Semistrong Edge Coloring of Graphs

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Abstract: Weakening the notion of a strong (induced) matching of graphs, in this paper, we introduce the notion of a semistrong matching. A matching M of a graph G is called semistrong if each edge of M has a vertex, which is of degree one in the induced subgraph $G[M]$. We strengthen earlier results by showing that for the subset graphs and for the Kneser graphs the sizes of the maxima of the strong and semistrong matchings are equal and so are the strong and semistrong chromatic indices. Similar properties are conjectured for the n-dimensional cube.

Keywords: *induced matchings; Kneser graphs; subset graphs; cubes*

1. INTRODUCTION

Assume that M is a matching in a graph G, i.e., M consists of pairwise disjoint edges of G. A vertex covered by M is said to be *strong* vertex if it has degree one in the graph induced in G by the vertex set covered by M. A *strong* (or induced) *matching* M is a matching in which every vertex covered by M is strong.

A *semistrong matching* M is defined by requiring that each edge of M has a strong vertex. The maximal sizes of a matching, strong matching, and semistrong matching of G are denoted by $\nu(G)$, $\nu_s(G)$, and $\nu_{ss}(G)$, respectively. Clearly $\nu(G) \geq \nu_{ss}(G) \geq \nu_s(G)$ for every graph G. The minimal numbers of classes needed to partition the edge set of G into matchings, strong matchings, and semistrong matchings are the chromatic index $q(G)$, the strong chromatic index $q_s(G)$, and the semistrong chromatic index $q_{ss}(G)$. These parameters clearly satisfy $q(G) \leq q_{ss}(G) \leq q_s(G)$. The parameters $\nu(G)$ and $q(G)$ are among the most extensively studied graph invariants. The study of $\nu_s(G)$ and $q_s(G)$ was started by Erdős and Nesetřil (see [6]) and continued in [1], [4], [5], [7], [8], and [13]; most of the results concern special graphs like cubes, subset graphs, and the Kneser graphs (see [7], [11]).

The notion of a semistrong matching, which seems to be new, arose from a special case of the following question. Assuming that H is a union of paths and even cycles, determine the smallest n for which H is an induced subgraph of Q^n. Here Q^n is the n-dimensional cube, the graph whose vertices are the 0–1 vectors of length n, and in which two vertices are joined if they differ in exactly one coordinate.

For certain graphs H (independent set, matching, union of cycles of length four, union of cycles of length eight), this question can be answered easily, but for the cycle, it is the Snake-in-the-box problem (see [12], [14]), which is considered to be very difficult.

Let us draw attention to the special case of the question above, in which H consists of several identical components F. For example, given a path or even cycle F, determine the largest k such that kF is an induced subgraph of Q^n. As a variant of this problem, consider the maximum number k for which $H = kP_3$ is an induced subgraph of Q^n with the additional property that each component of H has an edge in the same direction of Q^n. This problem can be reduced to finding the maximum semistrong matching of the cube by showing that $k = \nu_{ss}(Q^{n-1})$ (see [9]).

It is easy to prove that, for $n \geq 5$, every induced subgraph G of Q^n with more than 2^{n-1} vertices has a vertex of degree at least 3. Therefore for $n \geq 5$, every induced graph H of Q^n with $\Delta(H) \leq 2$ has at most 2^{n-1} vertices. (A result of [4] gives more information for large n: G must have a vertex of degree at least $(\frac{1}{2}) \log n - (\frac{1}{2}) \log \log n + (\frac{1}{2})$. Note, however that this gives a vertex of degree 3 only for $n \geq 44$.)

2. SUBSET GRAPHS, KNESER GRAPHS, AND CUBES

Everywhere in this paper by *coloring* we mean *edge coloring*. The *size* of a matching is the number of its edges.

For the m element ground set $[m] = \{1, 2, \ldots, m\}$ and $0 \leq k \leq l \leq m$, the subset graph $S_m(k, l)$ is the bipartite graph whose vertex classes are the k- and l-subsets of the ground set. Two vertices (subsets) are *adjacent* if and only if one of them is contained in the other.

It was conjectured by Brualdi and Quinn [3] that $q_s(S_m(k, l)) = \binom{m}{l-k}$. This conjecture was proved by Quinn and Benjamin in [11]. In this section, we shall prove a more general result (Theorem 1).

The main tool in our proofs is the ordered version of a theorem of Bollobás for pairs of sets [2] that was proved by Lovász [10].

