Combinatorics, Probability and Computing (2005) 14, 435–438. © 2005 Cambridge University Press
doi:10.1017/S0963548305006759 Printed in the United Kingdom

PROBLEM SECTION

Connected matchings and Hadwiger's conjecture

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Hadwiger's well known conjecture (see the survey of Toft [9]) states that any graph G has a $K_{\chi(G)}$ minor, where $\chi(G)$ is the chromatic number of G. Let $\alpha(G)$ denote the independence (or stability) number of G, namely the maximum number of pairwise nonadjacent vertices in G. It was observed in [1], [4], [10] that via the inequality $\chi(G) \geqslant \frac{|V(G)|}{\alpha(G)}$, Hadwiger's conjecture implies

Conjecture 1.1. *Any graph G on n vertices contains a $K_{\lceil \frac{n}{\alpha(G)} \rceil}$ as a minor.*

A popular question over the past five years has been to consider Conjecture 1.1 for graphs G with $\alpha(G) = 2$:

Conjecture 1.2. *Suppose G is a graph with n vertices and with $\alpha(G) = 2$. Then G contains $K_{\lceil \frac{n}{2} \rceil}$ as a minor.*

† Research partially supported by Hungarian National Science Foundation Grants OTKA T032452 and T037846 and by NSF grant DMS 0140692.
‡ Research partially supported by the Hungarian Foundation for Scientific Research Grant (OTKA) Nos. T037846 and T046376.

Duchet and Meyniel [1] proved that every graph G with n vertices has a $K_{\lceil \frac{n}{2\alpha(G)-1} \rceil}$ minor, and so the statement of Conjecture 1.2 is true if $n/2$ is replaced by $n/3$ (for follow-ups and for some improvements see [4], [5], [2], [6]). The problem of improving $n/3$ is attributed to Seymour [7]:

Conjecture 1.3. *There exists $\epsilon > 0$ such that every graph G with n vertices and with $\alpha(G) = 2$ contains $K_{\lceil (\frac{1}{3}+\epsilon)n \rceil}$ as a minor.*

Conjecture 1.3 has a fairly interesting reformulation with some 'Ramsey flavor'. A set of pairwise disjoint edges $e_1, e_2, \ldots, e_t$ of G is called a *connected matching of size t* ([8]) if for every $1 \leqslant i < j \leqslant t$ there exists at least one edge of G connecting an endpoint of e_i to an endpoint of e_j.

Conjecture 1.4. *There exists some constant c such that every graph G with ct vertices and with $\alpha(G) = 2$ contains a connected matching of size t.*

Conjecture 1.4 has probably been discovered independently by several people working on Conjecture 1.3. Thomassé [8] notes that Conjectures 1.4 and 1.3 are equivalent (a proof is in [2]).

This note risks the stronger conjecture that $f(t)$, the minimum n such that every graph G with n vertices and $\alpha(G) = 2$ must contain a connected matching of size t, is equal to $4t - 1$. The lower bound $f(t) \geqslant 4t - 1$ is obvious, as shown by the union of two copies of K_{2t-1}.

Conjecture 1.5. *Every graph G with $4t - 1$ vertices and with $\alpha(G) = 2$ contains a connected matching of size t.*

In modest support of Conjecture 1.5, we have the following.

Theorem 1.6. $f(t) = 4t - 1$ *for* $1 \leqslant t \leqslant 17$.

Proof. Assume G is a graph with $4t - 1$ vertices and with $\alpha(G) = 2$. Suppose, first, that the maximum degree of $\overline{G}$ is at least $t - 1$ and let v be a maximum degree vertex in $\overline{G}$. Let $A \subset V(G)$ consist of t (or all if there are only $t - 1$) non-neighbors of v (in G), thus $t - 1 \leqslant |A| \leqslant t$. Consider the bipartite subgraph $H = [A, B]$ of G, where $B = V(G) \setminus (A \cup \{v\})$. If H contains a matching of size t then it is a connected matching, since A induces a clique in G. Also, if $|A| = t - 1$ and H contains a matching of size $t - 1$, it can be extended by an edge incident to v to a connected matching of size t. If the required matching does not exist, by König's theorem, there is a $T \subset V(G)$ with $|T| \leqslant t - 1$ (or $|T| \leqslant t - 2$ if $|A| = t - 1$) meeting all edges of H. As $|B| \geqslant 3t - 2$, this implies that there exists a vertex in $A \setminus T$ nonadjacent to at least $2t$ vertices of G. Thus $K_{2t} \subset G$ which clearly contains a connected matching of size t.

