One-sided coverings of colored complete bipartite graphs *

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Abstract

Assume that the edges of a complete bipartite graph $K(A, B)$ are colored with r colors. In this paper we study coverings of B by vertex disjoint monochromatic cycles, connected matchings, and connected subgraphs. These problems occur in several applications.

1 Introduction

Some problems for edge colored complete graphs naturally lead to edge colored complete bipartite graphs. For example, in [?] it was proved that in every r-coloring of

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the edges of K_n there is a connected monochromatic subgraph of order at least $\frac{n}{r-1}$. The proof was based on the result that in every $(r-1)$-coloring of the edges of a complete bipartite graph of order n there is a connected monochromatic subgraph of order at least $\frac{n}{r-1}$. (We remark here that later Füredi [**?**] obtained an important result on fractional matchings of hypergraphs which also implies the cited result.) As another example, in [**?**] it was proved that the vertex set of an r-colored complete graph can be covered by at most $cr^2 \log r$ vertex disjoint monochromatic cycles. The proof used the following "one-sided" covering lemma for bipartite graphs. If $G = K(A, B)$ is an r-colored complete bipartite graph with $|A| \geq r^3|B|$ then B can be covered by the vertices of at most r^2 vertex disjoint monochromatic cycles. This lemma was strengthened in [**?**] by showing that at most $(6r\lceil\log r\rceil+2r)$ vertex disjoint monochromatic cycles suffice to cover B if $|A| \geq r^2|B|$. This result has been used to improve the result cited above as follows: the vertex set of an r-colored complete graph can be covered by at most $100r \log r$ vertex disjoint monochromatic cycles. In these improvements, as in this paper, the Regularity Lemma played a major role.

In this paper one-sided coverings of colored complete bipartite graphs are explored further. The main result is the following improved form of the one-sided covering lemma for cycles.

Theorem 1. *For every fixed r there exists $n_0 = n_0(r)$ such that the following is true. Assume that the edges of a complete bipartite graph $K(A, B)$ are colored with r colors, where $|A| \geq n_0$. If $|A| \geq 2r|B|$, then B can be covered by at most $3r$ vertex disjoint monochromatic cycles.*

Note that this is a significant improvement over the above cited result from [**?**], where the statement is proved with $(6r\lceil\log r\rceil + 2r)$ cycles instead of $3r$ cycles for $|A| \geq r^2|B|$.

One tool of the proof, interesting in its own, is Theorem **??** which has an easy elementary proof. It says that the condition $|A| \geq r|B|$ ensures that in an r-colored complete bipartite graph $K(A, B)$, B can be covered by at most r vertex disjoint monochromatic connected matchings, in fact one can require that each matching has a distinct color. Here a monochromatic (say red) connected matching is a matching that lies in the same red connected component. Note that monochromatic connected matchings also played an important role in [**?**], [**?**]. Luczak [**?**] realized (through the Regularity Lemma) that the Ramsey numbers of monochromatic connected matchings and paths are about the same. Using this method the same set of authors [**?**] determined exactly the three color Ramsey numbers for paths which was an open problem for more then twenty years.

Theorem **??** is close to best possible: there are infinitely many r-colored complete bipartite graphs $K(A_m, B_m)$ such that $|A| = |B|(r-1-\frac{r-1}{mr!})$ and B can not be covered by the vertices of at most r vertex disjoint connected monochromatic matchings (Corollary **??**).

We also prove that the (much) weaker condition $|B| < e^{|A|/r^{r+3}} - |A|$ is enough to ensure a covering of B with at most r vertex disjoint monochromatic connected subgraphs (Corollary **??**). This result is obtained through Theorem **??**, a generalization of a result of Haxell and Kohayakawa ([**?**]). Notice that for $|B| \geq r$ one can color $K(A,B)$ by defining a partition of B into r nonempty parts and color all edges between A and the i-th part by color i. This coloring shows that in one sided coverings of complete bipartite graphs at least r monochromatic subgraphs are needed.

