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PACKING TREES OF DIFFERENT ORDER INTO K_n

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We say that the (finite, undirected, loopless) graphs $G_1, G_2, \dots, G_k$ have a *packing* into K_n (the complete graph of n vertices) if K_n admits an edge-disjoint decomposition into graphs isomorphic to $G_1, G_2, \dots, G_k$. The following question was posed by A. Gyárfás: T_i denotes a tree with i vertices. Does an arbitrary sequence of trees $T_1, T_2, T_3, \dots, T_{n-1}, T_n$ have a packing into K_n? The present paper affirmatively answers two special cases.

Theorem 1. *The trees* $T_1, T_2, \dots, T_n$ *can be packed into* K_n *if all but two are stars.*

Proof. We use induction on n. The cases $n = 1$ and $n = 2$ are obvious. If $n \geqslant 3$ we consider three cases:

Case A. T_n is a star. $T_{n-1}, T_{n-2}, \dots, T_2, T_1$ can be packed into K_{n-1} by induction and T_n can be placed at a new vertex added into K_{n-1}. ∎

Case B. T_{n-1} is a star. T'_{n-1} is defined as the tree we get from T_n after removing a vertex of degree one together with the edge incident to it. $T'_{n-1}, T_{n-2}, \ldots, T_2, T_1$ can be packed into K_{n-1} by the inductive hypothesis. Adding a new vertex, we can complete T'_{n-1} with an edge to the new vertex and we can place the star T_{n-1} as well.∎

Case C. Neither T_n nor T_{n-1} is a star. We reduce T_n and T_{n-1} to T'_{n-2} and T'_{n-3} by removing two vertices of degree one together with the two independent edges incident to them. (Any tree has such a vertex-pair if it is not a star.) We decompose K_{n-2} into $T'_{n-2}, T'_{n-3}, T_{n-4}, \ldots, T_2, T_1$ by the inductive hypothesis. Adding two new vertices we can always complete T'_{n-2} and T'_{n-3} with two disjoint pairs of independent edges to form trees isomorphic to T_n and T_{n-1}. The stars T_{n-2} and T_{n-3} can be placed also at the new vertices.∎∎∎

Theorem 2. *The trees* $T_1, T_2, \ldots, T_n$ *can be packed into* K_n *if there is no* T_i *which is different from a path or a star.**

Proof. We shall use the following simple lemma.

Lemma. *Let* $V' = \{v_1, v_2, \ldots, v_m\}$ *be a proper subset of the vertices of a directed graph* $G(V, E)$ *where the outdegree of* v_i *is at least* i. *A* V'*-path is an elementary path with vertices from* V' *except its end-vertex which lies in* $V - V'$. *There exists a system of disjoint* V'*-paths covering all vertices of* V'.

Proof of the lemma. We use induction on m. The case $m = 1$ is obvious. Let $V'' = V' - \{v_m\}$. There is a covering of V'' with V''-paths $p_1 \ldots q_1, p_2 \ldots q_2, \ldots, p_t \ldots q_t$ guaranteed by the inductive hypothesis. Two cases are possible:

1. $v_m = q_{i_0}$ for some $1 \leqslant i_0 \leqslant t$. The set

$$A = \Big(\bigcup_{i \neq i_0} q_i \cup V''\Big) - \bigcup_{i \neq i_0} p_i$$

has $m - 1$ elements and $v_m \notin A$. The outdegree of v_m is at least m so there is an edge starting from v_m to a vertex outside A. This edge always completes the covering system of V'' into a covering system of V'.

*We have recently learned that S. Zaks independently discovered this theorem. His proof is much more simple. We hope, however, that our method can be applied for more general cases, too.

2. $v_m \notin \bigcup_{i=1}^{t} q_i$. In this case the set $B = \left(\bigcup_{i=1}^{t} q_i \cup V''\right) - \bigcup_{i=1}^{t} p_i$ has $m-1$ elements and $v_m \notin B$. There is an edge starting from v_m to a vertex outside B and it always completes the covering system of V'' into a covering system of V'.∎

Now we proceed with the proof of Theorem 2. We suppose that $T_n, T_{n-1}, \ldots, T_{k+1}$ are paths, T_k is a star, $T_{k-1}, \ldots, T_{l+1}$ are paths and T_l is a star. To put it into other words we can say that T_k is the first and T_l is the second star in the sequence. If we apply the convention that T_2 and T_1 are considered as stars, we have always a unique decomposition of $T_n, T_{n-1}, \ldots, T_2, T_1$. We call that decomposition *canonical.* The sequence $T_n, T_{n-1}, \ldots, T_{k+1}$ is called the sequence of *leading paths.* We will prove that the packing required by Theorem 2 exists with the additional condition that the *endpoints of the leading paths are representable,* which means that we can choose different vertices $X_n, X_{n-1}, \ldots, X_{k+1}$ from K_n so that X_i is an endpoint of the path (isomorphic to) T_i. We prove that by induction on n.

