \leq |B|$. Then H is a connected subgraph of G_n and contains a matching of size at least $|A| - \sqrt{\varepsilon}n$. Moreover, if only $2\sqrt{\varepsilon}n < |B|$ and $A \neq \emptyset$ is assumed then there is a subgraph H' which is connected and covers A and all but at most $\sqrt{\varepsilon}n$ vertices of B.*

Proof: Two vertices in A (respectively in B) have a common neighbor in B (respectively in A). Also if $a \in A, b \in B$ then any neighbor b' of a and b has a common neighbor with b in A. Thus H is a connected subgraph. Moreover any maximum matching M misses fewer than $\sqrt{\varepsilon}n$ vertices of A. The statement about H' follows by fixing a vertex $a \in A$ and H' is obtained by deleting from B the vertices nonadjacent to A. □

A monochromatic (say red) matching in a colored complete or almost complete graph is called *connected* if its edges are all in the same monochromatic connected red component. For example, if K_4 is three-colored so that each color class has two disjoint edges (factorization of K_4) then the largest monochromatic matching has two edges, but the largest connected monochromatic matching has only one edge.

The behavior of the Ramsey numbers for monochromatic matchings is perfectly well described:

Theorem 2. *[4](Cockayne and Lorimer, 1975) Assume that $n_1, \ldots, n_t \geq 1$ are integers and $n_1 = max(n_1, \ldots, n_t)$. Then*

$$R(n_1K_2, n_2K_2, \ldots, n_tK_2) = n_1 + 1 + \sum_{i=1}^{t}(n_i - 1). \tag{12}$$

In particular, we have

Corollary 1. $R(nK_2, nK_2, nK_2) = 4n - 2$

Theorem 3. *For nonnegative integers n_1, n_2, n_3 with $n_1 = max(n_1, n_2, n_3)$ and for any nonnegative integer s*

$$R(n_1K_2, n_2K_2, n_3K_2, K_{1,s}) \leq s + n_1 + 1 + \sum_{\{i:n_i \geq 1\}}(n_i - 1). \tag{13}$$

Proof: Induction on $\sum_1^3 n_i$ is combined with Lemma 8. For any s and $n_1 = n_2 = n_3 = 0$ the statement is obvious. We consider the proof finished when a path or a star appears in the coloring with three distinctly colored edges: removing the four vertices of this configuration induction can be applied. We think of the fourth color as the complement of a graph G. Assume that a 3-coloring of the edges of a graph G with

$N = |V(G)| = s + n_1 + 1 + \sum_{i=1}^{3}(n_i - 1)$ is given such that there is no matching of size n_i in color i, no 3-edge subtrees are colored with distinct colors and each vertex of G has degree at least $N - s$. Select a maximum matching $M_1 = \{e_1, \ldots, e_{k_1}\}$ of G in color 1. By Lemma 8 (applied to the graph of the edges of color 1) select the endpoints $\{x_1, \ldots, x_{k_1}\} = X$. Let H denote the subgraph of G spanned by $V(G) \setminus V(M_1)$.

Let M_2, M_3 be maximal matchings of the subgraph of G spanned by $V(H) \cup X$ such that M_i has edges of color i and both intersect X in as many vertices as possible. Set $|M_2| = k_2, |M_3| = k_3$.

Assume that a vertex, say, $x_1 \in X$ is not covered by $M_2 \cup M_3$. This implies that x_1 is adjacent to at most one vertex of H in color 1, at most k_2 vertices of H in color 2 and at most k_3 vertices of H in color 3 and there are at most $s - 1$ vertices of H nonadjacent to x_1. Thus

$$N \leq 2k_1 + 1 + k_2 + k_3 + s - 1 \leq 2(n_1 - 1) + (n_2 - 1) + (n_3 - 1) + s < N, \tag{14}$$

a contradiction.

Therefore each vertex, say, $x_1 \in X$ is covered by an edge of M_2 or M_3. Assume w.l.o.g. that $k_2 \geq k_3$. Suppose that a vertex, say, x_1, is covered by an edge $(x_1, y_1) \in M_3$. By Lemma 8, applied to the subgraph of the color 3 edges in the graph spanned by $V(H) \cup X$, select one vertex from each edge of M_3, in particular z from (x_1, y_1). Observe that no edge of color 2 is incident to z because no three-edge paths or stars are colored with distinct colors. Therefore, using the property of z, we get that z is adjacent to at most one vertex of H in color 1, at most $2k_3$ in color 3, none in color 2 and not adjacent to at most $s - 1$ vertices of H. Thus

$$N \leq 2k_1 + 1 + 2k_3 + s - 1 \leq 2k_1 + k_2 + k_3 + s \leq$$
$$\leq 2(n_1 - 1) + (n_2 - 1) + (n_3 - 1) + s < N, \tag{15}$$

a contradiction.

We conclude that all vertices of X are covered by M_2. Applying Lemma 8 to the subgraph of the color 2 edges in the graph spanned by $V(H) \cup X$, we select one vertex from each edge of M_2, in particular z from edge $e = (x_1, y_1) \in M_2$. Like before, no edge of color 3 is incident to z because there is no 3-colored star or path. So - using the property of z - we get that z is adjacent to at most one vertex of H in color 1, to at most $2k_2 - k_1 + 1$ in color 2 (because all vertices of X are covered by M_2), and to none in color 3 and not adjacent to at most $s - 1$ vertices of H. Thus

$$N \leq 2k_1 + 1 + 2k_2 - k_1 + 1 + s - 1 = k_1 + 2k_2 + s + 1 \leq n_1 - 1 + 2(n_2 - 1) + s + 1 =$$
$$= n_1 + 2n_2 - 2 + s \leq 2n_1 + n_2 - 2 + s < N \tag{16}$$

provided that $n_3 > 0$, a contradiction. □

The behavior of Ramsey numbers for connected components is also well understood (for most general results and references see [7]). Here we cite only the following easy result which was a forerunner of the conjecture of Faudree and Schelp (and that of Theorem 1):

Theorem 4. *([1], [8]) The minimum m for which every 3-coloring of K_m contains a monochromatic connected component with at least n vertices is $2n-1$ for odd n and $2n-2$ for even n.*

A well-known remark of Erdős states that in any two-coloring of the edges of K_n there is a monochromatic component covering all vertices of K_n. The approximate version of this is the following.

Lemma 11. *Suppose that G_n is a two-colored graph, $\Delta(\overline{G}_n) < \sqrt{\varepsilon}n$. Then there is a monochromatic component C such that $|C| \geq (1-2\sqrt{\varepsilon})n$.*

Proof: Suppose that $|V(G_n)\backslash C| > 2\sqrt{\varepsilon}n$ for the maximal monochromatic component C. Then all edges of the bipartite graph $[C, V(G_n) \setminus C]$ are colored with the other color and it has a component covering all vertices of C and all but $\sqrt{\varepsilon}n$ vertices of $V(G_n) \setminus C$ by Lemma 10. This contradicts the maximality of C. □.

Lemma 12. *Suppose that $\sqrt{\varepsilon} < \frac{1}{18}$, G_n is a tripartite graph with vertex classes V_i, $|V_i| > 4\sqrt{\varepsilon}n$ and $\Delta(\overline{G}_n) < \sqrt{\varepsilon}n$. Then, for any two-coloring of the edges of G_n, either a monochromatic component C covers all but at most $6\sqrt{\varepsilon}n$ vertices of G_n or the coloring consists of just two non-trivial monochromatic components C_1, C_2 of distinct colors, such that both cover the same two partite classes of G_n (say $V_1 \cup V_2 \subset V(C_1) \cap V(C_2)$ and together they cover the third ($V_3 \subseteq V(C_1) \cup V(C_2)$).*

Proof: Select a largest monochromatic component C_1, say in color 1. Set $R_i = V_i \cap V(C_1), S_i = V_i \setminus V(C_1)$. If $V(C_1)$ covers two of the V_i-s then either $C = C_1$ is large enough to satisfy the first conclusion of the lemma or the color 2 edges of the bipartite graph $[V(C_1), V(G_n) \setminus V(C_1)]$ span the connected component C_2 so that C_1, C_2 satisfy the second conclusion of the lemma. Otherwise at least two of the S_i-s, say S_1, S_2 are nonempty. Furthermore, by the choice of C_1, $|C_1| \geq \frac{n}{3}$.