Therefore the maximum degree of $\overline{G}$ is at most $t - 2$. Now let A_v denote the set of non-neighbors and B_v the set of neighbors of v in G. Some vertex $w \in B_v$ is nonadjacent to at most

$$\frac{|A_v|(t-3)}{|B_v|} \leqslant \frac{(t-2)(t-3)}{3t} \tag{1}$$

vertices of A_v. The right hand side of (1) is less than 4 if $t \leqslant 16$. If $t = 17$ then, as all vertices cannot have odd degree, v can be selected as a vertex nonadjacent to at most 14 vertices and the estimate (1) still gives a $w \in B_v$ nonadjacent to at most $142/51 < 4$ vertices of A_v. Thus we have found an edge vw in G such that the set $C \subset V(G)$ nonadjacent to both v and w satisfies $|C| \leqslant 3$. This allows us to carry out the inductive proof: removing v, w and two further vertices (as many from C as possible) the remaining graph has a connected matching of size $t-1$ and the edge vw extends it to a connected matching of size t. (Of course, it is trivial to start the induction with $f(1) = 3$.) □

An obvious upper bound for $f(t)$ comes from the Ramsey function: $f(t) \leqslant R(3, 2t)$ (which has order of magnitude $\frac{t^2}{\log t}$: see [3] and the references therein). Using the proof method of Theorem 1.6 we give a better bound for $g(t) \geqslant f(t)$ where $g(t)$ is the minimum n such that every graph G with n vertices and with $\alpha(G) = 2$ contains a '2-connected matching of size t': a set of pairwise disjoint edges $e_1, e_2, \ldots, e_t$ of G such that for every $1 \leqslant i < j \leqslant t$ there exists at least *two* edges of G connecting an endpoint of e_i to an endpoint of e_j.

Theorem 1.7. *Every graph G with $\alpha(G) = 2$ and with at least $2^{3/4}t^{3/2} + 2t + 1$ vertices contains a 2-connected matching of size t.*

Proof. Set $c = 2^{5/4}$ which is the positive root of $\frac{4}{c} = \frac{c\sqrt{2}}{2}$. We want to establish the recursive bound $g(t) \leqslant g(t-1) + ct^{1/2} + 2$, for the function $g(t)$ $(t \geqslant 2, g(1) = 3)$. Then (using the inequality between the arithmetic and quadratic means)

$$g(t) \leqslant c\left(\sum_{i=2}^{t} i^{1/2}\right) + 2(t-1) + g(1) \leqslant c\frac{(\sqrt{2})}{2}t^{3/2} + 2t + 1 = 2^{3/4}t^{3/2} + 2t + 1,$$

the theorem follows (for $t = 1$ it holds vacuously).

Using the argument of Theorem 1.6, let N be the smallest integer satisfying $N \geqslant 2^{3/4}t^{3/2} + 2t + 1$, let G be a graph with N vertices and with $\alpha(G) = 2$. Assuming G has no 2-connected matching of size t, any $v \in V(G)$ is nonadjacent to at most $2t-1$ vertices of G. Using the argument from the proof of Theorem 1.6, for any $v \in V(G)$ there is a $w \in B_v$ such that there are at most $M = \frac{(2t-1)(2t-2)}{N-2t}$ vertices of G nonadjacent to both v and w. Therefore, it is possible to remove at most $M + 2$ vertices of G so that the remaining graph does not contain 2-connected matchings of size $t-1$. Thus,

$$g(t) < g(t-1) + \frac{(2t-1)(2t-2)}{N-2t} + 2. \tag{2}$$

Notice that $\frac{(2t-1)(2t-2)}{N-2t} \leqslant ct^{1/2}$ because otherwise we get

$$N < \left(\frac{4}{c}\right)t^{3/2} + 2t = 2^{3/4}t^{3/2} + 2t$$

implying

$$2^{3/4}t^{3/2} + 2t + 1 \leqslant N < 2^{3/4}t^{3/2} + 2t,$$

contradiction. Thus (2) gives the claimed recursive bound for $g(t)$. □

It is natural to conclude this note by introducing $h(t)$, the minimum n such that every graph G with n vertices and with $\alpha(G) = 2$ contains a 3-connected matching of size t: a set of pairwise disjoint edges $e_1, e_2, \ldots, e_t$ of G such that for every $1 \leqslant i < j \leqslant t$ there exists at least *three* edges of G connecting an endpoint of e_i to an endpoint of e_j.

Problem 1.8. *Separate the functions* $f \leqslant g \leqslant h \leqslant R(3, 2t)$.

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