Lemma 1. *Let $S = \{(A_i, B_i) | 1 \leq i \leq r\}$ be a set-pair collection with $|A_i| = a$, $|B_i| = b$ satisfying the following conditions:*

(a) $A_i \cap B_i = \emptyset$ for $1 \leq i \leq r$;
(b) $A_i \cap B_j \neq \emptyset$ for $1 \leq i < j \leq r$.
Then $r \leq \binom{a+b}{a}$.

Lemma 2.

$$\nu_s(S_m(k, l)) = \nu_{ss}(S_m(k, l)) = \binom{m - l + k}{k}.$$

Proof. Recall that a matching M is semistrong if every edge contains at least one strong vertex. Fix one strong vertex on every edge of M and call the other vertices of M *weak*. Denote the classes of $S_m(k, l)$ by

$$L = \{A_i : A_i \subset [m], |A_i| = l\} \quad \text{and} \quad K = \{B_j : B_j \subset [m], |B_j| = k\}.$$

Let M be a semistrong matching in $S_m(k, l)$ and let $\{(A_1, B_1), (A_2, B_2), \ldots, (A_r, B_r)\}$ be the edge set of M with $A_i \in L$ and $B_i \in K$ for $1 \leq i \leq r$. Without loss of generality, we may assume that for some index p the vertices $B_1, B_2, \ldots, B_p$ and $A_{p+1}, A_{p+2}, \ldots, A_r$ are the weak vertices of M. Denoting $\overline{A_i} = [m] \setminus A_i$, observe that

$$\overline{A_i} \cap B_i = \emptyset \quad \text{for} \quad 1 \leq i \leq r$$

and

$$\overline{A_i} \cap B_j \neq \emptyset \quad \text{for} \quad 1 \leq i < j \leq r.$$

Hence, $S = \{(\overline{A_i}, B_i) | 1 \leq i \leq r\}$ satisfies the conditions of Lemma 2 and therefore, $r \leq \binom{m-l+k}{k}$. We have shown that

$$\nu_{ss}(S_m(k, l)) \leq \binom{m - l + k}{k}.$$

To show the reverse inequality, consider the following set-pair system. Fix a set $T \subset [m]$ of size $l - k$. Take all the sets $B_i \in K$ such that $B_i \cap T = \emptyset$. For every such B_i, define $A_i = B_i \cup T$. It is clear that this set-pair system $\{(A_i, B_i) | 1 \le i \le \binom{m-l+k}{k}\}$ determines a strong matching in $S_m(k, l)$, and thus

$$\binom{m-l+k}{k} \le \nu_s(S_m(k, l)) \le \nu_{ss}(S_m(k, l))$$

finishing the proof. ■

Clearly,

$$q_{ss}(S_m(k, l)) \le q_s(S_m(k, l)) \le \binom{m}{l-k}$$

since one can define a strong coloring on $S_m(k, l)$ with $\binom{m}{l-k}$ colors by assigning to every edge $(A_i, B_j) \in E(S_m(k, l))$ the set $A_i \setminus B_j$ as a color (this coloring is from [11]). Since

$$\nu_{ss}(S_m(k, l)) = \binom{m-l+k}{k}$$

and

$$|E(S_m(k, l))| = \binom{m}{l}\binom{l}{k} = \binom{m-l+k}{k}\binom{m}{l-k}$$

we get

$$\binom{m}{l-k} \le q_{ss}(S_m(k, l)).$$

Therefore, we obtain the following result.

Theorem 1.

$$q_s(S_m(k, l)) = q_{ss}(S_m(k, l)) = \binom{m}{l-k}.$$

For $m > 2n$, the *Kneser graph* $KN(m, n)$ is the graph whose vertices are the n-subsets of an m element ground set with two vertices connected if and only if the corresponding sets are disjoint. It has been proved in [7] that $q_s(KN(m, n)) = \binom{m}{2n}$. Using the same technique, we will show that (more generally) $q_{ss}(KN(m, n)) = \binom{m}{2n}$.