Call a set small if it has less than $2\sqrt{\varepsilon}n$ elements, otherwise it is large. The condition on $|V_i|$ implies that at least one of $|R_i|, |S_i|$ is large. Lemma 10 gives that $[R_i, S_j]$ is connected in color 2 if both R_i, S_j are large. Else it has a color 2 component covering all but at most $\sqrt{\varepsilon}n$ vertices of the larger part if only one of them is large and the other is nonempty. With these remarks in mind we have the following cases.

If all the three S_i-s are small then C_1 misses only these small sets, thus

$$|C_1| \geq (1-6\sqrt{\varepsilon})n \tag{17}$$

and C_1 works as C.

If exactly one S_i, say, S_1 is large then R_2, R_3 are both large. Lemma 10 implies that $C_2 = S_1 \cup R_2 \cup R_3$ is connected in color 2. If R_1 is small then C_2 works as C with the same estimate as (17). If R_1 is large then it is joined to C_2 through S_2 or through S_3, whichever is nonempty. Thus $C_2 = C$ works: (17) holds with reserve.

If exactly two S_i-s are large, say, S_1, S_2, then S_3 is small implying that R_3 is large. Lemma 10 ensures that $C_3 = S_1 \cup S_2 \cup R_3$ is connected in color 2. Then, applying Lemma 10 repeatedly, R_1, R_2 join to C_3. Thus $C = C_3$ works in color 2.

If all S_i-s are large we use that some R_i, say, R_1 is large,

$$|R_1| \geq \frac{1}{3}|C_1| \geq \frac{n}{9} \geq 2\sqrt{\varepsilon}n$$

since $\sqrt{\varepsilon} < \frac{1}{18}$. Then $R_1 \cup S_1 \cup S_2 \cup S_3$ is connected in color 2 and $R_2 \cup R_3$ is absorbed into that component (using Lemma 10 again). Thus, in this case G_n is connected in color 2 and the lemma is proved. □

5 Proof of Lemma 1

We address two main cases and several subcases. Using Lemma 9, we work in the subgraph $G = G_N$ where $N = (1 - \sqrt{\varepsilon})n$ and $\Delta(\overline{G_N}) < \sqrt{\varepsilon}n$. Throughout the proof we assume (1) for η, ε.

5.1 Colorings with 4-partitions

A coloring has a *4-partition* if $V(G_N)$ can be partitioned into $S, X_1, \dots, X_4$ so that $|S| \leq 4\sqrt{\varepsilon}n$ and all edges of $[X_1, X_2], [X_3, X_4]$ are colored with color 1, all edges of $[X_1, X_3], [X_2, X_4]$ are colored with color 2, all edges of $[X_1, X_4], [X_2, X_3]$ are colored with color 3. Moreover, all X_i-s are nonempty, all but at most one satisfy $|X_i| > 2\sqrt{\varepsilon}n$. The next lemma shows that we may always assume having a 4-partition if a coloring has no "large" monochromatic components.

Lemma 13. *Assume that G_N has a multicoloring with three colors such that all monochromatic components have less than $(\frac{3}{4} - \eta - 6\sqrt{\varepsilon})n$ vertices. Then either there is a monochromatic connected matching of size $(\frac{1}{4} + \eta)n$ or the coloring has a 4-partition.*

Proof: Assume that C_1 and C_2 are two monochromatic components of distinct colors such that their intersection is nonempty and their union is as large as possible. Suppose C_1 is red, C_2 is blue, set $D = V(G_N) \setminus C_1$, $X_1 = C_1 \cap C_2$, $X_2 = C_1 \setminus C_2$,

$X_3 = C_2 \cap D = C_2 \setminus C_1$ and $X_4 = D \setminus C_2$. Observe that all edges of the bipartite graphs $[X_1, X_4]$ and $[X_2, X_3]$ are green.

If $|X_4| \leq 4\sqrt{\varepsilon}n$ then by using the assumption $|C_1|, |C_2| < (\frac{3}{4} - \eta - 6\sqrt{\varepsilon})n$, $|X_2|, |X_3| \geq (\frac{1}{4} + \eta + \sqrt{\varepsilon})n$, it follows so Lemma 10 gives a connected green matching of size $(\frac{1}{4} + \eta)n$ proving Lemma 13.

Therefore $|X_4| > 4\sqrt{\varepsilon}n$ holds. Apply Lemma 10 for the green subgraph $[X_1, X_4]$ which gives a connected green subgraph $[X_1, X_4^*]$ where $|X_4| - |X_4^*| \leq \sqrt{\varepsilon}n$. By the definition of C_1, C_2 follows that $|X_2|, |X_3| > 3\sqrt{\varepsilon}n$ otherwise C_1 or C_2 could be replaced by $[X_1, X_4^*]$ to get a larger union. Thus $X = X_1 \cup X_2 \cup X_3 \cup X_4^*$ is vertex covered by two connected green subgraphs which - since there are no large monochromatic subgraphs by the assumptions of the lemma - implies that all edges of $[X_1, X_3]$ and $[X_2, X_4^*]$ are blue and all edges of $[X_1, X_2]$ and $[X_3, X_4^*]$ are red. Applying Lemma 10 for these blue and red subgraphs and deleting at most $\sqrt{\varepsilon}n$ vertices from X_2, X_3 (because $|X_1|$ has no lower bound) we find that in the subgraph of G_N spanned by the union of $X_1^* = X_1, X_2^*, X_3^*, X_4^*$, each $[X_i^*, X_j^*]$ $(1 \leq i < j \leq 4)$ is monochromatic, connected and $(1 - \sqrt{\varepsilon})$-dense. Also, $|X_i^*| > 2\sqrt{\varepsilon}n$ for $i = 2, 3, 4$ and $X_1^* \neq \emptyset$, $S = V(G_N) \setminus \cup_{i=1}^4 X_i^*$ satisfies $|S| \leq 4\sqrt{\varepsilon}n$. □

Case 1. All monochromatic components have less than $(\frac{3}{4} - \eta - 6\sqrt{\varepsilon})n$ vertices. (This, by Lemma 13 ensures the existence of a 4-partition of $V(G_N)$.

Subcase 1.1: $|X_i| \geq \left(\frac{1}{4} - 10\eta\right) n$, $i = 1, \ldots, 4$.

Adding the (at most $4\sqrt{\varepsilon}n$) vertices of $V(G_N) \setminus S$ to the largest X_i, we have $EC1$ with parameter $\frac{\alpha}{2} = 40\eta$. (Notice that X_i-s play the role of A, B, C, D and the densities are suitable.)

Subcase 1.2: $\exists i : |X_i| < \left(\frac{1}{4} - 10\eta\right) n$, and $10\eta n \leq |X_i|, i = 1, \ldots, 4$.

Set $m_i = |X_i|$ for $1 \leq i \leq 4$ and suppose w.l.o.g. that m_4 is the largest, m_1 is the smallest among the m_i-s. Since

$$m_1 + m_2 + m_3 + m_4 \geq (1 - 5\sqrt{\varepsilon})n > (1 - \eta/2)n \quad \text{and} \quad m_1 < \left(\frac{1}{4} - 10\eta\right) n,$$

it follows that

$$m_4 \geq \left(\frac{1}{4} + 3\eta\right) n.$$

If

$$m_3 \geq \left(\frac{1}{4} + 2\eta\right) n,$$

then by Lemma 10 there is a monochromatic connected matching between X_4 and X_3 of size

$$(m_3 - \sqrt{\varepsilon})n \geq \left(\frac{1}{4} + 2\eta - \sqrt{\varepsilon}\right) n > \left(\frac{1}{4} + \eta\right) n.$$

