Finding a monochromatic subgraph or a rainbow path

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Abstract

For simple graphs G and H, let $f(G, H)$ denote the least integer N such that every coloring of the edges of K_N contains either a monochromatic copy of G or a rainbow copy of H. Here we investigate $f(G, H)$ when $H = P_k$. We show that even if the number of colors is unrestricted when defining $f(G, H)$, the function $f(G, P_k)$, for $k = 4$ and 5, equals the $(k - 2)$– coloring diagonal Ramsey number of G.

1 Introduction

For simple graphs G and H, let $f(G, H)$ denote the least integer n such that every coloring of the edges of the complete graph K_n contains either a monochromatic copy of G (all its edges have the same color) or a rainbow copy of H (no two edges have the same color).

In [7] Jamison, Jiang, and Ling observe that as a corollary of the Erdős–Rado Canonical Ramsey Theorem [1] $f(G, H)$ is finite if and only if either G is a star or H has no cycles. For the case when G and H are trees having s and t edges, respectively, the upper bound $f(G, H) \leq (s - 1)(t^2 + 3t)$ is proved, and the existence of an absolute constant α is conjectured such that $f(G, H) \leq \alpha st$. They also ask whether $f(G, H)$ is maximized by $f(P_{s+1}, P_{t+1})$.

For the particular case when G is a tree with $s \geq 2$ edges and H is a path of length $t \geq 2$, that is $H = P_{t+1}$, Wagner [10] proves the bound $f(G, P_t) \leq 224(s - 1)^2 t$.

Here we investigate $f(G, P_k)$ for $k = 4, 5$. It turns out that these numbers are related to the 2– and 3– coloring Ramsey numbers of G. Let $R(G, G)$ /or $R(G, G, G)$/ be the minimum N such that if the edges of K_N are colored with two /or three/ colors, then there is always a monochromatic copy of G.

We also use the concept of local k–coloring introduced in Gyárfás et al. [5]. An edge coloring of a graph (using any number of colors) is called a local k–coloring if the set of all edges incident with any given vertex are colored by at most k colors.

In Section 2 we show that $f(G, P_4) = R(G, G)$ for any graph G of order at least 5 (Theorem 2). Section 3 deals with $f(G, P_5)$. It is shown that $f(G, P_5) = R(G, G, G)$ if $G = P_n$, or $G = C_n$, or G is non-bipartite and connected (Theorems 6, 10, and 8). In Section 4 we prove $f(G, P_5) = f(G, T_5)$ where G is one of the graphs listed above and T_5 is a star $K_{1,3}$ with one edge subdivided. In Section 5 related questions are proposed for further study.

2 Rainbow P_4

We need the characterization of local 2– colorings of a clique given in [5] as is described in the following lemma. We assume that the colors in a coloring are consecutive natural numbers, and two colorings of a graph are the same if they differ only in a permutation of colors.

Lemma 1. *Given a local 2–coloring of a clique, let $V = \cup A(i,j)$ be an arbitrary partition of its vertex set, where each vertex of $A(i,j)$ is incident with edges colored i or j. Then the following two cases are possible.*

(i) Three colors are used and none of them is present at all vertices so that $V = A(1,2) \cup A(2,3) \cup A(3,1)$.

(ii) There exists a color, say 1, that is present at all vertices so that $V = \cup_j A(1,j)$. □

Theorem 2. *For every graph G of order $n \geq 5$, $f(G, P_4) = R(G,G)$.*

Proof. Let $N = R(G,G)$. Any 2-coloring of K_{N-1} without a monochromatic G is clearly free of a rainbow P_4 requiring three distinct colors. Hence $N \leq f(G, P_4)$. To verify the reverse inequality, $f(G, P_4) \leq N$, take an arbitrary coloring of K_N with any number of colors. We shall verify that if no rainbow P_4 exists then there is a monochromatic G.

First we assume that the coloring is not a local 2–coloring. Then there exist a vertex x_0 and incident edges x_0x_1, x_0x_2, and x_0x_3 colored with distinct colors, say with $1, 2$, and 3, respectively. Because there is no rainbow P_4, the color of x_2x_3, x_1x_3, and x_1x_2 must be $1, 2$, and 3, respectively. Notice that any further edge yx_0 would create a rainbow P_4 with two appropriate vertices among x_1, x_2, x_3. Hence $N = 4$, contradicting the condition $N \geq n \geq 5$.