Lemma 3.

$$\nu_s(KN(m, n)) = \nu_{ss}(KN(m, n)) = \frac{1}{2}\binom{2n}{n}.$$

Proof. Take a semistrong matching M in $KN(m, n)$. Fix one strong vertex on every edge of M and call the other vertex of the edge *weak*. Take two copies of M

and assign to them a set-pair system. On the first copy of M, for every edge assign A_i as the set corresponding to its strong vertex and assign B_i as the set corresponding to its weak vertex. The set system constructed is denoted by $\{(A_1, B_1), \ldots, (A_r, B_r)\}$. On the second copy of M, consider the following set system $\{(A_{r+1}, B_{r+1}), \ldots, (A_{2r}, B_{2r})\}$. For every index $r+i$, let $A_{r+i} = B_i$ and $B_{r+i} = A_i$ (i.e., change the order in each ordered pair of the previous set-pair system). It is easy to check that the system

$$S = \{(A_1, B_1), \ldots, (A_{2r}, B_{2r})\}$$

satisfies the conditions of Lemma 2 and hence

$$2r \leq \binom{2n}{n} \quad \text{and} \quad |M| = r \leq \frac{1}{2}\binom{2n}{n}.$$

Thus,

$$\nu_s(KN(m,n)) \leq \frac{1}{2}\binom{2n}{n}.$$

Let N be a subset of size $2n$ of the ground set. Consider the collection of unordered pairs

$$T = \{\{A_i, N \backslash A_i\} : A_i \subset N, |A_i| = n\}.$$

Observe that T determines a strong matching in $KN(m,n)$ and

$$|T| = \frac{1}{2}\binom{2n}{n}.$$

Thus,

$$\frac{1}{2}\binom{2n}{n} \leq \nu_s(KN(m,n)) \leq \nu_{ss}(KN(m,n))$$

finishing the proof. ■

Consider the coloring of $KN(m,n)$, where to every edge (A_i, B_i), we assign the set $A_i \cup B_i$ as color. Note that in this coloring, every color class determines a strong matching, and we have used $\binom{m}{2n}$ colors. Hence,

$$q_{ss}(KN(m,n)) \leq q_s(KN(m,n)) \leq \binom{m}{2n}.$$

Since

$$|E(KN(m,n))| = \frac{1}{2}\binom{m-n}{n}\binom{m}{n} = \frac{1}{2}\binom{2n}{n}\binom{m}{2n}$$

and

$$\nu_{ss}(KN(m,n)) = \frac{1}{2}\binom{2n}{n};$$

we get

$$\binom{m}{2n} \leq q_{ss}(KN(m,n))$$

and obtain the following.

Theorem 2.

$$q_s(KN(m,n)) = q_{ss}(KN(m,n)) = \binom{m}{2n}.$$

Properties known for strong matchings are not always true for semistrong matchings. For instance, a possible method of proving that $q_s(KN(m,n)) \geq \binom{m}{2n}$ was to find a subgraph of size $\binom{m}{2n}$ in $KN(m,n)$ such that any two edges of it must be colored with different colors in a strong coloring. Such subgraphs were called "antimatchings" in [7]. An *antimatching* is a subset F of edges such that between any two disjoint edges of F there is at least one further edge of G. A *semistrong antimatching* is a subset F of edges such that between any two disjoint edges of F, there are at least two further edges of G.

Let $am_s(G)$ and $am_{ss}(G)$ denote the largest numbers of edges in a strong and semistrong antimatching of G, respectively. In contrast with antimatchings, the size of semistrong antimatching cannot achieve $q_{ss}(KN(m,n))$, in fact $am_{ss}(KN(m,n)) \leq \binom{m-1}{2n-1}$. We do not prove this, but show a better upper bound due to Oleg Pikhurko.

Lemma 4.

$$am_{ss}(KN(m,n)) \leq 2\binom{\lfloor \frac{m}{2} \rfloor}{n}\binom{\lceil \frac{m}{2} \rceil}{n}.$$

Proof. Let $E = \{(A_i, B_i)\}$ be the set of edges in a semistrong antimatching of $KN(m,n)$, where A_i and B_i (for every i) are disjoint n-subsets of the m element ground set S. For every $x \in S$ and $y \in S$, $x \neq y$ define

$$V_{xy} = \cup\{A_i \setminus \{x\} : (A_i, B_i) \in E, x \in A_i, y \in B_i\}.$$

Suppose that $z \in V_{xy} \cap V_{yx}$. Then there exist edges $e_i = (A_i, B_i)$ and $e_j = (A_j, B_j)$ such that $x \in A_i \cap B_j$, $y \in A_j \cap B_i$, $z \in A_i \cap A_j$. Therefore, at most, one edge is spanned between e_i and e_j, namely (B_i, B_j). This contradicts the definition of the semistrong antimatching. Thus $V_{xy} \cap V_{yx} = \emptyset$ implying that

$$|\{(A_i, B_i) \in E : x \in A_i, y \in B_i\}| \leq \binom{V_{xy}}{n-1}\binom{V_{yx}}{n-1} \leq \binom{\lfloor \frac{m-2}{2} \rfloor}{n-1}\binom{\lceil \frac{m-2}{2} \rceil}{n-1}.$$