Therefore we may assume that

$$m_2 \leq m_3 < \left(\frac{1}{4} + 2\eta\right) n.$$

Apply Theorem 3 with $n_i = (\frac{1}{4} + 2\eta)n - m_i > 0$ for $i = 1, 2, 3$ and with $s = \sqrt{\varepsilon}n$. From the assumption $m_1 \geq 10\eta n > 8\eta n + 6\sqrt{\varepsilon}n$

$$\begin{aligned} m_4 &> m_4 + 8\eta n + 6\sqrt{\varepsilon}n - m_1 && (18)\\ &\geq 2\left(\left(\frac{1}{4} + 2\eta\right) n - m_1\right) + \left(\left(\frac{1}{4} + 2\eta\right) n - m_2\right) && (19)\\ &+ \left(\left(\frac{1}{4} + 2\eta\right) n - m_3\right) + \sqrt{\varepsilon}n = s + n_1 + \sum_{i=1}^{3} n_i \geq R(n_1, n_2, n_3, s), && (20) \end{aligned}$$

it follows that in X_4 we have either a monochromatic matching M_i of size $n_i = (\frac{1}{4} + 2\eta)n - m_i$ for some $1 \leq i \leq 3$ or a vertex nonadjacent to at least $\sqrt{\varepsilon}n$ vertices. The latter is impossible by the assumption on G_N. Now apply Lemma 10 for $[X_4 \setminus V(M_i), X_i]$ to find there a matching N_i of size at least $m_i - \sqrt{\varepsilon}n$. Clearly, the matching $M_i \cup N_i$ is monochromatic. It is connected because all edges in $[X_4, X_i]$ have the same color. Also

$$|M_i \cup N_i| \geq \left(\frac{1}{4} + 2\eta\right) n - m_i + m_i - \sqrt{\varepsilon}n > \left(\frac{1}{4} + \eta\right) n.$$

For $\eta \geq \sqrt{\varepsilon}$ this gives the desired matching.

Subcase 1.3: $\exists i : |X_i| < 10\eta n$.

Delete the vertices of X_i, say X_1, of the smallest cardinality. Clearly $|X_2| + |X_3| + |X_4| > n - 5\sqrt{\varepsilon}n - 10\eta n \geq n - 11\eta$. Assume w.l.o.g. that $|X_2| \leq |X_3| \leq |X_4|$. Notice that $X_2 \geq \left(\frac{1}{4} - 11\eta\right) n$, otherwise through Lemma 10 - since X_3 is nonempty and $|X_4|$ is large - the monochromatic bipartite graph $[X_3, X_4]$ would be connected but $|V([X_3, X_4])| > 3n/4 > (\frac{3}{4} - \eta - 6\sqrt{\varepsilon})n$ contradicting the assumption of Case 1.

If $|X_3| \geq \left(\frac{1}{4} + 2\eta\right) n$ then by Lemma 10 there is a monochromatic connected matching between X_4 and X_3 of size

$$|X_3| - \sqrt{\varepsilon}n \geq \left(\frac{1}{4} + 2\eta - \sqrt{\varepsilon}\right) n > \left(\frac{1}{4} + \eta\right) n.$$

Therefore, $X_4 > n - 11\eta - 2\left(\frac{1}{4} + 2\eta\right) n = n/2 - 15\eta n$. Thus we obtained

$$\left(\frac{1}{4} - 11\eta\right) n \leq |X_2| \leq \left(\frac{1}{4} + 2\eta\right) n,$$

$$\left(\frac{1}{4} - 11\eta\right) n \leq |X_3| \leq \left(\frac{1}{4} + 2\eta\right) n,$$
$$\left(\frac{1}{2} - 15\eta\right) n \leq |X_4| \leq \left(\frac{1}{2} + 22\eta\right) n,$$

and all edges in $[X_2, X_4]$ are colored 1, all edges in $[X_3, X_4]$ are colored 2, all edges in $[X_2, X_3]$ are colored 3. If X_4 contains $13\eta n^2$ edges of color 1, then it contains a matching M_1 with $13\eta n$ edges in color 1 covering $V(M_1) \subseteq X_4$. By Lemma 10 there is a monochromatic connected matching M_2 in $[X_4 \setminus V(M_1), X_2]$ of size $(|X_2| - \sqrt{\varepsilon})n \geq \left(\frac{1}{4} - 11\eta - \sqrt{\varepsilon}\right) n > \left(\frac{1}{4} - 12\eta\right) n$. Clearly, $M_1 \cup M_2$ is a monochromatic connected matching of size

$$(1/4 - 12\eta + 13\eta)n = (1/4 + \eta)n.$$

So we may assume that X_4 contains less then $13\eta n^2$ edges of color 1. By the same argument X_4 contains less then $13\eta n^2$ edges of color 2, i.e., all but $26\eta n^2$ edges in X_4 have color 3. Redistributing the exceptional vertices (i.e. those not in the X_i) and choosing $\frac{\alpha}{2} = 26\eta$ we obtain $EC2$ with parameter $\frac{\alpha}{2}$.

5.2 Colorings with a large component

We shall use Lemma 14 which says that a relaxed variant of $EC2$ ($WEC2$) suffices to finish the proof of Lemma 1:

Weak extremal coloring 2 ($WEC2$)(with some small positive parameter β): There exists $A \cup B \cup C \subseteq V(G)$ and $Q \subset A \cup B$ such that

- $|A|, |B|, |C| \geq (1-\beta)\frac{|V(G)|}{4}$, $|Q| < (1+\beta)\frac{|V(G)|}{4}$,
- All edges of the bipartite graph $((A \cup B) \times C)$ have colors i or j (but they are not necessarily monochromatic)
- The bipartite graph $(A \times B)$ is colored with color i except the edges within Q which are colored with either i or j. Again, these edges are not necessarily monochromatic.

Lemma 14. *For a given β and ε, η satisfying (1), there is an α such that the following is true. If an $(1-\varepsilon)$-dense G_n has $WEC2$ with parameter β, then G_n has either a connected matching of size at least $(\frac{1}{4} + \eta)n$ or has $EC2$ with parameter α.*

The proof of Lemma 14 is postponed to the last subsection. Notice that - through Lemma 13 - in the previous subsection we covered all cases when all monochromatic components have less than $(\frac{3}{4} - \eta - 6\sqrt{\varepsilon})n$ vertices. Thus we may assume the following.

Case 2: There is a monochromatic component with at least $(\frac{3}{4} - \eta - 6\sqrt{\varepsilon})n > (\frac{3}{4} - 2\eta)n$ vertices.

Let C_1 be the largest monochromatic, say green component, $|C_1| = (\frac{3}{4} - 2\eta + \gamma)n$ (with $0 \leq \gamma \leq \frac{1}{4} + 2\eta$). Let M_1 be a largest matching of C_1, note that M_1 is connected by its definition so we may assume $|M_1| = k_1 = (\frac{1}{4} + \eta - \rho_1)n$ with some $0 < \rho_1 \leq \frac{1}{4} + \eta$. Apply Lemma 8 to select the strong points of M_1, $V_1 = \{x_1, \dots, x_{k_1}\}$. Set $V_0 = V(M_1) \setminus V_1$, $V_2 = V(C_1) \setminus V(M_1)$, $V_3 = V(G_N) \setminus V(C_1)$. Let H be the subgraph defined by the almost complete tripartite graph with vertex classes V_i, $i = 1, 2, 3$ and by the edges of the almost complete subgraph spanned by V_2. Notice that - after deleting at most one green edge of H from each vertex of V_1 - every edge of H is either red or blue. We have

$$|V_1| = k_1 = (\frac{1}{4} + \eta - \rho_1)n, \tag{21}$$

$$|V_2| = (\frac{1}{4} + \gamma - 4\eta + 2\rho_1)n, \tag{22}$$

$$|V_3| = (\frac{1}{4} + 2\eta - \gamma)n. \tag{23}$$

If $|V_3| \leq 4\sqrt{\varepsilon}n$ then Lemma 11 is applied to the subgraph of H spanned by V_2, to select a monochromatic, say blue component, covering all but at most $\sqrt{\varepsilon}|V_2|$ vertices of V_2. This component is extended to a blue component C_2 of H. Otherwise, when $|V_3| > 4\sqrt{\varepsilon}n$, we apply Lemma 12 to define C_2, either the component covering all but at most $6\sqrt{\varepsilon}n$ vertices of H or the larger of the two components covering two V_i-s. To ensure that Lemma 12 is applicable, we need that V_1, V_2 are large enough, i.e. that

$$\frac{1}{4} + \eta - \rho_1 \geq 4\sqrt{\varepsilon}, \quad (\frac{1}{4} + \gamma - 4\eta + 2\rho_1) \geq 4\sqrt{\varepsilon}, \tag{24}$$

and the inequalities in (1) (together with the upper bound (31) on ρ_1 coming up later) ensure that. As a last step, C_2 is extended to the whole graph G by adding those vertices of V_0 that can be reached by blue paths from C_2 and will be referred to as the blue component.