The cases $n = 1$ and $n = 2$ are obvious. Let us suppose that $n \geqslant 3$. Let us consider T_k in the canonical decomposition of $T_n, T_{n-1}, \ldots, T_2, T_1$. If $k = n$ then the set of leading paths is empty. The inductive hypothesis gives a packing of $T_{n-1}, \ldots, T_2, T_1$ into K_{n-1} and the star T_n can be placed at a new vertex without any difficulty. We assume therefore that $k \neq n$, i.e. T_n is a path.

We define a sequence of trees as follows: $T'_{n-2}, T'_{n-3}, \ldots, T'_l$ are paths and $T'_{l-1}, T'_{l-2}, \ldots, T'_2, T'_1$ are identical with $T_{l-1}, T_{l-2}, \ldots, T_2, T_1$. Applying the inductive hypothesis, the sequence $\{T'_i\}_1^{n-2}$ can be packed into K_{n-2} with the additional condition that the endpoints of $T'_{n-2}, \ldots, T'_l$ are representable. Let $\{X_{n-2}, \ldots, X_l\}$ be a representing set of the endpoints of the paths $T'_{n-2}, \ldots, T'_l$. We define

$$X' = \{X_{n-3}, \ldots, X_{k-1}\}, \quad X'' = \{X_{k-2}, \ldots, X_l\} \cup \{X_{n-2}\}.$$

Let A and B be two new vertices. We plan to complete the packing of $\{T'_i\}$ into K_{n-2} to form a packing of $\{T_i\}$ into K_n according to the following:

– The paths $T'_{n-3}, \dots, T'_{k-1}$ will be increased by two from their endpoints in X', using two edges between K_{n-2} and $\{A, B\}$, – we get the paths $T_{n-1}, T_{n-2}, \dots, T_{k+1}$ in this way.

– The paths $T'_{k-2}, \dots, T'_l$ will be increased by one vertex A or B from their endpoint in X'', using the edges between K_{n-2} and $\{A, B\}$. – We obtain the paths $T_{k-1}, T_{k-2}, \dots, T_{l+1}$ in this way.

– The path T'_{n-2} will be increased by two from its endpoint X_{n-2} using the edge AB and one of the edges $X_{n-2}A$ or $X_{n-2}B$. – The path T_n is constructed in that way.

– T_k and T_l will be placed with their centers placed at A and B using the edges between K_{n-2} and $\{A, B\}$.

If we succeed in doing that completion properly then we have a packing of $T_n, T_{n-1}, \dots, T_2, T_1$ into K_n. "Properly" means that

(i) we have to use different edges for completing different paths.

(ii) we have to be sure that the leading paths are representable. We shall assure it by completing $T'_{n-2}, T'_{n-3}, \dots, T'_{k-1}$ into different vertices.

(iii) T_k and T_l are to be placed into A and B – therefore the number of edges between K_{n-2} and $\{A, B\}$ not used for increasing the paths, must be distributed between A and B properly, i.e. $k-1$ and $l-1$.

Now we define the increment-procedure in details. If X' is empty that is $n-1=k$ then we are in an easy position: T'_{n-2} is increased with edges $X_{n-2}B$ and AB. $T'_{k-2}, \dots, T'_l$ are increased with edges $X_{k-2}B, X_{k-3}B, \dots, X_lB$. The number of edges not used between A and K_{n-2} is $n-2=k-1$ therefore we can place T_k at the vertex A and consequently T_l at vertex B; (i)-(iii) are satisfied.