2 One-sided covers of bipartite graphs

In certain covering or partition problems one may require that all monochromatic objects have distinct colors, i.e. color repetition is not allowed. For example, it is not known whether every 3-colored complete graph can be covered by three monochromatic paths but there are examples when there is no cover if we want paths of distinct colors. Another example is the result of Haxell and Kohayakawa proving that every r-colored complete graph can be partitioned into at most r monochromatic trees of distinct colors. In this section we prove two lemmas about one-sided coverings where the colors of the objects are all different.

2.1 Covering B by monochromatic connected matchings

Theorem 2. *Assume that the edges of a complete bipartite graph $K(A,B)$ are colored with r colors, $|A| \geq r|B|$. Then there are vertex disjoint monochromatic connected matchings, all of different color, such that their union covers each vertex of B.*

Proof: We define by iteration r-colored complete bipartite graphs $G_i = K(A \setminus A_i, B)$, $\overline{G_i} = K(A_i, B)$ and sets $X_i \subseteq A_i, Y_i \subseteq B$, such that $A_i = \cup_{j=0}^{i} X_j$. Initially $G_0 = G, A_0 = X_0 = Y_0 = \emptyset$.

The general step is to select an arbitrary vertex $a \in A \setminus A_{i-1}$ and consider the partition $\mathcal{P}$ of B by putting two vertices $p, q \in B$ into the same class if and only if the colors of ap, aq are the same and label the class by the color of ap. Let E be defined as the set of those edges ab of G_{i-1} whose color is the same as the label of the class of $\mathcal{P}$ containing b. Observe that the existence of a matching of B to A_{i-1} using edges of E proves the theorem - then the procedure stops. Therefore we may assume that such a matching does not exist. By Hall's theorem there are sets $X_i \subseteq A_{i-1}, Y_i \subseteq B$ such that $|X_i| < |Y_i|$ and all edges of E incident to Y_i are incident to X_i (i.e. X_i is the set of E-neighbors of Y_i). Set $A_i = A_{i-1} \cup X_i$ and let G_i be the complete bipartite subgraph of G spanned by $[A \setminus A_i, B]$. Notice that $a \in X_i$ thus at least one new vertex is added to A_i. This finishes the definitions for step i.

Since at each step $|A_i| > |A_{i-1}|$, the procedure terminates with $A_m = A$ (and $G_m = \emptyset$) for some m. We show that this leads to a contradiction, thus the procedure must terminate with finding the required cover of B.

Assume that a vertex $b \in B$ in $\overline{G_m}$ is covered by k of the sets Y_i, w.l.o.g by $Y_1, Y_2, \dots, Y_k$. Then there are k distinct colors such that all edges incident to b in one of these colors go to $\cup_{i=1}^{k} X_i$. Therefore b is incident to edges of at most $r-k$ colors in G_k implying $k \le r$. Assuming that the procedure takes m steps, consider the hypergraph on vertex set B with edges Y_i,

$$r|B| \ge \sum_{x \in B} d(x) = \sum_{j=1}^{m} |Y_j| \ge \sum_{j=1}^{m} (|X_j| + 1) = |A| + m > |A| \tag{1}$$

contradicting the assumption of the theorem. □

A bipartite graph $G(k,l)$ is γ-**dense** if it contains at least γkl edges. We will need the following $(1-\varepsilon)$-dense version of Theorem **??** as well.

Theorem 3. *For some $0 < \varepsilon < 1/4$ assume that the edges of a $(1-\varepsilon)$-dense bipartite graph $G(A,B)$ are colored with r colors, $|A| \ge 2r|B|$. Then there are vertex disjoint monochromatic connected matchings, each of a different color, such that their union covers at least $(1-\sqrt{\varepsilon})$-fraction of the vertices of B.*

Proof: First we "trim" $G(A,B)$, we keep only the high degree vertices. For this purpose we use the following fact.

Fact 1. *Let $G(A,B)$ be a $(1-\varepsilon)$-dense bipartite graph. Then there is a subset $A' \subseteq A$ with $|A'| \ge (1-\sqrt{\varepsilon})|A|$ such that $deg(a,B) \ge (1-\sqrt{\varepsilon})|B|$ for all $a \in A'$.*

Indeed we get A' by removing those vertices from A that have degree less than $(1-\sqrt{\varepsilon})|B|$ in G. The number of these vertices is at most $\sqrt{\varepsilon}|A|$ from the density condition.