Next we assume that the local 2–coloring has a rainbow triangle (x_1, x_2, x_3), say the color of x_1x_2, x_2x_3, and x_3x_1 is $3, 1$, and 2, respectively. Then any further vertex x_0 would be incident with three distinct colors. Indeed, x_0x_1, x_0x_2, and x_0x_3 must be colored by $1, 2$, and 3, respectively, to avoid rainbow P_4, contradicting the local 2–coloring.

Using Lemma 1 we conclude that the local 2–coloring of K_N obeys property (ii). If three colors were used by that coloring, then there are edges xx' and yy' colored with 2 and 3, respectively, such that $x \in A(1,2)$ and $y \in A(1,3)$. Then the color of xy is 1, and hence (x', x, y, y') was a rainbow P_4. Thus we conclude that the coloring of K_N must use two colors, and thus contains a monochromatic copy of G. □

3 Rainbow P_5

In order to extend Theorem 2 for $f(G, P_5)$ we need the following key lemma.

Lemma 3. *For $N \geq 3$, every edge coloring of K_N with at least four colors and without a rainbow P_5 contains a locally 2–colored clique of order at least $N - 2$.*

Proof. The claim is obvious for $N \leq 5$, thus we assume $N \geq 6$. Suppose that the coloring of K_N is not a local 2–coloring. Then there exists a rainbow star $K_{1,3}$, that is a vertex x_0 and incident edges x_0x_1, x_0x_2, and x_0x_3 colored with distinct colors, say $1, 2$, and 3, respectively. Let $A = \{x_0, x_1, x_2, x_3\}$ and $B = \{V(K_N) \setminus A\}$. In the cases below we shall discuss the occurrence of the fourth color, say 4.

Case 1: *Edges in A are 3-colored with colors $1, 2,$ and 3.*

Subcase 1.1: *Some edge between the sets $A \setminus \{x_0\}$ and B has a new color, say x_1y is colored 4 for some $y \in B$.*

Because there is no rainbow P_5 starting with edge yx_1 and including the vertices of A, it follows that the color of x_2x_3, x_1x_3, and x_1x_2 must be $1, 2$, and 3, respectively. Then every edge yz with $z \in B$ must have color 4 to avoid rainbow P_5 running into A. This immediately implies that every edge between $A \setminus x_0$ and B is also colored with 4. Therefore $V(K_N) \setminus \{x_0, x_3\}$ induces a clique with one edge only (namely x_1x_2) having a color different from 4.

Subcase 1.2: *yw for some $y, w \in B$ is colored 4.*

The colors of yx_1, yx_2, and yx_3 must be $1, 2$, and 3, respectively, to avoid a rainbow P_5. Furthermore, for the same reason, the color of x_2x_3, x_1x_3, and x_1x_2 must NOT be $1, 2$, and 3, respectively. We assume by symmetry that x_1x_2 has color 1. Then in the path (x_1, x_2, x_3, y, w) the color of x_2x_3 can not be $1, 2$ or any new color. Thus x_2x_3 has color 3. The color of x_1x_3 is not new, it is either 1 or 3. In each case the path (w, y, x_2) extends to a rainbow P_5 by including vertices x_1 and x_2 in appropriate order.

Subcase 1.3: *x_0y has color 4 for some $y \in B$.*

We suppose that Subcases 1.1 and 1.2 do not apply, so that all edges induced by B and between the sets $A \setminus x_0$ and B are colored with colors $1, 2$, or 3. Let wy have color 1 for some $w \in B$ (no loss of generality, by symmetry). Then x_2x_3 must have color 1 which implies that both x_1x_2 and x_1x_3 are also have color 1. It follows that all edges induced by B and incident with y have color 1. Thus we conclude easily that B induces a clique that is monochromatic in color 1.

For $N = 6$ edge wx_1 must have color 1 implying that all edges induced by $V(K_N) \setminus \{x_0, y\} = \{x_1, x_2, x_3, w\}$ have color 1.

For $N \geq 7$ we choose a vertex $z \in B \setminus \{y, w\}$. Because yz has color 1, all edges between $A \setminus x_0$ and $B \setminus \{y\}$ must be colored with 1. Therefore, $V(K_N) \setminus \{x_0, y\}$ induces a clique in color 1.

