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COLORING THE MAXIMAL CLIQUES OF GRAPHS*

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Abstract. In this paper we are concerned with the so-called clique-colorations of a graph, that is, colorations of the vertices so that no maximal clique is monochromatic. On one hand, it is known to be NP-complete to decide whether a perfect graph is 2-clique-colorable, or whether a triangle-free graph is 3-clique-colorable; on the other hand, there is no example of a perfect graph where more than three colors would be necessary. We first exhibit some simple recursive methods to clique-color graphs and then relate the chromatic number, the domination number, and the maximum cardinality of a stable set to the clique-chromatic number. We show exact bounds and polynomial algorithms that find the clique-chromatic number for some classes of graphs and prove NP-completeness results for some others, trying to find the boundary between the two. For instance, while it is NP-complete to decide whether a graph of maximum degree 3 is 2-clique-colorable, $K_{1,3}$-free graphs without an odd hole turn out to be always 2-clique-colorable by a polynomial algorithm. Finally, we show that "almost" all perfect graphs are 3-clique-colorable.

Key words. clique-coloring, hypergraph

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1. Introduction.

A *hypergraph* $\mathcal{H}$ is a pair $(V, \mathcal{E})$, where V is the set of *vertices* of $\mathcal{H}$, and $\mathcal{E}$ is a family of nonempty subsets of V called *edges* of $\mathcal{H}$. In this paper graphs are always undirected, that is, they are hypergraphs where every edge has two elements. A *k-coloration* of $\mathcal{H} = (V, \mathcal{E})$ is a mapping $c : V \to \{1, 2, \ldots, k\}$ such that for all $e \in \mathcal{E}$, $|e| \geq 2$, there exist $u, v \in e$ with $c(u) \neq c(v)$. The *chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ is the smallest k for which $\mathcal{H}$ has a k-coloration. In other words, a k-coloration of $\mathcal{H}$ is a partition $\mathcal{P}$ of V into at most k parts such that no edge of cardinality at least 2 is contained in some $P \in \mathcal{P}$.

As usual, $K_{i,j}$ $(i, j \in \mathbb{N})$ denotes the complete bipartite graph with classes of cardinality i and j; K_n is the complete graph on n vertices, and C_n is a graph on n vertices and n edges forming a circuit. The graph $K_{1,3}$ is also called a *claw*, and $K_3 = C_3$ a *triangle*. A *hole* is an induced chordless cycle with at least five vertices. A *cobipartite graph* is the complement of a bipartite graph.

A graph is called *H-free*, where H is an arbitrary fixed graph, if it does not contain H as an induced subgraph.

In this paper we consider hypergraphs arising from graphs: for a given graph $G = (V, E)$, the *clique-hypergraph of G* is defined as $\mathcal{H}(G) = (V, \mathcal{E})$, where $\mathcal{E} = \{K \subseteq V : K \text{ is a maximal clique of G}\}$. (A set $K \subseteq V$ of vertices is a *clique* if $ab \in E$ holds for all distinct $a, b \in K$, and K is a maximal clique if it is not properly contained in

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any other clique.) A hypergraph $\mathcal{H}$ will be called a *clique-hypergraph* if $\mathcal{H} = \mathcal{H}(G)$ for some graph G defined on the vertices of $\mathcal{H}$.

A k-coloration of $\mathcal{H}(G)$ will also be called a *k-clique-coloration* of G, and the chromatic number of $\mathcal{H}(G)$ the *clique-chromatic number of* G. We hope it will not be confusing to use in parallel the usual terms *k-coloration* and *chromatic number* $\chi(G)$ of G where $c(u) \neq c(v)$ is required for every edge $uv \in E$. As usual, the maximum size of a clique in G is denoted by $\omega = \omega(G)$ and the maximum size of a stable set (a set of vertices not containing any induced edge) by $\alpha = \alpha(G)(= \omega(\bar{G}))$. We will also use the shorthand notations $\kappa := \kappa(G) := \chi(\mathcal{H}(G))$, $\bar{\kappa} := \kappa(\bar{G})$, $\bar{\chi} := \chi(\bar{G})$.

Note that what we call k-clique-coloration here is called strong k-division by Hoàng and McDiarmid in [7]. The main objective of [7] is to find a k-coloration of the hypergraph of *maximum* cliques, which leads for most part to problems of a different nature from those studied here. However, the theorems of [7] on strong k-divisions are related to some of our results, and we will point out the connections that we have understood.