The proof of Theorem **??** is similar to that of Theorem **??**, but we always select the vertex a from the set A' and then we get a partition $\mathcal{P}$ of the ($\ge (1-\sqrt{\varepsilon})|B|$) neighbors of a in B. Then a matching covering these neighbors gives the desired covering with monochromatic connected matchings. The contradiction in (**??**) is similar, since we stop if there are no more A' vertices in the leftover:

$$r|B| \ge \ldots > (1-\sqrt{\varepsilon})|A|,$$

which is a contradiction for $0 < \varepsilon < 1/4$ and $|A| \ge 2r|B|$. □

To see that Theorem **??** can not be improved too much, let $G_1 = K(A, B)$ be the following r-colored complete bipartite graph. Set $A = [r]$ and each vertex of B is associated with a permutation of $[r]$. Vertex $i \in A$ is adjacent to a permutation in B in the color which is the i-th element of the permutation.

Lemma 1. *Assume that $\mathcal{S} = \{S_1, S_2, \dots S_t\}$ are monochromatic stars of G_1 with their centers in A and such that the union of their leaves cover B. Then $t \geq r$ with equality if and only if : (i) all centers coincide and all colors are different, or (ii) all centers are different and all colors are the same.*

Proof: Suppose that $X_i \subseteq [r]$, $i \in [r]$ is the set of colors (we always color by colors $1, 2, \dots, r$) appearing on the members of $\mathcal{S}$ with center at $i \in A$. The sets $\overline{X_i} = [r] \setminus X_i$ have no distinct representatives. Indeed, the existence of such a set of representatives is equivalent to the existence of a vertex of B uncovered by the leaves of the stars, contradicting the assumption. Thus, by Hall's theorem, there exists a set $A^* \subseteq A$ such that $|A^*| = j$ and $|\cup_{i \in A^*} \overline{X_i}| \leq j - 1$ implying that $|\cap_{i \in A^*} X_i| \geq r - j + 1$. Therefore

$$t \geq \sum_{i \in A^*} |X_i| \geq |A^*|| \cap_{i \in A^*} X_i| \geq j(r - j + 1) \geq r$$

with equality in the last inequality if and only if $j = 1$ or $j = r$ giving cases (i) and (ii) in the lemma. □

The following corollary shows that r can not be essentially lowered in the condition $|A| \geq r|B|$ of Theorem **??**.

Corollary 1. *For every fixed r there are infinitely many r-colored complete bipartite graphs $[A_m, B_m]$ such that $|A_m| = |B_m|(r - \frac{1}{m(r-1)!})$ and B_m can not be covered by the vertices of vertex disjoint connected monochromatic matchings, each having a different color.*

Proof: Consider the graph $G_1 = K(A, B)$ and replace each vertex of B by a set of m vertices, each vertex of A by a set of $mr! - 1$ vertices. This gives an r-colored complete bipartite graph $G_1^m = K(A_m, B_m)$ with $|B_m| = mr!$, $|A_m| = r(mr! - 1) = |B_m|(r - \frac{1}{m(r-1)!})$ for every positive integer m. Since for any $x \in B$ two edges of G_1 incident to x are always colored with different color, a connected monochromatic matching in G_m corresponds (can be contracted) to a monochromatic star in G_1 with center in A. Thus the required covering of B_m with disjoint monochromatic matchings corresponds to a star-cover as in Lemma **??**. Applying Lemma **??**, the only possibility to cover B_m is coming from (i), i.e. all monochromatic matchings are using the vertices of a replacement of a single vertex of $A \subseteq V(G_1)$. Since any vertex of A is replaced by $mr! - 1$ vertices there is no matching from that set to B_m since $|B_m| = mr!$. □

If one does not require that all monochromatic connected matchings have distinct colors we have only a weaker construction:

Corollary 2. *For every fixed r there are infinitely many r-colored complete bipartite graphs $K(A_m, B_m)$ such that $|A_m| = |B_m|(r - 1 - \frac{r-1}{mr!})$ and B_m can not be covered by the vertices of at most r vertex disjoint connected monochromatic matchings.*

Proof: It is similar to the proof of Corollary **??**. The only difference is that here we use G_1^* obtained from G_1 by deleting an arbitrary vertex of A. Then, using the same replacements as in the proof of Corollary **??**, possibility (ii) of an r-covering is eliminated from Lemma **??** and the proof follows. □

2.2 Covering B by monochromatic cycles

In this section we prove our main result, Theorem **??**. We will use the bipartite r-color version of the Regularity Lemma (for an extensive survey on different variants of the Regularity Lemma see [**?**]). For this purpose we will need some definitions. For non-empty A and B,

$$d_G(A, B) = \frac{e_G(A, B)}{|A||B|}$$

is the **density** of the graph between A and B.

Definition 1. *The bipartite graph $G = (A, B, E)$ is (ε, G)-**regular** if*

$$X \subset A,\ Y \subset B,\ |X| > \varepsilon|A|,\ |Y| > \varepsilon|B| \quad \textit{imply} \quad |d_G(X, Y) - d_G(A, B)| < \varepsilon,$$

*otherwise it is (ε, G)-**irregular**. Furthermore, (A, B, E) is (ε, δ, G)-**super-regular** if it is (ε, G)-regular and*

$$deg_G(a) > \delta|B| \ \ \forall\, a \in A, \qquad deg_G(b) > \delta|A| \ \ \forall\, b \in B.$$

Proof of Theorem ??: Consider a r-edge coloring $(G_1, G_2, \ldots, G_r)$ of $K(A, B)$. We apply the bipartite r-color version of the Regularity Lemma with a sufficiently small ε. By standard arguments we may assume that for each cluster that is not V_0, all vertices of the cluster belong to the same partite class. Thus we get a partition $A = V_A^0 + V_A^1 + \ldots V_A^{l_A}$, $B = V_B^0 + V_B^1 + \ldots V_B^{l_B}$, where $|V_A^{j_1}| = |V_B^{j_2}| = m$, $1 \le j_1 \le l_A$, $1 \le j_2 \le l_B$ and $|V_A^0| \le \varepsilon|A|$, $V_B^0| \le \varepsilon|B|$. We define the reduced graph G^R: The vertices of G^R are $A^R = \{p_A^{j_1} \mid 1 \le j_1 \le l_A\}$ and $B^R = \{p_B^{j_2} \mid 1 \le j_2 \le l_B\}$, and we have an edge between vertices $p_A^{j_1}$ and $p_B^{j_2}$, if the pair $\{V_A^{j_1}, V_B^{j_2}\}$ is (ε, G_s)-regular for $s = 1, 2, \ldots, r$. Thus we have a one-to-one correspondence $f : \{p_A^j, p_B^j\} \rightarrow \{V_A^j, V_B^j\}$ between the vertices of G^R and the non-exceptional clusters of the partition.

Then $G^R = (A^R, B^R)$ is a $(1-\varepsilon)$-dense bipartite graph. Define an r-edge coloring $(G_1^R, G_2^R, \ldots, G_r^R)$ of G^R in the following way. The edge between the clusters $V_A^{j_1}$ and $V_B^{j_2}$ is colored with a color s that contains the most edges from $K(V_A^{j_1}, V_B^{j_2})$, thus clearly

$$\left|E_{G_s}(V_A^{j_1}, V_B^{j_2})\right| \geq \frac{1}{r}|V_A^{j_1}||V_B^{j_2}|.$$

Applying Theorem **??** to G^R we get at most r vertex disjoint monochromatic connected matchings that cover at least $(1-\sqrt{\varepsilon})$-fraction of the vertices of B^R. The clusters not covered by these monochromatic connected matchings are placed into the exceptional set V_B^0. With standard techniques, going back to the original graph, from these monochromatic connected matchings we can construct monochromatic cycles that cover most of the clusters belonging to these connected matchings. Indeed, let us take a monochromatic connected matching M, say M is in G_1^R and has size $|M| = l_1$. We will make this connected matching into a cycle in G_1.