Case 2: *Set A induces an edge with a new color, say x_2x_3 has color 4.*

Step 0: K_N is colored with exactly four colors $1, 2, 3,$ and 4.

Note that any edge leading from B to $A \setminus \{x_0\}$ with a new color 5 would be extendable with the vertices of A into a rainbow P_5 . Similarly, any edge yz of color 5, with $y, z \in B$, would result in a rainbow P_5 on the vertices $\{y, z, x_0, x_2, x_3\}$. Thus a new color 5 might be assigned either to an edge x_0y, $y \in B$, or to an edge induced by A, say x_1x_2. In the first case $B \cup \{x_2, x_3\}$ must be a clique in color 4, since otherwise there is a rainbow P_5. In the second case, for each color $1, 2, 3$, or 4 assigned to yx_1, one obtains a rainbow P_5 in $A \cup \{y\}$ (the obvious details are omitted).

From now on we assume that K_N is colored with $1, 2, 3$, and 4. Furthermore, for any set $A \subset V(K_N)$, $|A| = 4$, if A contains a rainbow star $K_{1,3}$, then either all the four colors are present on the edges of A, or otherwise, Case 1 does apply.

Step 1: B is locally 2–colored.

Assume to the contrary that there is a vertex $y \in B$ incident with three distinct colors a, b, and c. Let d be the color of the edge yx_3. Note that $d \neq 3$ because of the emerging rainbow path (x_1, x_0, x_2, x_3, y).

We show that actually $d = 2$. W.l.o.g. we assume that among a, b, and c the color c is different from 3 and 4. Let c be assigned to the edge yz ($z \in B$). Then $d \neq 4$ because (z, y, x_3, x_0, x_i) is a rainbow path for $i = 1$ or 2. Finally if d was 1 then we have $3, 4 \notin \{a, b, c\}$; hence among a, b, and c one color, say c, is different from both 1 and 2, resulting in the rainbow path (z, y, x_3, x_0, x_2).

We conclude that the color of yx_3 is 2, and similarly, the color of yx_2 is 3. Furthermore, $\{a, b, c\} = \{1, 2, 3\}$, say yz_i has color i ($z_i \in B\setminus\{y\}$), for $i = 1, 2, 3$

If the color of x_0y is $e = 4$, then Case 1 applies with the rainbow star on $A' = (A \setminus \{x_1\}) \cup \{y\}$ centered at y. In any other case there is a rainbow P_5: (z_2, y, x_0, x_3, x_2) for $e = 1$, (z_1, y, x_0, x_3, x_2) for $e = 2$, and (z_1, y, x_0, x_2, x_3) for $e = 3$.

Step 2: $B \cup \{x_2\}$ and $B \cup \{x_3\}$ are locally 2–colored. Moreover,
(i) if there exist two distinct colors incident with x_2 then they are 1 and 3;
(ii) if there exist two distinct colors incident with x_3 then they are 1 and 2.

By symmetry, it is enough to deal with the set $B \cup \{x_2\}$. Let $y \in B$ be a vertex incident with distinct colors $\{a, b, c\}$ when including the vertex x_2. Assume that a is the color of edge yx_2. Note that $a \neq 2$ because of the emerging rainbow path (y, x_2, x_3, x_0, x_1).

Suppose that $a = 1$. Among b and c, one color is different from 3; let the color of zy be $b \notin \{1, 3\}$ ($z \in B$). Whatever is color b there is always a rainbow P_5: (z, y, x_2, x_3, x_0) for $b = 2$, and (z, y, x_2, x_0, x_3) for $b = 4$.

Suppose that $a = 4$. Among b and c, one color is different from 2; let the color of zy be $b \notin \{4, 2\}$ ($z \in B$). Whatever is color b there is always a rainbow P_5: (z, y, x_2, x_0, x_1) for $b = 3$, and (z, y, x_2, x_0, x_3) for $b = 1$.

