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Note

Hall ratio of the Mycielski graphs

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Abstract

Let $n(G)$ denote the number of vertices of a graph G and let $\alpha(G)$ be the independence number of G, the maximum number of pairwise nonadjacent vertices of G. The *Hall ratio* of a graph G is defined by

$$\rho(G) = \max\left\{\frac{n(H)}{\alpha(H)} : H \subseteq G\right\},$$

where the maximum is taken over all induced subgraphs H of G. It is obvious that every graph G satisfies $\omega(G) \leqslant \rho(G) \leqslant \chi(G)$ where ω and χ denote the clique number and the chromatic number of G, respectively. We show that the interval $[\omega(G), \rho(G)]$ can be arbitrary large by estimating the Hall ratio of the Mycielski graphs.
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1. Introduction

The chromatic number of a graph G, $\chi(G)$, is certainly at least the number of vertices of G, $n(G)$, divided by its independence number, $\alpha(G)$. Therefore

$$\rho(G) = \max\left\{\frac{n(H)}{\alpha(H)} : H \subseteq G\right\},$$

the *Hall ratio* of G , is a natural lower bound for $\chi(G)$. The Hall ratio is so named because of its connection with Hall's condition, which is of interest in the study of list-colorings; see [2,4–6]. It is immediate that $\omega(G) \leqslant \rho(G)$ where $\omega(G)$ is the *clique number*, the maximum number of pairwise adjacent vertices in G. The Hall ratio is also related to a well-known parameter, $\chi_f(G)$, the *fractional chromatic number* that has many equivalent definitions; see [10], where one can implicitly find the inequality $\rho(G) \leqslant \chi_f(G)$. Thus we have

$$\omega(G) \leqslant \rho(G) \leqslant \chi_f(G) \leqslant \chi(G). \tag{1}$$

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The interval $[\rho(G), \chi(G)]$ can be arbitrary large. In fact it follows from the discussions in [10] (Chapter 3) concerning the Kneser graphs that for every $\varepsilon > 0$ and integer $k \geqslant 2$ there is a Kneser graph G with $\chi(G) = k$ and $\rho(G) < 2 + \varepsilon$. However, the interval $[\omega(G), \rho(G)]$ is bounded for all Kneser graphs G, because $\rho(G) = \chi_f(G) \leqslant \omega(G) + 1$.

In this paper we show that for the well known sequence of the Mycielski graphs, M_k, the length of both intervals $[\omega(M_k), \rho(M_k)]$ and $[\rho(M_k), \chi(M_k)]$ tends to infinity with k. For the second interval this follows from a result of Larson et al. [8] (see also in [10]) establishing the recurrence

$$\chi_f(M_k) = \chi_f(M_{k-1}) + \frac{1}{\chi_f(M_{k-1})}. \tag{2}$$

The recurrence (2) implies that $\chi_f(M_k)$ is $\Theta(\sqrt{k})$; actually the bounds $\sqrt{2k} < \chi_f(M_k) < \sqrt{2k + \frac{1}{2}\log k}$ are derived in [9] (Problem 60). Because $\chi(M_k) = k$ and $\rho(M_k) \leqslant \chi_f(M_k)$, the length of $[\rho(M_k), \chi(M_k)]$ tends to infinity with k.

To see that the intervals $[\omega(M_k), \rho(M_k)]$ are getting large as well, we shall prove here a simple but perhaps surprising property of the Mycielski graphs in Theorem 1. This result combined with the lower bounds on the Ramsey number $R(3, m)$ will give an estimate of $\rho(M_k)$ in Theorem 2.

2. The Hall ratio of the Mycielski graphs

The Mycielski graphs M_k form a sequence of triangle-free k-chromatic graphs defined recursively starting with $M_2 = K_2$ and M_k is obtained from M_{k-1} by adding an independent set of vertices of size $n(M_{k-1})$ that twin those in M_{k-1} (i.e., their neighbors are exactly the neighbors of their mate in M_{k-1}), then adding one further vertex, v_k, which is adjacent to each of the vertices in the added independent set. We observe the following remarkable property of Mycielski graphs [1].

Theorem 1. *Every connected triangle-free graph with n vertices is an induced subgraph of the Mycielski graph M_n.*

Proof. Let G be a connected triangle-free graph of order n. We shall prove the existence of an embedding of G into M_n by induction on n. Clearly the result holds for $n = 2$ when $G = K_2$. Assume that any connected triangle-free graph with $n - 1$ vertices has an embedding into M_{n-1} and so into M_n. Let v be a vertex of G such that $G - v$ is still connected. By the induction hypothesis, M_{n-1} has an induced subgraph isomorphic to $G - v$, thus there is an embedding G' of $G - v$ into M_n. Begin replacing each vertex of G' in the neighborhood set of v by its twin in M_n. Since G is triangle-free, the neighbors of v form an independent set, so the resulting subgraph of M_n is isomorphic to $G - v$ as well. The additional vertex v_n of M_n is adjacent to each of the twins, thus by identifying v_n with v we obtain a required embedding of G into M_n. □

Let G be a Ramsey graph, more precisely a connected triangle-free graph with $\alpha(G) \leqslant m - 1$ whose order is one less than the Ramsey number $R(3, m)$. ($R(3, m)$ is the smallest integer s for which every graph of s vertices contains either a triangle or a set of m independent vertices, see [3].) It follows from Theorem 1 that G is an induced subgraph of M_k for $k = R(3, m) - 1$. A well-known result of Kim [7] is the lower bound $R(3, m) \geqslant cm^2/\log m$ for some constant c (upper bound of the same order of magnitude was known before). This implies $\rho(M_k) \geqslant \rho(G) \geqslant (R(3, m) - 1)/(m-1) \geqslant cm/\log m$ for m sufficiently large. Combining this with the fact that the asymptotic of $\chi_f(M_k)$ is $\sqrt{2k}$ we obtain

Theorem 2. *Assume that $k = R(3, m) - 1 (=\Theta(m^2/\log m))$. Then $c_1 m/\log m \leqslant \rho(M_k) \leqslant c_2 m/\sqrt{\log m}$ (where c_1, c_2 are constants).*

Since $\omega(M_k) = 2$, Theorem 2 implies that the length of the interval $[\omega(M_k), \rho(M_k)]$ tends to infinity with k.

We know very little about exact values. It is easy to verify that $\rho(M_2) = 2$, $\rho(M_3) = \frac{5}{2}$. The subgraphs of M_4 achieving its Hall ratio, $\rho(M_4) = \frac{8}{3}$, are Ramsey graphs (eight-vertex triangle-free graphs with independence number 3). The graph M_5 has at least two non-isomorphic subgraphs that achieve its Hall ratio, $\rho(M_5) = \frac{15}{5} = \frac{18}{6}$.

From (2) $\chi_f(M_4) = \frac{29}{10}$ and $\chi_f(M_5) = \frac{941}{290}$, these graphs yield examples where the fractional chromatic number and Hall ratio are unequal. But how unequal can they be? Even the more modest question is unanswered, a favorite of Pete Johnson's (personal communication): Is $\chi_f(G)/\rho(G)$ bounded?

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