If X' is non-empty, we define a directed graph on the vertices of K_{n-2}. We draw an edge from $X_i \in X'$ to the vertices which are not on the path T'_i. (We have two structures on K_{n-2}: the packing of T'_i and the directed graph defined just now. These are considered as distinct structures and the one we are referring to will be clear from the context.) The outdegrees of $X_{n-3}, X_{n-4}, \ldots, X_{k-1}$ are at least $1, 2, \ldots, n-k-1$ respectively. X' is a proper subset of $V(K_{n-2})$ therefore we can apply our lemma which guarantees a covering of X' with disjoint paths. We can increase T_i' from $X_i \in X'$ (through A or B) to the vertex which follows X_i on the path covering X_i – since that vertex is not on T_i'. We note that (ii) will be satisfied by doing that. If P is a path from the covering of X' then $l(P)$ denotes the length of this path – it is the number of vertices on P lying in X'. A path P in the covering is classified as

type I, if $l(P)$ is even and the endpoint of P is in X'';

type II, if $l(P)$ is odd and the endpoint of P is in X'';

type III, if $l(P)$ is even and the endpoint of P is not in X'';

type IV, if $l(P)$ is odd and the endpoint of P is not in X''.

For each path P of the covering of X', the vertices A and B will be used *alternatively* to increase the paths by two vertices. Condition (i) is assured by that. The degrees of A and B after increasing the paths T_i' along the vertices of P will be:

$l(P)$ and $l(P) + 1$, if P is of type I or II,

$l(P)$, if P is of type III,

$l(P) - 1$ and $l(P) + 1$, if P is of type IV.

We give an *algorithm* assuring that the *degree of* A will be $|X'|$ after increasing all the paths T_i' according to our above plan. In this case $(n-2) - |X'| = (n-2) - (n-3) - (k-1) + 1 = k-1$ edges are not used between A and K_{n-2}, and T_k can be packed using these edges. The packing of T_l with center B is straightforward after that. The algorithm is the following:

Step A. Choose an even number of paths of type IV from the covering, as much as possible. A is alternatively assigned to be a vertex of degree $l(P) - 1$ or $l(P) + 1$. During this process $d(A) = \sum l(P)$ is true for the paths P processed so far. (d dentoes the degree of a vertex.)

Step B. If the number of type IV paths is even then continue with step C. Otherwise we have one path of type IV not processed, it is denoted by P'.

Step B1. If there is an $X_i \in X''$ which is not used by the covering of X' then we increase T_i by one vertex using the edge X_iA and along the path P' we increase the paths T_i' so that the degree of A would increase by $l(P) - 1$ – this manoeuvre preserves the property $d(A) = {}$ $= l(P)$ on the paths processed so far. Continue with step C. If the condition is not true then continue with step B2.

Step B2. If there is no $X_i \in X''$ which is not used by the paths covering X' then there exists a path P'' of type I or type II because X'' is non-empty. The increases of the T_i'-s along P' and P'' is made so that the degree of A is increased by $l(P) - 1$ and $l(P) + 1$ respectively. The property $d(A) = \sum l(P)$ is clearly preserved on the paths processed so far.

Step C. Along the paths of the cover of X' not processed during the previous steps we can assign A so that the degree of A is increased by $l(P)$ because we have only paths of type I, II, or III which are not processed yet. Obviously $d(A) = \sum l(P)$ holds where the summation is made over all paths of the cover of X'.

Step D. If there are some $X_i \in X''$ so that the path T_i' is not increased yet during the previous steps then it is increased by one using the edge X_iB. Finally the edge AB is added to the path T_{n-2} which was increased only by one during the previous steps. ($X_{n-2} \in X''$).

It is clear that (i), (ii) and (iii) hold after the algorithm therefore the desired packing is constructed.■■■

We remark that our paper was motivated by the following conjecture of P. Erdős and V.T. Sós [1]: a graph of m vertices and at least $\left[\frac{1}{2}(n-1)m\right]$ edges contains T_n if $m \geqslant n + 1$. A very special case of that conjecture can be formulated as:

Proposition 1. *T_n and T_{n-1} can be packed into K_n.*

The proof follows at once from Theorem 1. A more general special case of the conjecture is:

Proposition 2. *If G is a graph with n vertices and at most $n-1$ edges then G and T_{n-1} can be packed into K_n.*

We do not prove this because B. Bollobás informed us that he had proved a more general result to be published in the Journal of Combinatorial Theory [2].

REFERENCES

[1] P. Erdős, Extremal problems in graph theory, *Theory of graphs and its applications,* Proc. Symp. Smolenice, Publ. Czech. Acad., Prague 1964, 29-36.

[2] B. Bollobás – S.E. Eldridge, Packing of graphs and applications to computational complexity, *J. of Comb. Theory* (B), to appear.

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