Suppose that $a = 3$. Then $\{b, c\} = \{1, 2\}$, otherwise we find easily a rainbow P_5. Let i be the color of z_iy ($z_i \in B$) for $i = 1, 2$. If the color of z_1z_2 is $d \neq 3$, then we always find a rainbow P_5: (z_1, z_2, y, x_2, x_3) for $d = 1$, (z_2, z_1, y, x_2, x_3) for $d = 2$, and (z_2, z_1, y, x_2, x_0) for $d = 4$. Thus we conclude that yx_2 has color $d = 3$, therefore $A' = \{y, z_1, z_2, x_2\}$ induces a rainbow $K_{1,3}$ centered at y. Because A' must be 4–colored and z_1z_2 has color 3, edge z_ix_2 must have color 4 for $i = 1$ or for $i = 2$. In each case there is a rainbow P_5: (y, z_1, x_2, x_0, x_3) or $(z_1, z_2, x_2, x_0, x_1)$.

Therefore there are at most two distinct colors incident with y in $B \cup \{x_2\}$. To conclude the proof of Step 2 we show that the same is true for x_2.

Let $z, w \in B$ such that zx_2 and wx_2 have distinct colors b and c, respectively. We show that $\{b, c\} = \{1, 3\}$. Because of the paths (z, x_2, x_3, x_0, x_1) and

(w, x_2, x_3, x_0, x_1) we have $2 \notin \{b, c\}$. Suppose to the contrary that $4 \in \{b, c\}$, say $c = 4$. Note that in this case the color of zw cannot be 1 or 3, because of the emerging rainbow paths (z, w, x_2, x_0, x_3) or (z, w, x_2, x_0, x_1), respectively. If zw has color 4, then we find a rainbow P_5 for the possible values of b: (w, z, x_2, x_0, x_3) works for $b = 1$, and (w, z, x_2, x_0, x_1) works for $b = 3$. Thus we obtain that the color of zw must be 2. This implies easily that $b = 3$. Then any color chosen for x_1x_3 creates a rainbow P_5. Therefore $\{b, c\} = \{1, 3\}$, and claim (i) follows as well.

Step 3: $B \cup \{x_2, x_3\}$ is locally 2–colored.

Assume on the contrary that $y \in B$ is the center of a rainbow $K_{1,3}$ in a four element set $A' \subseteq B \cup \{x_2, x_3\}$. By Step 2, $A' = \{y, x_2, x_3, z\}$ for some $z \in B$. Let a, b, and c be the colors of x_2y, x_3y and zy, respectively,

Note that $a \neq 2$ and $b \neq 3$ follows by considering the paths (y, x_2, x_3, x_0, x_1) and (y, x_3, x_2, x_0, x_1), respectively.

First we show that $a \neq 4$; then by symmetry, we will also have $b \neq 4$. If a was 4, then the cases $c = 1$ and $c = 3$ are immediately excluded by considering the paths (z, y, x_2, x_0, x_3) and (z, y, x_2, x_0, x_1), respectively. Thus we obtain $c = 2$ and $b = 1$. Then the set $A' = \{y, x_2, x_3, z\}$ induces a rainbow star centered at y. By Step 2 (i), the color of zx_2 must be different from 3, and by Step 2 (ii), the color of zx_3 must be different from 3. Hence A' is 3–colored and Case 1 applies.

Suppose $a = 1$. The path (z, y, x_2, x_0, x_3) shows that $c \neq 4$, therefore $b = 2$ and $c = 3$. If the color of x_0z is $d \neq 3$, then there is a rainbow P_5: (y, z, x_0, x_2, x_3) for $d = 1$, (x_0, z, y, x_2, x_3) for $d = 2$, and (x_1, x_0, z, y, x_3) for $d = 4$.

Then the set $A' = \{x_0, x_1, x_2, z\}$ induces a rainbow star centered at x_0. The color of zx_2 is different from 4 because P_5: (y, z, x_2, x_0, x_1) would be rainbow. The color of x_1x_2 and that of x_1z cannot be 4 either because of the paths (x_1, x_2, y, x_3, x_0) and (x_0, x_1, z, y, x_3), respectively. Hence A' is 3–colored and Case 1 applies. Thus $a \neq 1$ and by symmetry, $b \neq 1$.