Denote the matching $M = \{e_1, e_2, \ldots, e_{l_1}\}$ between the two sets of end points $U_A \subseteq A^R$ and $U_B \subseteq B^R$. Furthermore, let $f(e_i) = (V_A^i, V_B^i)$ for $1 \leq i \leq l_1$ where V_A^i and V_B^i are the clusters assigned to the endpoints of e_i.

We need to do some preparations on the matching M. First we will find connecting paths between the edges of the matching M. Since M is a connected matching in G_1^R we can find l_1 connecting paths P_i^R in G_1^R from $f^{-1}(V_B^i)$ to $f^{-1}(V_A^{i+1})$ for every $1 \leq i \leq l_1$ (for $i = l_1$ we go from $f^{-1}(V_B^{l_1})$ back to $f^{-1}(V_A^1)$). Note that these paths in G_1^R may not be internally vertex disjoint. From these paths P_i^R in G_1^R we can construct vertex disjoint connecting paths P_i in G_1 connecting a typical vertex v_B^i of V_B^i to a typical vertex v_A^{i+1} of V_A^{i+1}. More precisely we construct P_1 with the following simple greedy strategy. Denote $P_1^R = (p_1, \ldots, p_t), 2 \leq t \leq l_A + l_B$, where according to the definition $f(p_1) = V_B^1$ and $f(p_t) = V_A^2$. Let the first vertex u_1 $(= v_B^1)$ of P_1 be a vertex $u_1 \in V_B^1$ for which $deg_{G_1}(u_1, f(p_2)) \geq (1/r - \varepsilon)m$ and $deg_{G_1}(u_1, V_A^1) \geq (1/r-\varepsilon)m$. By (ε, G_1)-regularity most of the vertices satisfy this in V_B^1. The second vertex u_2 of P_1 is a vertex $u_2 \in (f(p_2) \cap N_{G_1}(u_1))$ for which $deg_{G_1}(u_2, f(p_3)) \geq (1/r - \varepsilon)m$. Again by (ε, G_1)-regularity most vertices satisfy this in $f(p_2) \cap N_{G_1}(u_1)$. The third vertex u_3 of P_1 is a vertex $u_3 \in (f(p_3) \cap N_{G_1}(u_2))$ for which $deg_{G_1}(u_3, f(p_4)) \geq (1/r - \varepsilon)m$. We continue in this fashion, finally the last vertex u_t $(= v_A^2)$ of P_1 is a vertex $u_t \in (f(p_t) \cap N_{G_1}(u_{t-1}))$ for which $deg_{G_1}(u_t, V_B^2) \geq (1/r - \varepsilon)m$.

Then we move on to the next connecting path P_2. Here we follow the same greedy procedure, we pick the next vertex from the next cluster in P_2^R. However, if the cluster has occurred already on the path P_1^R (or on any other connecting paths later in the procedure), then we just have to make sure that we pick a vertex that has not been used so far. Since the total number of vertices on the connecting paths will be a constant, this is feasible.

We continue in this fashion and construct the vertex disjoint connecting paths P_i in G_1, $1 \leq i \leq l_1$. These will be parts of the final cycle in G_1. We remove the internal vertices of these paths from G_1. Furthermore, we remove some more vertices from each $(V_A^i, V_B^i), 1 \leq i \leq l_1$ to achieve super-regularity in all of these pairs. From V_A^i we remove all exceptional vertices v_A for which

$$deg_{G_1}(v_A, V_B^i) < \left(\frac{1}{r} - \varepsilon\right) m,$$

and from V_B^i all exceptional vertices v_B for which

$$deg_{G_1}(v_B, V_A^i) < \left(\frac{1}{r} - \varepsilon\right) m.$$

(ε, G_1)-regularity guarantees that at most εm vertices are removed from each cluster. By doing this we may create some discrepancies in the cardinalities of the clusters of this connected matching. We remove some more vertices from clusters V_A^i and V_B^i to assure that now we have the same number of vertices left in each cluster of the matching. For simplicity we still keep the notation $f(e_i) = (V_A^i, V_B^i)$ for the modified clusters. The removed vertices are added to the exceptional set V_B^0.