Hence,

$$n^2|E| \leq \sum_{x,y} \binom{V_{xy}}{n-1}\binom{V_{yx}}{n-1} \leq \binom{m}{2}\binom{\lfloor\frac{m-2}{2}\rfloor}{n-1}\binom{\lceil\frac{m-2}{2}\rceil}{n-1}.$$

Therefore,

$$|E| \leq 2\binom{\lfloor\frac{m}{2}\rfloor}{n}\binom{\lceil\frac{m}{2}\rceil}{n}. \quad \blacksquare$$

A maximal complete bipartite subgraph is obviously a semistrong antimatching in $KN(m,n)$ and gives a lower bound $\binom{\lfloor\frac{m}{2}\rfloor}{n}\binom{\lceil\frac{m}{2}\rceil}{n}$ for $am_{ss}(KN(m,n))$. By adding edges of maximal complete bipartite graphs M_1 and M_2 to each of its partite classes, respectively, one can increase the number of edges preserving the antimatching property. This procedure can be iterated by adding edges of a maximal complete bipartite graphs to partite classes of M_1 and M_2 and so on. The number of edges in the resulting graph G is at least

$$\binom{\lfloor\frac{m}{2}\rfloor}{n}\binom{\lceil\frac{m}{2}\rceil}{n} + 2\binom{\lfloor\frac{m}{4}\rfloor}{n}^2 + 4\binom{\lfloor\frac{m}{8}\rfloor}{n}^2 + \cdots$$

and at most

$$\binom{\lfloor\frac{m}{2}\rfloor}{n}\binom{\lceil\frac{m}{2}\rceil}{n} + 2\binom{\lceil\frac{m}{4}\rceil}{n}^2 + 4\binom{\lceil\frac{m}{8}\rceil}{n}^2 + \cdots.$$

Thus, the number of edges in G is $(1+\varepsilon)\binom{\lfloor\frac{m}{2}\rfloor}{n}\binom{\lceil\frac{m}{2}\rceil}{n}$ for some small positive ε. This with Lemma 2 yields the following.

Theorem 3.

$$(1+\varepsilon)\binom{\lfloor\frac{m}{2}\rfloor}{n}\binom{\lceil\frac{m}{2}\rceil}{n} \leq am_{ss}(KN(m,n)) \leq 2\binom{\lfloor\frac{m}{2}\rfloor}{n}\binom{\lceil\frac{m}{2}\rceil}{n}.$$

Observe that semistrong antimatchings in bipartite graphs must be complete bipartite subgraphs. Thus, for bipartite graphs, a largest complete bipartite subgraph is a largest semistrong antimatching. With this observation, it is easy to verify that $am_{ss}(S_m(k,l)) = \binom{m-k}{l-k}$. Recall that $q_{ss}(S_m(k,l)) = \binom{m}{l-k}$. So the size of the maximal semistrong antimatching of $(S_m(k,l))$ is smaller than $q_{ss}(S_m(k,l))$.

It was proved in [7] that $\nu_s(Q^n) = 2^{n-2}$, $q_s(Q^n) = 2n$, and $am_s(Q^n) = 2n$. Note that $am_{ss}(Q^n) = n$, since Q^n is bipartite and the n-star is its maximum complete bipartite subgraph.

Claim. $2n \geq q_{ss}(Q^n) \geq (5/4)n$.

Proof. For a semistrong coloring of Q^n, consider the subgraph that is formed by two of its color classes. Observe that each component of this subgraph is an alternating path of order at most 5. Therefore, any two color classes together have at most $\frac{4}{5}2^n$ edges, and hence $q_{ss}(Q^n) \geq (5/4)n$. The upper bound comes from the fact that $q_s(Q^n) \geq q_{ss}(Q^n)$. ■

We do not know whether a semistrong matching of Q^n can exceed the size of 2^{n-2} when n is large. However, it is proved in [9] that $\nu_{ss}(Q^n) = 2^{n-2}$ for $2 \leq n \leq 12$. We conjecture that it is true for all $n \geq 2$.

Conjecture 1. $\nu_{ss}(Q^n) = 2^{n-2}$.

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