Suppose $a = 3$. The path (z, y, x_2, x_0, x_1) shows that $c \neq 4$, therefore $b = 2, c = 1$. Also note that in the rainbow star centered at x_2 and induced by the set $\{x_2, x_0, x_3, y\}$ the edge x_0y has color 1, since otherwise the set is three colored and Case 1 applies. If the color of x_1x_2 is $d \neq 1$, then there is a rainbow P_5: (x_1, x_2, x_3, x_0, y) for $d = 2$, (x_1, x_2, x_3, y, z) for $d = 3$, and (x_2, x_1, x_0, x_3, y) for $d = 4$. Then the set $A' = \{x_2, x_0, x_1, y\}$ induces a rainbow star centered at x_2. The color of yx_1 cannot be 4 due to the path (y, x_1, x_2, x_0, x_3). Because x_0x_1 and x_0y both have color 1, A' is three colored and Case 1 applies.

It remains to show that in the subgraph induced by $B \cup \{x_2, x_3\}$ at most two colors appear at x_2 and at x_3. By symmetry, it suffices to verify this for x_2. Assume to the contrary that a, b, c are distinct colors incident with x_2. By Step 2, we may assume $a = 4, b = 1$, and $c = 3$. Let $z, w \in B$ such that x_2z and x_2w

have color 1 and 3, respectively. Note that zw has color different from 2 due to the path (w, z, x_2, x_3, x_0). Also observe that in the rainbow star centered at x_2 and induced by the set $\{x_2, x_3, z, w\}$ the edge x_3w or x_3z has color 2, since otherwise the set is three colored and Case 1 applies.

First we consider the case when x_3w has color 2. The color of zw and that of x_0z is different from 4 due to the paths (z, w, x_2, x_0, x_1) and (x_0z, x_2, w, x_3), respectively. If x_0w is not colored 4, then the set $\{x_2, x_0, z, w\}$ is three colored; if x_0w has color 4, then the set $\{x_3, x_0, x_2, w\}$ is three colored, and Case 1 applies.

Next we consider the case when x_3z has color 2 and x_3w has color 1. (No other possibilities remain by Step 2.) The color of x_0w is not 4 due to the path (w, x_0, x_3, z, x_2). By symmetry, the color of x_0z is not 4 either. The color of zw is also different from 4 because of the path (z, w, x_2, x_0, x_1). Then set $\{x_2, x_0, z, w\}$ containing the rainbow star with center x_2 is three colored and Case 1 applies. □

Let $R^2_{loc}(G)$ be the smallest integer n such that every local 2–coloring of a clique K_n contains a monochromatic copy of graph G.

Lemma 4. *If $R^2_{loc}(G) + 2 \le R(G, G, G)$, then*

$$f(G, P_5) = R(G, G, G)\ .$$

Proof. Any 3-coloring of a clique without monochromatic copy of G has no rainbow P_5 that requires four distinct colors. Hence $f(G, P_5) \ge R(G, G, G)$. To verify the reverse inequality $f(G, P_5) \le R(G, G, G)$, take a coloring of a clique of order $N = R(G, G, G)$. If $k \le 3$ colors are used, then obviously there is a monochromatic G. For $k \ge 4$, if no rainbow P_5 exists, then by Lemma 3, K_N contains a locally 2-colored clique of order at least $N - 2$. Because $N - 2 \ge R^2_{loc}(G)$ is guaranteed by the condition, there exists a monochromatic G, and $f(G, P_5) \le N$ follows. □

To obtain $f(G, P_5)$ for a few graphs G we shall apply Lemma 4. The inequality between the Ramsey numbers stated in the lemma will be verified as a proposition preceding the corresponding result.

Proposition 5. *For $n \ge 5$, $R^2_{loc}(P_n) + 2 \le R(P_n, P_n, P_n)$.*

Proof. Let $A(1), A(2), B(1)$, and $B(2)$ be pairwise disjoint sets. Consider the clique on vertex set $A(1) \cup A(2) \cup B(1) \cup B(2)$ and color its edges as follows. Edges induced by $A(1) \cup A(2)$ and those induced by $B(1) \cup B(2)$ are colored 1; edges between $A(1)$ and $B(1)$ and those between $A(2)$ and $B(2)$ are colored 2; edges between $A(1)$ and $B(2)$ and those between $A(2)$ and $B(1)$ are colored 3.