Let M_2 be a largest blue matching in $C_2 \cap H$, it is automatically connected, we may assume that

$$|M_2| = k_2 = (\frac{1}{4} + \eta - \rho_2)n \tag{25}$$

with some positive ρ_2. Apply Lemma 8 to select the strong points of M_2. The set of strong points is denoted by U, the set of other end points is denoted by T. For $i = 1, 2, 3$, set $U_i = U \cap V_i$, $T_i = T \cap V_i$, let M_{ij} denote the edges of M_2 going from V_i to V_j, $m_{ij} = |M_{ij}|$. Set $W_i = V_i \setminus (U_i \cup T_i)$.

Notice that

$$\sum_{i=1}^{3}|T_i| = \sum_{i=1}^{3}|U_i| = m_{12} + m_{13} + m_{23} + m_{22} = k_2 \tag{26}$$

Lemma 8 and the definitions of the green and blue matchings M_1, M_2 imply (with the convention that the exceptional blue edge from each $u \in U_i$ to W_j and from each $u \in U_2$ to W_2 are deleted) that the following bipartite subgraphs of H have only red edges:

$$\begin{aligned}&[U_1, W_2], [U_1, W_3], [U_2, W_1], [U_2, W_3], [U_3, W_1], [U_3, W_2],\\ &[U_2, W_2], [W_1, W_2], [W_1, W_3], [W_2, W_3],\end{aligned} \tag{27}$$

Let H^* denote the subgraph of H defined by the union of the bipartite subgraphs defined in (27). Note that all edges of H^* are red. Next we establish inequalities to handle the largest matching of H^*, called *the red matching* for simpler reference. It is easy to see that the red matching is connected. Indeed, since $|W_1| + |W_2| + |W_3| = (1 - \sqrt{\varepsilon})n - k_1 - 2k_2$, for some i, $|W_i| > 4\sqrt{\varepsilon}n$ if ε is small enough. Then Lemma 10 ensures that all red edges of H^* belong to the same component. This component is extended to the whole graph and will be referred to as the red component C_3.

5.2.1 Critical cutsets of H^*

Notice that $|V(H^*)| = n - k_1 - k_2 = (\frac{1}{2} - 2\eta + \rho_1 + \rho_2)n$ so even a perfect matching of H^* is not large enough, we have to cover $4\eta - (\rho_1 + \rho_2)$ more vertices to get a matching of size $(\frac{1}{4} + \eta)n$. To find the largest (connected) matching in H^* and its possible extensions to the required larger matching, we need to estimate the maximum of $c_o(H^* \setminus S) - |S|$ in the Tutte - Berge formula. In fact, we estimate the maximum of a larger quantity, $cr(S) = c(H^* \setminus S) - |S|$ since in our special graph H^* most odd components are isolated points (c is the number of components). We call a set $S \subseteq V(H^*)$ critical, if it maximizes $cr(S)$. We shall prove that (apart from the empty set) critical sets are close to the five sets described in the next lemma.

Lemma 15. *Set $S_1 = W_1 \cup W_2 \cup W_3$, $S_2 = W_2 \cup W_3$, $S_3 = W_1 \cup W_2$, $S_4 = U_2 \cup U_3 \cup W_2 \cup W_3$, $S_5 = U_1 \cup U_2 \cup W_1 \cup W_2$. Then*

$$cr(S_1) = (|U_1| + |U_2| + |U_3|) - (|W_1| + |W_2| + |W_3|) \leq (5\eta - (\rho_1 + 3\rho_2))n$$

$$cr(S_2) \leq |U_1| - (|W_2| + |W_3|) \leq (5\eta - 2(\rho_1 + \rho_2))n - |T_1|$$

$$cr(S_3) \leq |U_3| - (|W_1| + |W_2|) \leq (5\eta - (\gamma + \rho_1 + 2\rho_2)n - |T_3|$$

$$cr(S_4) = (|U_1| + |W_1|) - (|U_2| + |U_3| + |W_2| + |W_3|) \leq (5\eta - (3\rho_1 + \rho_2))n - 2|T_1|$$

$$cr(S_5) = (|U_3| + |W_3|) - (|U_1| + |U_2| + |W_1| + |W_2|) \leq (5\eta - (2\gamma + \rho_1 + 2\rho_2))n - 2|T_3|$$

Proof: The components of $H^* \setminus S_1$ are the vertices of $U_1 \cup U_2 \cup U_3$ and the estimate of $cr(S_1$ comes from (21)-(23), (25) and (26). Rearranging (26) and using (22),(23), we have

$$m_{12} + m_{13} = k_2 - (m_{12} + m_{23} + 2m_{22}) + k_2 - (m_{13} + m_{23}) =$$

$$|V_2| - (m_{12} + m_{23} + 2m_{22}) + +|V_3| - (m_{13} + m_{23}) + (4\eta - 2(\rho_1 + \rho_2))n =$$
$$= |W_2| + |W_3| + (4\eta - 2(\rho_1 + \rho_2))n \quad (28)$$

and this gives

$$|U_1| = m_{12} + m_{13} - |T_1| = |W_2| + |W_3| - |T_1| + (4\eta - 2(\rho_1 + \rho_2))n.$$

The components of $H^* \setminus S_2$ are the vertices of U_1 and possibly the components of $U_2 \cup U_3 \cup W_1$. The latter (from Lemma 10) has at most $6\sqrt{\varepsilon}n \leq \eta n$ components which yields the estimate for $cr(S_2)$. The estimate for $cr(S_3)$ comes similarly so it is omitted.

The components of $H^* \setminus S_4$ are the vertices of $U_1 \cup W_1$. Since $|U_1| + |W_1| = k_1 - |T_1|$ and $|U_2| + |W_2| + |U_3| + |W_3| = |V(H^*)| - (|U_1| + |W_1|)$, the estimate for $cr(S_4)$ follows from (21):

$$(|U_1|+|W_1|)-(|U_2|+|U_3|+|W_2|+|W_3|) = 2k_1-2|T_1|-|V(H^*)| = (4\eta-(3\rho_1+\rho_2))n-2|T_1|.$$

The calculation of S_5 is similar. □

Now we show that - up to a small error - critical sets are determined by the five sets S_i treated in Lemma 15.

Lemma 16. *For all $S \subseteq V(H^*)$, $cr(S) \leq max_{0 \leq i \leq 5}\{cr(S_i)\} + 24\sqrt{\varepsilon}n$, where $S_0 = \emptyset$ (and thus $cr(S_0) \leq 1$).*

Proof: Let S be an arbitrary subset of $V(H^*)$. Partition S into six parts, $S \cap U_i$, $S \cap W_i$ and let $S^* = S \cup M$ where M is the union of those U_i-s and W_i-s that satisfy $|U_i \setminus S| < 2\sqrt{\varepsilon}n$ or $|W_i \setminus S| < 2\sqrt{\varepsilon}n$. Then we have

$$|S^*| \leq |S| + 6 \times 2\sqrt{\varepsilon}n. \quad (29)$$

Claim $cr(S^*) < max_{0 \leq i \leq 5}\{cr(S_i)\}$.
Proof of Claim . Call U_i (respectively W_i) full, if $S^* \cap U_i = U_i$ (respectively $S^* \cap W_i = W_i$). Observe that if $i \neq j$ or if $i = j = 2$ and neither U_i nor W_j are full then $|U_i \setminus S^*| \geq 2\sqrt{\varepsilon}n$, $|W_j \setminus S^*| \geq 2\sqrt{\varepsilon}n$. Thus, from Lemma 10, the bipartite graph $B = [U_i \setminus S^*, W_j \setminus S^*]$ is connected. The same argument shows that $[W_i \setminus S^*, W_j \setminus S^*]$ is connected for $i \neq j$ whenever W_i, W_j are not full. This argument shows that there

is at most one nontrivial component, all other components of $H^* \setminus S^*$ are trivial, i.e. isolated vertices. Hence removing vertices of S^* from components that are not full can not change the number of components of $H^* \setminus S^*$. Therefore we may assume all sets U_i, W_i are either full or empty (i.e. $U_i \cap S^*, W_i \cap S^*$ are empty). This reduces the claim to check the maximum of $cr(S)$ for the weighted graph on six vertices, the skeleton of H^*, defined with vertices u_i, w_i, $1 \leq i \leq 3$ and edges $(u_i, w_j), (w_i, w_j)$, $1 \leq i < j \leq 3$ and (u_2, w_2), and vertex-weights $|U_i|, |W_i|$. A moment of reflection gives that nonnegative $cr(S)$ may come from $S_0 = \emptyset$ - when $cr(S_0)$ is zero or one, or from the five sets S_i of Lemma 15, proving the claim. □