To get the final cycle in G_1 will use the following property of (ε, δ, G)-super-regular pairs.

Lemma 2. *For every $\delta > 0$ there exist an $\varepsilon > 0$ and m_0 such that the following holds. Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2| = m \geq m_0$, and let the pair (V_1, V_2) be (ε, δ, G)-super-regular. Then for every pair of vertices $v_1 \in V_1, v_2 \in V_2$, G contains a Hamiltonian path connecting v_1 and v_2.*

A lemma somewhat similar to Lemma **??** is used by Łuczak in [**?**] and by Haxell in [**?**]. Lemma **??** is a special case of the much stronger Blow-up Lemma (see [**?**] and [**?**]).

Applying Lemma **??** for $1 \leq i \leq l_1$, we get a path in $G_1|_{f(e_i)}$ connecting v_A^i and v_B^i that contains all of the remaining vertices of $f(e_i)$ (in case of $i = 1$ we just select a Hamiltonian path of $f(e_1)$ starting from v_B^1 and in case of $i = l_1$, we select a Hamiltonian path of $f(e_{l_1})$ starting from $v_A^{l_1}$). These paths together with the connecting paths give us the desired G_1 cycle.

We repeat this procedure for all the at most r monochromatic connected matchings. This gives us a covering of B with at most r vertex disjoint monochromatic cycles that cover B apart from at most $2\sqrt{\varepsilon}|B|$ vertices. For the covering of these remaining vertices we can apply the following lemma from [**?**] (Lemma 8 in [**?**]).

Lemma 3. *There exists a constant n_0 such that the following is true. Assume that the edges of the complete bipartite graph $K(A, B)$ are colored with r colors. If $|A| \geq n_0$,*

$|B| \le |A|/(8r)^{8(r+1)}$, *then* B *can be covered by at most* $2r$ *vertex disjoint monochromatic cycles.*

Indeed we can apply this lemma as ε is sufficiently small. Thus altogether we covered B with at most $r + 2r = 3r$ vertex disjoint monochromatic cycles, and thus finishing the proof of Theorem **??**. $\square$

2.3 Covering B by monochromatic connected subgraphs

We show here that a covering of B with vertex disjoint connected monochromatic subgraphs is possible if $|B|$ is not too large compared to $|A|$. To achieve that, we need a generalization of the following result.

Theorem 4. *(Haxell, Kohayakawa, [**?**]) Let* $r \ge 1$ *and* $n \ge 3r^4 r!(1 - 1/r)^{3(1-r)} \log r$ *be integers, and suppose the edges of* K_n *are colored with* r *colors. Then* K_n *contains* $t \le r$ *monochromatic trees* $T_1, \ldots, T_t$ *of radius at most* 2*, each of different color, such that their vertex sets* $V(T_i)$ $(1 \le i \le t)$ *partition the vertex set of* K_n*.*

We shall prove that Theorem **??** remains true even if there is a not too large "hole" in K_n. More precisely, let $H = H(A, B)$ be the graph whose vertex set is partitioned into A and B and contains all edges except the ones inside B.

Theorem 5. *Let* $r \ge 1$ *and suppose the edges of* $H = H(A, B)$ *are colored with* r *colors, where* $|A| = n$, $|B| < e^{n/5r^{r+3}} - n$ *(in particular,* n *sufficiently large). Then* H *contains* $t \le r$ *vertex disjoint monochromatic trees* $T_1, \ldots, T_t$ *of radius at most* 2*, each of different color, such that their vertex sets* $V(T_i)$ $(1 \le i \le t)$ *partition the vertex set of* H*.*