For n odd, let $|A(1)| = |A(2)| = |B(1)| = |B(2)| = (n - 1)/2$; for n even let $|A(1)| = |A(2)| = |B(1)| = n/2 - 1$ and $|B(2)| = n/2$. It is easy to check that in the first case a clique K_{2n-2} and in the second case a clique K_{2n-3} is colored with three colors in such a way that the longest monochromatic paths have $n - 1$ vertices. Therefore,

$$R(P_n, P_n, P_n) \ge \begin{cases} 2n - 1 & \text{if } n \text{ is odd} \\ 2n - 2 & \text{if } n \text{ is even} \end{cases}\ .$$

It is proved in [5] that

$$R^2_{loc}(P_n) = \begin{cases} R(P_n, P_n) + 1 & \text{if } n \text{ is odd} \\ R(P_n, P_n) & \text{if } n \text{ is even} \end{cases} .$$

Furthermore, it is proved in [4] that

$$R(P_n, P_n) = \begin{cases} (3n-1)/2 - 1 & \text{if } n \text{ is odd} \\ 3n/2 - 1 & \text{if } n \text{ is even} \end{cases} .$$

For $n \geq 5$, odd in the first row, and for $n \geq 6$ even in the second row, we have the inequalities

$$R^2_{loc}(P_n) + 2 = \left\{ \begin{array}{lll} R(P_n, P_n) + 3 = & (3n+3)/2 & \leq & 2n-1 \\ R(P_n, P_n) + 2 = & 3n/2 + 1 & \leq & 2n-2 \end{array} \right\} \leq R(P_n, P_n, P_n) .$$

□

Theorem 6. *For $n \geq 3$, $f(P_n, P_5) = R(P_n, P_n, P_n)$.*

Proof. For $n = 3$ and 4, one easily obtains $R(P_3, P_3, P_3) = 5 = f(P_3, P_5)$, and $R(P_4, P_4, P_4) = 6 = f(P_4, P_5)$. For $n \geq 5$ the condition $R^2_{loc}(P_n) + 2 \leq R(P_n, P_n, P_n)$ is established by Proposition 5 and the claim follows from Lemma 4. □

Proposition 7. *If G is a connected non-bipartite graph then $R^2_{loc}(G) + 2 \leq R(G, G, G)$.*

Proof. Let $N = R(G, G) - 1$. Consider a coloring of the clique K_N with colors 1 and 2 such that no monochromatic copy of G exists. Take two disjoint copies and add all edges between them in color 3. Because G is connected no monochromatic G exists in color 1 or 2; and since G is not bipartite, the coloring of K_{2N} has no monochromatic G in color 3. Hence $R(G, G, G) \geq 2N+1 = 2R(G, G)-1$.

It is proved in [5] that $3n/2 - 1 \leq R^2_{loc}(G) < \frac{3}{2}R(G, G)$ provided G is a connected graph of order n. Because $R(G, G) \geq 6$ for any non-bipartite G,

$$R(G, G, G) \geq 2R(G, G) - 1 \geq \frac{3}{2}R(G, G) + 2 > R^2_{loc}(G) + 2 .$$

□

Theorem 8. *If G is a connected non-bipartite graph then $f(G, P_5) = R(G, G, G)$.*

Proof. By Proposition 7, $R^2_{loc}(G)+2 \leq R(G, G, G)$, so that the theorem follows from Lemma 4. □

Proposition 9. *For $n \geq 4$ even integer, $R^2_{loc}(C_n) + 2 \leq R(C_n, C_n, C_n)$.*

Proof. For $n = 4, R(C_4, C_4, C_4) = 11$ and $R^2_{loc}(C_4) = 6$ (see [9]). For $n \geq 6$ even, $R^2_{loc}(C_n) = R(C_n, C_n) = 3n/2 - 1$ is proved in [5], and a lower bound $2n-2 \leq R(P_n, P_n, P_n) \leq R(C_n, C_n, C_n)$ is obvious from the proof of Proposition 5. Hence $R^2_{loc}(C_n)+2 = R(C_n, C_n)+2 = 3n/2+1 \leq 2n-2 \leq R(C_n, C_n, C_n)$. □

Theorem 10. *For $n \geq 3$ $f(C_n, P_5) = R(C_n, C_n, C_n)$.*

Proof. For every $n \geq 3$ odd, the claim is a corollary of Theorem 8. For $n \geq 4$ even the claim follows from Proposition 9 and Lemma 4. □

4 Rainbow T_5

The method developed for rainbow P_5 in Lemmas 3 and 4 can be adapted for small rainbow trees. Here we deal with the case when P_5 is replaced with T_5, a star $K_{1,3}$ with one edge subdivided (two–fork on 5 vertices).