Observe that for $X \subseteq X^*$, $c(G \setminus X) - |X| \leq c(G \setminus X^*) - |X^*| + 2(|X^*| - |X|)$. Using this observation, the claim and (29), we get

$$cr(S) = c(H^* \setminus S) - |S| \leq c(H^* \setminus S^*) - |S^*| + 2(|S^*| - |S|) =$$
$$= cr(S^*) + 2(|S^*| - |S|) \leq max_{0 \leq i \leq 5}\{cr(S_i)\} + 2 \times 2 \times 6\sqrt{\varepsilon}n$$

for any $S \subseteq V(H^*)$, proving the lemma. □

From Lemmas 7 and 16 we have

Corollary 2. $2\nu(H^*) \geq |V(H^*)| - max_{0 \leq i \leq 5}\{cr(S_i)\} - 24\sqrt{\varepsilon}n$.

It depends on several parameters which $cr(S_i)$ is the maximum. For example, if $cr(S_0)$ is maximum then H^* has a (red) matching covering at least

$$|V(H^*))| - 1 - 24\sqrt{\varepsilon}n = (\frac{1}{2} - 2\eta + \rho_1 + \rho_2 - 24\sqrt{\varepsilon})n - 1 \tag{30}$$

vertices. In particular, if $\rho_1 + \rho_2 \geq 5\eta$ then $cr(S_i)$ are negative for $1 \leq i \leq 5$ and Corollary 2 and (30) ensures a red matching of size $(\frac{1}{4} + \eta)n$. Thus we assume

$$\rho_1 + \rho_2 < 5\eta \tag{31}$$

Since no $cr(S_i)$ is larger than $5\eta n$, we also observe the following fact.

Fact 1. *A connected red matching with size $\Delta = 10\eta n$ larger than the estimate of Corollary 2 is a connected red matching of size at least $(\frac{1}{4} + \eta)n$.*

One way to apply Fact 1 is to join a red matching of size Δ to $V(H^*)$. For this we need the following lemma.

Lemma 17. *Suppose that Z is a subset of U_i or a subset of W_i, $|Z| = \Delta$ and we have a red matching F of Δ edges in $[Z, V(G_N) \setminus V(H^*)]$. If $|T_1| \geq 2\Delta$ then the values of $cr(S_i)$ for $i = 2, 4$ in the graph $H^* \setminus Z$ are negative. If $\gamma n \geq 2\Delta$ or $|T_3| \geq 2\Delta$ then the values of $cr(S_i)$ for $i = 3, 5$ in the graph $H^* \setminus Z$ are negative. Furthermore, if all $cr(S_i)$, $1 \leq i \leq 5$, are negative or do not increase in the graph $H^* \setminus Z$, then there is a connected red matching of size at least $(\frac{1}{4} + \eta)n$.*

Proof: Obvious from inspecting the formulas in Lemma 15 and from Fact 1. □

With these preparations we are ready to address the subcases of case 2.

5.2.2 Subcases of Case 2

Subcase 2.1: $|U_3| + |W_3| \le 2\Delta$ or $|V_3| \le 9\Delta$.

If $|T_1| \le 2\Delta$ then using (31)

$$|T_2 \cup T_3| \ge k_2 - |T_1| \ge (\frac{1}{4} + \eta - \rho_2)n - 2\Delta \ge (\frac{1}{4} - 24\eta)n,$$

$$|U_1 \cup W_1| = |V_1| - |T_1| \ge (\frac{1}{4} + \eta - \rho_1)n - 2\Delta \ge (\frac{1}{4} - 24\eta)n,$$

$$|U_2 \cup W_2| = |V_2| - |T_2| = |V_N| - (2k_1 + |V_3| + |T_2|) \ge |V_N| - (2k_1 + |T_3| + 2\Delta + |T_2|)$$
$$\ge |V_N| - (2k_1 + k_2 + 2\Delta) \ge (\frac{1}{4} - 24\eta)n,$$

$$|U_1 \cup U_2| \le k_2 = (\frac{1}{4} + \eta - \rho_2)n$$

(In the third inequality we used the first condition $|U_3| + |W_3| \le 2\Delta$ through $|V_3| \le |T_3| + 2\Delta$; obviously one can also estimate directly $|V_3|$ from the second condition.) Therefore we can apply Lemma 14 with $C = T_2 \cup T_3, A = U_1 \cup W_1, B = U_2 \cup W_2, Q = U_1 \cup U_2$ for the red and blue colors (with $\beta = 24\eta$).

Now $|T_1| > 2\Delta$. Assume there is a red matching of size Δ in the bipartite graph $[U_2, V(G) \setminus V(H^*)]$. Notice that after deleting a set Z from U_2, i.e. decreasing $|U_2|$ by Δ, $cr(S_0), cr(S_2)$ are not affected and $cr(S_1)$ decreases. Since $|T_1| > 2\Delta$, $cr(S_4)$ is negative in $H^* \setminus Z$ from Lemma 17. We show that the condition of Lemma 17 holds for $cr(S_5)$ as well: if $|T_3| \le 2\Delta$ then then we get from (23) that

$$|V_3| = (\frac{1}{4} + 2\eta - \gamma)n = |T_3| + |U_3| + |W_3| \le 4\Delta$$

and this leads to $\gamma n \ge (\frac{1}{4} + 2\eta)n - 4\Delta \ge 2\Delta$ (if η, ε are small enough). Thus in the presence of a red matching of size Δ in $[U_2, V(G) \setminus V(H^*)]$ Lemma 17 would give a new red matching covering at least $(\frac{1}{2} + 2\eta)n$ vertices and the proof ends here. (Notice that the connectivity of the new matching is maintained.) Otherwise, by König's theorem, the red edges of both $[U_2, T_1], [U_2, T_2]$ have transversals of at most Δ points. In this case we call $[U_2, T_1], [U_2, T_2]$ almost complete bipartite graphs (in the blue color). Arguing similarly, we may assume that $[U_1, T_2]$ is almost complete in blue, $[V_0, U_2], [V_0, U_1]$ is almost complete in blue or green.

Next we look at $Z = [U_1, U_2]$. Assume first that Z has Δ independent red edges. Deleting the vertices of this matching from U_1, U_2, we shall proceed similarly as before. Using that T_1 and either T_3 or γ is large, we can apply Lemma 17 so it follows that either $[W_1, T_2]$ has Δ independent red edges giving the required red matching (and finishing the proof) or it is almost complete in color blue. The same argument also gives that $[W_1, T_3], [W_2, T_1], [W_2, T_2], [W_2, T_3]$ are almost complete in blue, and that $[W_1, V_0], [W_2, V_0]$ are almost complete in colors blue or green. It follows now that $V_1 \cup V_2$ spans a connected subgraph in blue. However, $(\frac{1}{4}+\eta)n-k_2 = \rho_2 n$ independent blue edges of $[W_1, V_0] \cup [W_2, V_0]$ would extend M_2 to a connected blue matching of size $(\frac{1}{4}+\eta)n$, finishing the proof. Thus $[W_1, V_0], [W_2, V_0]$ are almost complete in color green. Also, $[U_1, V_0], [U_2, V_0]$ are almost complete in color green, otherwise M_2 could be extended through the blue edges of $[T_1, W_2], [T_2, W_1], [T_2, W_2], [T_3, W_1], [T_3, W_2]$. It is obvious that deleting at most 4Δ vertices of each involved set, we get monochromatic (or two-colored) bipartite graphs instead of almost complete. Then we have a $WEC2$ with $A = U_1 \cup U_2, B = W_1 \cup W_2, C = V_0, Q = U_1 \cup W_1$ and the proof is finished through Lemma 14 applied to the red and green colors.