Corollary 3. *Let* $r \ge 1$ *and suppose the edges of the complete bipartite graph* $K(A, B)$ *are colored with* r *colors,* $|A| = n$*. If* $|B| < e^{n/5r^{r+3}} - n$ *(in particular,* n *sufficiently large) then* B *can be covered by the vertices of vertex disjoint monochromatic trees* $\{T_1, \ldots, T_t\}$, $t \le r$*, of radius at most* 2*, each of different color.*

Proof: Consider an arbitrary coloring of edges of $K(A, B)$ with r colors and color all $\binom{n}{2}$ edges inside A with a new color, say, $r + 1$. This is an $(r + 1)$-coloring of the edges of $H(A, B)$. Thus, by Theorem **??** it contains $t \le r + 1$ monochromatic trees $T_1, \ldots, T_t$ of radius at most 2, each of different color such that their vertex sets $V(T_i)$ $(1 \le i \le t)$ partition the vertex set of H. But color $r + 1$ can be used only to cover some subset of vertices in A. Therefore the trees whose color is not $r + 1$ have the required property. $\square$

Proof of Theorem ??: We may assume $r \geq 2$, otherwise the statement is trivial. We tailor the proof of Haxell and Kohayakawa [**?**] to our needs. For some k, $1 \leq k \leq r$, k-**anchor** is a k-edge colored complete bipartite graph $[X, Y]$ with $|X| = k$, $|Y| \geq s_k$ such that for $x_i \in X$ all edges of the form $[x_i, Y]$ are colored with color i ($i = 1, \ldots, k$). Let $s_i = n/r^i$, for $1 \leq i \leq r$ and $s_i = 0$ for $i > r$. Clearly, the sequence s_i is non-increasing. Let $\Gamma_i(v, V)$ be the neighborhood of the vertex v in color i in some subset of vertices V, $d_i(v, V) = |\Gamma_i(v, V)|$.

Consider an arbitrary r-edge coloring of $H(A, B)$, $|A| = n$, and a t-anchor $[X, Y]$ such that $X \subseteq A \cup B$, $Y \subseteq A$ and maximal in the sense that no $t+1$-anchor $[X_1, Y_1]$ with $Y_1 \subseteq A$ exists in this coloring. Set $|X| = \{x_1, \ldots, x_t\}$ and assume $\{1, \ldots, t\}$, are the colors of the t-anchor. Since a 1-anchor can be defined by selecting $x_1 \in B$ and defining color 1 as the majority color on $[x_1, A]$, t is well defined.

Now we proceed to prove that the vertices of $Z'_0 = (A \cup B) \setminus (X \cup Y)$ can be covered by vertex disjoint monochromatic stars with centers in Y. In fact we achieve this by applying the following greedy procedure in less than

$$\lfloor s_r/2r \rfloor \leq s_r/2r \leq s_t/2r \leq |Y| \tag{2}$$

steps.

Let $y_1 \in Y$ be the vertex which is adjacent to the most vertices in Z'_0 in some color $i_1 \in [t]$ (i.e., we pick a monochromatic star centered in Y containing the most leaves in Z'_0). Let $Z_1 \subseteq Z'_0$ be the set of the leaves just chosen, $Z'_1 = Z'_0 \setminus Z_1$. In general, assume that vertices $y_1, \ldots, y_q \in Y$, not necessarily different colors $i_1, \ldots, i_q \in [t]$, pairwise disjoint sets $Z_1, \ldots, Z_q$ and sets $Z'_1, \ldots, Z'_q$ are already defined. Let $Y_q = Y \setminus \{y_1, \ldots, y_q\}$. Select $y_{q+1} \in Y_q$ and $i_{q+1} \in [t]$ such that $d_{i_{q+1}}(y_{q+1}, Z'_q)$ is maximal, $Z_{q+1} = \Gamma_{i_{q+1}}(y_{q+1}, Z'_q)$, $Z'_{q+1} = Z'_q \setminus Z_{q+1} = Z'_0 \setminus (\cup_{i=1}^{q} Z_i)$.