Lemma 11. *For $N \geq 3$, every edge coloring of K_N with at least four colors and without a rainbow T_5 contains a locally 2–colored clique of order at least $N - 2$.*

Proof. Suppose that a vertex x_0 is incident with four distinct colors, say x_0x_i has color i, for $1 \leq i \leq 4$. Let $y \in V(K_N) \setminus \{x_0, x_1, x_2, x_3, x_4\}$. If the color of yx_1 is different from 1, say 2, then the set $\{y, x_1, x_0, x_3, x_4\}$ induces a rainbow T_5. Therefore the color of yx_i is i, for every $1 \leq i \leq 4$. Whatever is the color of the edge yx_0 we always find a rainbow T_5 induced by the set $\{y, x_1, x_0, x_3, x_4\}$. Thus the coloring of K_N is a local 3–coloring.

Suppose that x_0 is incident with three distinct colors, say x_0x_i has color i, for $1 \leq i \leq 3$. Let $A = \{x_0, x_1, x_2, x_3\}$ and $B = V(K_N) \setminus A$. Note that no edge between A and B has a fourth new color, since otherwise we get a rainbow T_5.

If yz is colored 4 for some $y, z \in B$, then let i be the color of x_0y (for some $i \in \{1, 2, 3\}$). Now the set $\{x_0, x_1, x_2, x_3, y, z\} \setminus x_i$ induces a rainbow T_5. Therefore every new color occurs inside A.

Suppose x_2x_3 has color 4. Then $V' = V(K_N) \setminus \{x_2, x_3\}$ is colored with $1, 2$, and 3. If it is not a local 2 coloring, then a set $A' \subseteq V'$ induces a rainbow three star. Then $B' = V(K_N) \setminus A'$ would contain a new color 4 which cannot happen as we justified above. □

Lemma 12. *If $R^2_{loc}(G) + 2 \leq R(G, G, G)$, then*

$$f(G, T_5) = R(G, G, G) \ .$$

Proof. Any 3-coloring of a clique without monochromatic copy of G has no rainbow T_5 that requires four distinct colors. Hence $f(G, T_5) \geq R(G, G, G)$. To verify the reverse inequality $f(G, T_5) \leq R(G, G, G)$, take a coloring of a clique of order $N = R(G, G, G)$. If $k \leq 3$ colors are used, then obviously there is a monochromatic G. For $k \geq 4$, if no rainbow T_5 exists, then by Lemma 11, K_N contains a locally 2-colored clique of order at least $N - 2$. Because $N - 2 \geq R^2_{loc}(G)$ is guaranteed by the condition, there exists a monochromatic G, and $f(G, P_5) \leq N$ follows. □

Theorem 13. *If $G = P_n$ or $C_n (n \geq 3)$ or G is non-bipartite and connected, then $f(G, T_5) = f(G, P_5)$.*

Proof. One easily obtains $f(P_3, P_5) = 5 = f(P_3, T_5)$ and $f(P_4, P_5) = 6 = f(P_4, T_5)$. For $n \geq 5$, $f(P_n, T_5) = f(P_n, P_5)$ is a corollary of Proposition 5 and Lemma 12. If G is a connected non-bipartite graph or a cycle then Propositions 7, 9, and Lemma 12 imply $f(G, T_5) = f(G, P_5)$. □

5 Concluding Remarks

5.1 Values of $f(G, P_4)$ and $f(G, P_5)$

The method and the results presented in the previous sections show how Ramsey numbers are related to our 'mono–multi' function $f(G, P_k)$, for $k = 4, 5$. Indeed, the known diagonal Ramsey numbers of a graph G for two and three colors can be used to determine $f(G, P_4)$ and $f(G, P_5)$ for particular graphs.

For the sake of examples we are going to mention a few here. The electronic survey by Radziszowski [9] contains an updated list of further Ramsey results and all references we did not include here. The two color Ramsey numbers are known for paths and cycles resulting in the following values of $f(G, P_4)$.