Finally, if Z is almost complete in blue, the role of blue and red can be reversed. More precisely, notice that removing 4Δ vertices from each T_i, U_i, the almost complete blue bipartite graphs

$$[T_1, U_2], [T_2, U_1], [T_3, U_1], [T_3, U_2], [T_2, U_2] \tag{32}$$

treated above become monochromatic in blue, their union is denoted by H^{**}. Note that

$$|V(H^{**})| = 2k_2 - 24\Delta \geq (\frac{1}{2} - 252\eta)n.$$

Critical sets of H^{**} can be analyzed in the same way as of H^* to support formally the arguments of the next paragraphs but this is omitted.

Choosing Δ^* large, say $\Delta^* = 600\eta$, either the blue matching of H^{**} can be enlarged by Δ^* to size $(\frac{1}{4}+\eta)n$, or $[T_2, W_1], [T_2, W_2], [T_1, W_2], [T_3, W_1], [T_3, W_2]$ are almost complete in red and this implies that $[T_1, V_0], [T_2, V_0], [T_3, V_0]$ are almost complete in green. Thus we have $WEC2$ coloring with $A = U_1 \cup U_2, B = T_1 \cup T_2 \cup T_3, C = V_0, Q = U_1 \cup T_1$ and the subcase is finished through applying Lemma 14 to the blue and green colors.

For the next two subcases we define $F = V_0 \setminus C_2, P = V_0 \setminus C_3$. ($C_2, C_3$ were defined at Case 2.)

Subcase 2.2: C_2 covers all but at most 4Δ vertices of H.

We may assume that $|U_3|+|W_3| > 2\Delta$ and $|V_3| > 9\Delta$ otherwise subcase 2.1 applies. Since $9\Delta < |V_3| \leq (\frac{1}{4} + 2\eta - \gamma)n$, we get that $\gamma n < (\frac{1}{4} + 2\eta)n - 9\Delta$. Using this and that C_1 is the largest component, $n - |F| - 4\Delta < |C_2| \leq |C_1| = (\frac{3}{4} - \eta - 6\sqrt{\varepsilon} + \gamma)n$ it follows that

$$|F| > (\frac{1}{4} + 2\eta - \gamma)n - 4\Delta > 2\Delta \tag{33}$$

(if η, ε are small enough). Notice that (33) implies $|T_3| \leq \Delta + \sqrt{\varepsilon}n$ otherwise - since F belongs to C_3 through the edges of $[F, U_3 \cup W_3]$ - using Lemma 10 for the red edges of $[F, T_3]$ we get an increment of the red matching by Δ. This gives that

$$|U_3| + |W_3| > 8\Delta - \sqrt{\varepsilon}n. \tag{34}$$

If $|T_1| \leq \Delta$ then we have $WEC2$ with

$$A = U_1 \cup W_1, B = U_2 \cup U_3 \cup W_2 \cup W_3, C = T_2, Q = U_1 \cup U_2 \cup U_3$$

and we are done through Lemma 14.

Therefore we may assume that $|T_1| > \Delta$. Observe that (34) and the definition of C_2 imply that either $I = V(C_2) \cap U_3$ or $J = V(C_2) \cap W_3$ is at least $2\Delta - \frac{1}{2}\sqrt{\varepsilon}n$. The edges of $[F, I], [F, J]$ are red. Suppose first that $|I| \geq 2\Delta - \frac{1}{2}\sqrt{\varepsilon}n$. Then we have Δ independent red edges in $[F, I]$, that increases the red matching to the required size, because only $cr(S_4)$ can increase but Lemma 17 is applicable ($|T_1| > \Delta$). Thus we may assume $|I| < 2\Delta - \frac{1}{2}\sqrt{\varepsilon}n$ and $J \geq 2\Delta - \frac{1}{2}\sqrt{\varepsilon}n$.

Now a simpler version of the argument used in subcase 2.1 is applied. We increase the red matching by adding $\frac{\Delta}{2}$ independent red edges from $[U_1, V(G) \setminus V(H^*)]$ and simultaneously from $[F, J]$. This does not affect any of the $cr(S_i)$ (decreasing $|U_1|$ and $|W_3|$ with the same quantity does not affect the bounds) therefore we can reach the required size. We conclude that almost all edges of $[U_1, T_2]$ must be blue, and almost all edges of $[U_1, V_0], [U_1, T_1]$ must be blue or green. The same argument is applied to prove that almost all edges of $[U_2, T_1], [U_2, T_2]$ are blue and almost all edges of $[U_2, V_0]$ are blue or green. However, here we have to argue with Lemma 17 since decreasing U_2 and W_3 simultaneously affects $cr(S_2), cr(S_4)$ (but $|T_1| > \Delta$). If $|P| \geq 2\Delta$ then by Lemma 10 the blue subgraph $[P, J]$ has a matching of size Δ vertices which extends M_2 to the required size. Therefore we may assume that $|P| < 2\Delta$. If the red subgraphs $B_1 = [T_1, V_0 \setminus P], B_2 = [T_2, V_0 \setminus P]$ have Δ independent edges then we can extend the red matching into a connected red matching of the required size. Otherwise these graphs are almost complete in blue or green and we have a $WEC2$ with

$$A = T_1 \cup T_2, B = U_1 \cup U_2, C = V_0, Q = T_1 \cup U_1$$

and we are done through Lemma 14, finishing this subcase.

Subcase 2.3: C_2 covers $V_2 \cup V_3$ (and splits V_1).

Set $Y = V_1 \setminus V(C_2)$, clearly $Y \subseteq W_1$. Let C_3 denote the red component covering $V_2 \cup V_3$, set $X = V_1 \setminus V(C_3)$, clearly $X \subseteq T_1$. We may assume that $|Y| > 4\Delta$ otherwise subcase 2.2 covers the case, and also that $|X| > 4\Delta$ since we defined C_2 as the larger of the two components splitting V_1. Observe that if $v \in V_0 \setminus (V(C_2) \cup V(C_3)$ then no color can be assigned to any edge vt for $t \in V_3$. Thus $|V_3| \leq \sqrt{\varepsilon}n$ and this is addressed in subcase 2.1. Therefore $C_2 \cup C_3$ cover all vertices of V_0.

Assume first that $|F| \geq 2\Delta$. Here the argument of subcase 2.2 works with some simplification. Notice that $|T_3| \leq \Delta + \sqrt{\varepsilon} n$ otherwise the edges of $[F, T_3]$ are red and from Lemma 10 we get an improvement of the red matching by Δ. This implies $|U_3|+|W_3| \geq 5\Delta$. If $|U_3| \geq 2\Delta$ then we can enlarge the red matching since all edges of $[U_3, F]$ are red and because only $cr(S_4)$ is affected and Lemma 17 is applicable ($|T_1| > \Delta$). If $|W_3| \geq \Delta$ then we increase the red matching by adding $\frac{\Delta}{2}$ independent red edges from $[U_1, V(G) \setminus V(H^*)]$ and simultaneously from $[F, W_3]$. This does not affect any of the $cr(S_i)$ (decreasing $|U_1|$ and $|W_3|$ with the same quantity does not affect the bounds) therefore we can reach the required size. We conclude that almost all edges of $[U_1, T_2]$ must be blue, and almost all edges of $[U_1, V_0], [U_1, T_1]$ must be blue or green. The same argument is applied to prove that almost all edges of $[U_2, T_1], [U_2, T_2]$ are blue and almost all edges of $[U_2, V_0]$ are blue or green. However, here we have to argue with Lemma 17 since decreasing U_2 and W_3 simultaneously affects $cr(S_2), cr(S_4)$ (but $|T_1| > \Delta$). If $|P| \geq 2\Delta$ then by Lemma 10 the blue subgraph $[P, W_3]$ has a matching of size Δ vertices which extends M_2 to the required size. Therefore we may assume that $|P| < 2\Delta$. If the red subgraphs $B_1 = [T_1, V_0 \setminus P], B_2 = [T_2, V_0 \setminus P]$ have Δ independent edges then we can extend the red matching into a connected red matching of the required size. Otherwise these graphs are almost complete in blue or green and we have a $WEC2$ with

$$A = T_1 \cup T_2, B = U_1 \cup U_2, C = V_0, Q = T_1 \cup U_1$$

and we are done through Lemma 14.