Consider the edges between the (yet uncovered) vertices in Z'_q and the (yet not used) vertices in Y_q (Y_q is nonempty because of (**??**)). We have

$$\sum_{z \in Z'_q} \sum_{1 \leq i \leq t} d_i(z, Y_q) > |Z'_q| \left(|Y| - q - (r - t)s_{t+1}\right).$$

Indeed, $|Y_q| = |Y| - q$, and a vertex $z \in Z'_q$ is adjacent to less then s_{t+1} vertices of Y in each color j, $t+1 \leq j \leq r$. Else, if $z \in Z'_q$, $Y^* \subset Y$, $|Y^*| \geq s_{t+1}$ exist such that all edges in $[z, Y^*]$ colored j, $t+1 \leq j \leq r$, then $\{z\} \cup X$ with Y^* would form a $(t+1)$-anchor, contradicting the choice of t. Therefore, by a standard averaging argument

$$\begin{aligned} d_{i_{q+1}}(y_{q+1}, Z'_q) &\geq \frac{1}{t} \frac{1}{|Y| - q} |Z'_q| \left(|Y| - q - (r - t)s_{t+1}\right) \\ &= \frac{1}{t} |Z'_q| \left(1 - \frac{(r - t)s_{t+1}}{|Y| - q}\right). \end{aligned}$$

Using (**??**) we have

$$\begin{aligned}
|Z'_{q+1}| &= |Z'_q| - |Z_{q+1}| \le |Z'_q| \left(1 - \frac{1}{t}\left(1 - \frac{(r-t)s_{t+1}}{|Y|-q}\right)\right) && (3)\\
&\le |Z'_q| \exp\left\{-\frac{1}{t}\left(1 - \frac{(r-t)s_{t+1}}{|Y|-q}\right)\right\} && (4)\\
&\le |Z'_q| \exp\left\{-\frac{1}{t}\left(1 - \frac{(r-1)s_{t+1}}{s_t - s_t/(2r)}\right)\right\} && (5)\\
&= |Z'_q| \exp\left\{-\frac{1}{(2r-1)t}\right\} \le |Z'_q| \exp\left\{-\frac{1}{2rt}\right\} \le |Z'_q| \exp\left\{-\frac{1}{2r^2}\right\}. && (6)
\end{aligned}$$

To obtain (**??**) we utilized $|Y| \ge s_t$ and (**??**), and (**??**) follows from (**??**) by $s_t = rs_{t+1}$. Summarizing,

$$|Z'_{q+1}| \le |Z'_q| \exp\left\{-\frac{1}{2r^2}\right\},$$

and we let our algorithm run for at most $\lfloor s_r/2r \rfloor$ steps. Therefore we shall cover all vertices in Z'_0 if

$$|Z'_0| \left(e^{-\frac{1}{2r^2}}\right)^{\left\lfloor \frac{s_r}{2r} \right\rfloor} \le |Z'_0| \left(e^{-\frac{1}{2r^2}}\right)^{\left(\frac{s_r}{2r}-1\right)} = |Z'_0| \left(e^{-\frac{1}{2r^2}}\right)^{\left(\frac{n}{2r^{r+1}}-1\right)} \le |Z'_0| e^{-\frac{n}{5r^{r+3}}} < 1,$$

which is satisfied by

$$|Z'_0| < |V(H)| < e^{n/5r^{r+3}}. \qquad (7)$$

Assume that we covered Z'_0 with monochromatic stars with centers $y_1, \dots, y_{q_0}$, colors $i_1, \dots, i_{q_0}$ and sets of leaves $Z_1, \dots, Z_{q_0}$. The partitioning trees $T_1, \dots, T_t$ of colors $1, \dots, t$ are defined as follows.

$$V(T_i) = \{x_i\} \cup \bigcup_{k \in [q_0]:\ i_k = i} \left(\{y_k\} \cup Z_k\right),$$

and

$$E(T_i) = \bigcup_{k \in [q_0]:\ i_k = i} \left(\{(x_i, y_k)\} \cup \{(y_k, z) : z \in Z_k\}\right).$$

Clearly, the vertices of $Y_{q_0} = Y \setminus \{y_1, \dots y_{q_0}\}$ can be added to, say, T_1. $\square$

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