Proposition 14. *$f(P_3, P_4) = 5$, and for $n \geq 4$, $f(P_n, P_4) = n + \lfloor n/2 \rfloor - 1$.* □

Proposition 15. *$f(C_3, P_4) = f(C_4, P_4) = 6$, and for $n \geq 5$,*

$$f(C_n, P_4) = \begin{cases} 2n - 1 & \text{for } n \text{ odd} \\ 3n/2 - 1 & \text{for } n \text{ even}. \end{cases}$$ □

Not too much is known about the three color Ramsey numbers. Even determining $R(K_4 - e, K_4 - e, K_4 - e)$ is a seemingly hard open problem. The Ramsey number $R(G, G, G)$ is known for every other graph G containing at most 5 edges and no isolated vertices. Thus we obtain the values $f(G, P_5)$ for any connected non-bipartite G among them different from a diamond (also called a 2–book).

5.2 Discussing $f(G, P_k)$, for $k \geq 5$

Let $R^c(G)$ be the diagonal c-color Ramsey number of graph G, that is the minimum integer such that in every c–coloring of a clique of that order there is always a monochromatic copy of G. Let $N = R^{k-2}(G))$. Any $(k-2)$-coloring of K_{N-1} without monochromatic G is clearly free of rainbow P_k that requires $k-1$ distinct colors. Hence $N = R^{k-2}(G) \leq f(G, P_k)$ follows. As we proved here we actually have $R^{k-2}(G) = f(G, P_k)$, for $k = 4$ and for $k = 5$ with particular instances of G (see Theorems 2, 6, 8, and 10).

It is a natural question whether the reverse inequality, $f(G, P_k) \leq R^{k-2}(G)$, remains true in general for $k \geq 6$. Even more generally one might ask for which trees T_k with k vertices is $f(G, T_k) \leq R^{k-2}(G)$. Our method of obtaining the results for $k = 4, 5$ uses the structure of local 2-colorings (see Lemmas 1, 3, and 4). Unfortunately, no concise characterization of local c-colorings is known for $c > 2$.

5.3 The maximum of f(S,T) and the 3-color Ramsey number of paths

In [7] Jamison, Jiang and Ling ask the following question:

Among all pairs of trees S, T with s and t edges, respectively, is $f(S,T)$ maximized by $f(P_{s+1}, P_{t+1})$?

The answer is negative. It was generally believed that $R(P_n, P_n, P_n) = (2 + o(1))n$, in fact its exact value was conjectured in [2] ($2n - 1$ for odd n, $2n-2$ for even n). Łuczak [8] conjectured that even $R(C_n, C_n, C_n) = (2+o(1))n$ holds true for even n. Recently this conjecture have been proved, [3]. In fact, for large enough n, the conjectured exact value of $R(P_n, P_n, P_n)$ have been proved recently [6]. We show how to use these results to demonstrate that $f(P_n, K_{1,4}) - f(P_n, P_5)$ tends to infinity with n.

To see this, set $N = 7k$ and consider the partition of the vertices of a clique K_N into the k element sets $V(i), i = 0, \dots, 6$. First we color all edges of the clique induced by $V(i)$ with color i, for every $i = 0, \dots, 6$. To color the remaining edges of K_N we shall use the lines of a Fano plane defined on the point set $\{V(i) : 0 \le i \le 6\}$. Let $L_\ell = \{V(\ell), V(\ell+1), V(\ell+3)\}$, $\ell = 0, \dots, 6$, be those seven lines (modular addition in $\mathbf{Z}_7$). For any edge of K_N between two distinct $V(i)$ and $V(j)$ we assign the color ℓ where L_ℓ is the unique line of the Fano plane containing both $V(i)$ and $V(j)$.

Because each point $V(i)$ of the Fano plane is incident with three lines, the 7-coloring obtained from the Fano plane is a local 3-coloring of K_N, that is every vertex in each $V(i)$ is incident with edges of three distinct colors. In particular, this coloring of K_N contains no rainbow star $K_{1,4}$. The largest monochromatic path is obviously P_{3k}, thus we conclude $f(P_{3k+1}, K_{1,4}) > N = 7k$.

On the other hand, Theorem 6 and $R(P_n, P_n, P_n) = (2 + o(1))n$ imply

$$f(P_{3k+1}, P_5) = R(P_{3k+1}, P_{3k+1}, P_{3k+1}) \le (6 + \epsilon)k$$

for any fixed positive ϵ and $k \ge k_0(\epsilon)$. To get just one small counterexample, one might take $k = 2$, and verify that $R(P_7, P_7, P_7) \le 14$ (a non-trivial exercise, the details are omitted).

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