If $|F| < 2\Delta$ then γ is large. Indeed, using that $N - |C_2| \geq N - |C_1|$, we get

$$2\Delta + (\frac{1}{4} + \eta - \rho_1)n - 4\Delta \geq |F| + |V_1| - |X| \geq |F| + |Y| \geq |V_3|$$

$$= (\frac{1}{4} + 2\eta - \gamma)n$$

showing that $\gamma n \geq 2\Delta + (\eta + \rho_1)n$. We also know that $|T_1| \geq |X| \geq 4\Delta$, so the conditions are present to apply Lemma 17 and follow the argument of subcase 2.1. We get that $[U_i, T_j]$ for $1 \leq i < j \leq 3$ and for $i = j = 2$ are almost complete in blue. Then, continuing the argument there, we look at $Z_{ij} = [U_i, U_j]$, for $1 \leq i < j \leq 3$. The assumption that Z_{ij} has Δ independent red edges implies that $[W_i, T_j]$ are almost complete in blue for $1 \leq i < j \leq 3$ and for $i = j = 2$. However, the edges of $[Y, T_2 \cup T_3]$ are red and - since $|Y| \geq 4\Delta$ - this is possible only if $|T_2 \cup T_3| \leq 2\Delta$. This implies that $|T_1| = k_2 - |T_2 \cup T_3| \geq (\frac{1}{4} + \eta - \rho_2)n - 2\Delta$ and then $|U_1 \cup W_1| = k_1 - |T_1| \leq 2\Delta + (\rho_2 - \rho_1)n$ follows. However, using the definition of Δ and (31),

$$40\eta n = 4\Delta \leq |Y| \leq |U_1 \cup W_1| \leq 2\Delta + (\rho_2 - \rho_1)n \leq 20\eta n + \rho_2 \leq 25\eta n$$

is a contradiction.

Therefore Z_{ij} is almost complete in blue (for all three pairs of indices)and we can follow the argument in the last paragraph of subcase 2.1.

Subcase 2.4: C_2 covers $V_1 \cup V_2$ (and splits V_3).

Set $Y = V_3 \setminus V(C_2)$, clearly $Y \subseteq W_3$. Set $X = V_3 \setminus V(C_3)$, clearly $X \subseteq T_3$ and $|X|, |Y| \geq 4\Delta$ otherwise we are in case 2.2. It follows also that $X = T_3$ and $Y = W_3$. It is easy to see that the sets $(C_2 \setminus C_3) \cap V_0, (C_3 \setminus C_2) \cap V_0$ are empty. Now $X_1 = V_0 \setminus V(C_2), X_2 = V(C_1) \cap V(C_2), X_3 = X, X_4 = Y$ define a 4-partition of $V(G_N)$. Moreover it is easy to check that $|X_i| > 10\eta n$ for $1 \leq i \leq 4$. ($|X_1| \leq 10\eta n$ would contradict the choice of C_1, $|X_2| > 10\eta n$ is trivial, $|X_3|, |X_4| \geq 4\Delta > 10\eta n$.) Therefore either subcase 1.1 or subcase 1.2 can be applied. This finishes subcase 2.4.

Our final note here is that C_2, C_3 can not split V_2 because no color could be assigned to the edges of $[(V_2 \cap C_2) \setminus C_3, (V_2 \cap C_3) \setminus C_2]$.

This finishes the proof of Lemma 1.

5.3 Reducing Weak Extremal Colorings - proof of Lemma 14

For convenience, set $m = \frac{n}{4}$, we shall use colors red and blue for colors i and j. By deleting vertices from A, B, C, without loss of generality we may assume that $|A|, |B|, |C|, |D| \leq (1 + 3\beta)m$, where $D = V(G_n) \setminus A \cup B \cup C$. We shall maintain a parameter p so that $(1 - p)m$ is a lower bound for the size of the sets A, B, C, D, initially $p = \beta$. (Although at all steps the adjustment of p affects at most two of A, B, C, D, we decrease all lower bounds to simplify the calculations.) From the definition of $WEC2$ (before Lemma 14) $|Q| < (1 + \beta)m$, thus we may remove at most $2\beta m$ vertices from $[A, B]$ to ensure that the remaining bipartite graph (which is still denoted by $[A, B]$) is connected in red, therefore we set $p = 3\beta$. Now any red matching of $[A, B]$ is connected. By a *greedy matching* we understand a matching defined by selecting at each step an edge which is disjoint from the previously selected ones. Let b denote the size of a maximum blue matching of $[A \cap Q, B \cap Q]$.

Case 1: $b \leq cm$ (we shall determine c later).

Removing at most cm vertices from $[A, B]$, and setting $p = 3\beta + c$, all edges of $[A, B]$ are red. Assume E_1, E_2 are red matchings of size $2dm$ in $[C, A]$ and in $[C, B]$, respectively (d will be determined later). Select $F_1 \subset E_1, F_2 \subset E_2$ so that $|F_1| = |F_2| = dm$ and $F_1 \cup F_2$ is a matching. Then select a greedy matching M in $[A \setminus V(F_1), B \setminus V(F_2)]$. Using that at most $\sqrt{\varepsilon}n \leq 4\eta m$ edges are missing from any vertex, $|M| \geq (1 - p - 4\eta - d)m$, therefore

$$|M \cup F_1 \cup F_2| \geq (d + 1 - p - 4\eta)m \geq (\frac{1}{4} + \eta)n,$$

provided that

$$d \geq 8\eta + 3\beta + c. \tag{35}$$

Thus we may assume that the red edges of $[A, C]$ or $[B, C]$ can be eliminated by deleting at most $(8\eta + 3\beta + c)m$ vertices. Assuming (by symmetry) the former and adjusting $p = 8\eta + 6\beta + 2c$, we get that all edges of $[A, C]$ are blue.

Subcase 1.a: $[B, C]$ have red and blue matchings M_1, M_2 of size dm.

Here we can assume that almost all edges of $[A, D]$ are green. Suppose on the contrary that we have a blue matching M_3 of size dm in $[A, D]$. Then, together with a greedy matching of $[A \setminus V(M_3), C \setminus V(M_2)]$ we have a connected blue matching of size at least $(1-p)m - 4\eta m + dm \geq (\frac{1}{4} + \eta)n$ provided that

$$d \geq 10\eta + 6\beta + 2c. \tag{36}$$

(The argument is similar for supposing a red matching of size dm in $[A, D]$.) The same argument also shows that we can find the required (of size $(\frac{1}{4} + \eta)n$) connected matching in red or blue if the subgraph spanned by A has a red or blue matching of size $\frac{dm}{2}$ provided that

$$d \geq 20\eta + 12\beta + 4c. \tag{37}$$

Adjusting $p = 40\eta + 24\beta + 8c$, all edges of $[A, D]$ and all edges inside A are green. Also, by the same argument and using that all edges inside A are green, we can find the required matching in green if either $[C, D]$ or $[B, D]$ has a green matching of size dm provided that (37) holds. Thus deleting $2dm$ vertices from B, C, D all edges of the tripartite graph T spanned by B, C, D are red or blue - of course we adjust $p = 80\eta + 48\beta + 16c$. Applying Lemma 12, either a monochromatic, say red component C^* almost covers $V(T)$ or a red and a blue component, C_1, C_2, cover two partite classes of T and together they cover the third. In the former case, $|C \setminus V(C^*)| \leq 6\sqrt{\varepsilon}n \leq 6\eta n = 24\eta m$ therefore a red matching of size $(p + 2\eta)m$ in $[C \cap V(C^*), D]$ would extend any greedy matching of $[A, B]$ into a connected matching of the required size. Thus, deleting at most $24\eta m + (p + 2\eta)4m$ vertices from C, D and adjusting p, all edges of $[C, D]$ are blue. This implies that $[B, D]$ can not have a blue matching of size $(p + 2\eta)n$, otherwise we get a connected blue matching of the required size by extending any greedy matching of $[A, C]$. Thus, with a final deletion and adjustment of p, all edges of $[B, D]$ are red. Now we have an $EC2$ coloring with parameter α being a linear combination of η, β, c. If C_1 is a red component covering two of B, C, D, the previous argument works in the same way.

Subcase 1.b: $[B, C]$ has no blue matching of size dm. (The case when $[B, C]$ has no red matching of size dm is symmetric).

Since the technique here is quite similar to subcase 1.a, we do not compute the adjustments of p. Deleting at most dm vertices from B, C, all edges of $[B, C]$ are red.

Using the technique of the previous argument, a red matching of size dn in $[C, D]$ or in $[A, D]$ would give a red matching of required size; a blue matching of size dn in both $[C, D], [A, D]$ would give a similar blue matching. Thus we can conclude that at least one of $[C, D], [A, D]$ has only green edges. If both have then we have $EC2$. If $[C, D]$ ($[A, D]$) has only green edges and $[A, D]$ ($[C, D]$) has a blue matching of size dm then - like in subcase 1.a - all edges inside C (A) must be green as well. Finally we conclude that the tripartite graph spanned by A, B, D (A, B, C) is colored with red or blue and we finish like in subcase 1.a.

To finish Case 1, we have to define d as the maximum of two similar quantities, a linear combination of η, β, c and the same same remark is true for the value of α.

Case 2: $b \geq cm$.

Set $A_1 = A \cap Q, B_1 = B \cap Q, A_2 = A \setminus A_1, B_2 = B \setminus B_1$. Using the assumptions about the sizes we get

$$(1 - 3\beta)m - |A_2| + |B_1| \leq |A_1| + |B_1| \leq (1 + \beta)m$$

and a similar inequality holds for B_2, A_1, leading to

$$|B_1| \leq |A_2| + 4\beta m, |A_1| \leq |B_2| + 4\beta m \tag{38}$$

Assume that there exist red matchings E_1, E_2 of size $2dm$ in $[A, C]$ and in $[B_1, C]$. Select $F_1 \subset E_1, F_2 \subset E_2$ so that $|F_1| = |F_2| = dm$ and $F_1 \cup F_2$ is a matching. Consider the matching $M = F_1 \cup F_2 \cup M_1 \cup M_2 \cup M_3$ where M_1, M_2 are greedy matchings of $[A_1 \setminus V(F_1), B_2], [B_1 \setminus V(F_2), A_2 \setminus V(F_1)]$ and M_3 is a greedy matching of $[A_2 \setminus (V(M_1) \cup V(M_2)), (B_2 \setminus V(M_1))]$.

We claim that $M_1 \cup M_2 \cup M_3$ covers $A \setminus V(F_1)$ or $B \setminus V(F_2)$ with an error of at most $(4\beta + 4\eta)m$. Observe that (by (38)) $A_1 \setminus V(F_1)$ is almost covered by M_1 (with an error of $4\beta m$). If $A_2 \setminus V(F_1)$ is covered by M_2, the claim is proved (with error $4\beta m$). Otherwise a vertex $x \in A_2 \setminus V(F_1)$ is not covered by M_2 and from the definition of M_2, $B_1 \setminus F_2$ is almost covered by M_2 (with an error of $\eta n = 4\eta m$). Finally, from the definition of M_3, either B_2 is almost covered by $M_1 \cup M_3$ or $A_2 \setminus V(F_1)$ is almost covered by $M_2 \cup M_3$ (with an error of $\eta n = 4\eta m$). Thus the claim is proved. Counting the edges of $M_1 \cup M_2 \cup M_3$ from the well covered side of $[A, B]$, say from A, we get

$$|M| = (|F_1| + |F_2|) + (|M_1| + |M_2| + |M_3|) \geq 2dm + |A \setminus V(F_1)| - 4\beta m - 4\eta m =$$

$$|A| + dm - 4\beta m - 4\eta m \geq (d + 1 - 7\beta - 4\eta)m \geq (\frac{1}{4} + \eta)4m$$

provided that

$$d \geq (8\eta + 7\beta)m \tag{39}$$

thus we have a red connected matching of the required size. We conclude that either $[A, C]$ or $[B_1, C]$ has no red matching of size $2dm$. By symmetry, either $[A_1, C]$ or $[B, C]$ has no red matching of size $2dm$. This gives two possibilities to be checked in Subcases 2.a and 2.b

Subcase 2.a: $[A, C]$ has no $2dm$ independent red edges. (The same statement for $[B, C]$ comes by symmetry.)

Deleting at most $2dm$ vertices from A, C and adjusting $p = 3\beta + 2(3\eta + 7\beta)$ all edges of $[A, C]$ are blue. Using the edges of $[A, C]$ and the blue matching of size at least cm in $[A_1, B_1]$, the presence of a blue matching of size dm in $[B, C]$ would allow to find the required blue matching - if c, d are large enough in terms of β, η. Thus - after adjusting p - we may suppose that all edges of $[B, C]$ are red. Repeating the same argument it follows that all edges of $[C, D]$ are green, then it follows that all edges inside C are green, then $[A, D]$ is blue, $[B, D]$ is red and we get $EC2$ if c, d are selected as a suitable linear combination of β, η. The parameter α of $EC2$ is the final value of p (it is also a linear combination of β, η).

Subcase 2.b: Neither $[A_1, C]$ nor $[B_1, C]$ have a red matching of size $2dm$.

Set $B^* = B_1 \cup A_1, A^* = B_2 \cup A_2$. Repeating the usual steps, all edges of $[A_1, C], [B_1, C]$ are blue, thus $[B^*, C]$ is blue.

Assume first that $|B^*| > (1-p-d-2\eta)m$. Then the usual step shows that $[A^*, C]$ is red because a blue matching of size dm in $[A^*, C]$ would give a blue matching of the required size (if b, c are large enough). Then it follows that $[C, D]$ is green. The presence of a red matching of size dm inside C gives a red matching of the required size. Assume that we have a blue matching of size dm inside C. Then a blue or red matching of size dn in $[B^*, D]$ would give a required matching in blue or red. Otherwise all edges of $[B^*, D]$ are green and it gives that all edges of $[A^*, B^*]$ are red and we get $EC2$. We conclude that all edges inside C are green. A green or red matching of size dm in $[B^*, D]$ would give the required matching. Thus $[B^*, D]$ is blue, then $[A^*, D]$ is red and again we have $EC2$, finishing the proof.

Thus we can assume that $|B^*| = |A_1| + |B_1| < (1 - p - d - 2\eta)m$. We claim that either $[A_2, C]$ or $[B_2, C]$ is blue. Indeed, by the usual procedure, suppose that $E_1 \cup E_2$ is a red matching, $E_1 \subset [A_2, C], E_2 \subset [B_2, C]$, $|E_1| = |E_2| = dm$. Since $|A|, |B| \geq (1-p)m \geq |A_1| + |B_1| + dm + 2\eta m$, it follows that $|A - A_1| = |A_2| \geq |B_1| + dm + 2\eta m$, $|B - B_1| = |B_2| \geq |A_1| + dm + 2\eta m$. Therefore in the bipartite graph $[A, B]$, all vertices of A_1 can be greedily matched to $B_2 \setminus V(E_2)$ with a red matching M_1, all vertices of B_1 can be greedily matched to $A_2 \setminus V(E_1)$ with a red matching M_2. Then let M_3 be the largest greedy matching of $[A_2 \setminus (V(E_1) \cup V(M_2)), B_2 \setminus (V(E_2) \cup V(M_1))]$. Then $M = M_1 \cup M_2 \cup M_3$ covers either A or B with error at most $\eta n = 4\eta m$. Counting M from the well-covered side of $[A, B]$, say from A,

$$|M| = (|E_1| + |E_2|) + (|M_1| + |M_2| + |M_3|) \geq 2dm + |A \setminus V(F_1)| - 4\eta m \geq$$

$$\geq dm + |A| - 4\eta m \geq (\frac{1}{4} + \eta)4m$$

provided that

$$d \geq (p + 4\eta)m \tag{40}$$

thus we have a red matching of the required size. Therefore the claim is proved, and by symmetry we assume $[A_2, C]$ is blue - implying that $[A, C]$ is blue. This - since $[B_1, C]$ is blue as well - allows to increase the blue matching to the required size. This finishes the proof of Lemma 14. □

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