Before explaining some connections between colorations and clique-colorations of graphs, let us show some essential differences concerning combinatorial properties as well as problem complexity.

1. A basic property of graph colorations is that they also provide proper colorations of all the subgraphs of the colored graph. This allows us to define various notions of "critical graphs" and is extensively used in coloring algorithms and proofs. On the contrary, a clique-coloration of G does not necessarily induce clique-colorations of the subgraphs of G; accordingly, the clique-chromatic number is not necessarily smaller for induced subgraphs.

For example, if G is a (nonempty) graph and G' is obtained from G by adding a vertex of full degree, then $\chi(G') = \chi(G) + 1$ while $\kappa(G') = 2$.

However, a k-clique-coloration of a graph can be defined with the k-coloration of a subgraph. This subgraph is not induced by a set of vertices, but arises by deleting edges and vertices of the graph (see after 3 below). Unfortunately a proper way of doing this depends on the clique-coloration itself: deleting or contracting monochromatic edges in a clique coloration does lead to properly colored graphs.

2. The hereditary property of colorations involves advantageous algorithmic behavior as well: one can color the vertices successively by giving to each new vertex a color different from those already assigned to its neighbors (rules can be defined for the order in which the vertices are colored and for the choice of the color). All vertex-colorations, including the optimal ones, can arise in this way.

A simple but very useful modification of this sequential coloring procedure is to combine it with "bichromatic exchanges" (see, for example, [13]). Such natural procedures do not show up for the clique-coloring number even if some sequential procedures will produce some results in what follows.

3. Some of the most basic problems that are completely trivial for coloring become intractable for clique-coloring: the problem of deciding whether a hypergraph given explicitly admits a 2-coloration is known to be NP-complete [11], even for clique-hypergraphs [10]. Furthermore, just to check whether a given set is a color class in some clique-coloration is NP-hard; see section 2.

Clearly, any k-coloration of G is a k-clique-coloration, whence $\kappa \leq \chi$. Typically κ is much smaller than χ. However, *a graph G has a k-clique-coloration if and only if it has a subgraph H such that*

- *for every maximal clique K of G, $|E(H) \cap E(K)| \geq 1$;*
- *H has a k-coloration.*

Indeed, a k-coloration of H can be arbitrarily extended to a k-clique-coloration of G. Conversely, the edges whose two endpoints have different colors in a k-clique-coloration of G define H with the claimed properties.

If G is triangle-free, then of course $\kappa(G) = \chi(G)$. Since the chromatic number of triangle-free graphs is known to be unbounded [17], we get that the same is true for the clique-chromatic number. Let us recall for further use Mycielski's triangle-free graphs with unbounded chromatic number:

- G_2 consists of two adjacent vertices.
- For any $k > 2$, the graph $G_k = (V_k, E_k)$ is defined by the following:
  - $V_k = V_{k-1} \cup S_k \cup \{x_k\}$, where $V_{k-1} = \{v_1, \ldots, v_{n_{k-1}}\}$ and $S_k = \{s_1, \ldots, s_{n_{k-1}}\}$;
  - the subgraph induced by V_{k-1} is isomorphic to G_{k-1}, and the subgraph induced by S_k is a stable set;
  - there exists an edge $s_i v_j$ if and only if there exists an edge $v_i v_j$;
  - x_k is adjacent to all vertices in S_k and to no other vertex.

It is easy to show by induction that G_k is triangle-free and $\chi(G_k) = k$ for all $k \geq 2$. It is also easy to check that $\chi(G_k \setminus \{e\}) = k - 1$ for every edge e of G_k.

The clique-chromatic number is unbounded already for the line-graphs of very particular graphs. Indeed, from the existence of Ramsey numbers we get that for any fixed k there exists $N_k \in \mathbb{N}$ so that for all $n \geq N_k$, every k-edge-coloration of K_n contains a monocolored triangle. A triangle of K_n is a maximal clique in the line-graph L_n of K_n. Therefore $\kappa(L_n) \geq k + 1$ if $n \geq N_k$.

However, in [4] (reported also in [8]), the following question is asked.

Question 1. Does there exist some constant C so that it is always possible to C-color the clique-hypergraph $\mathcal{H}(\mathcal{G})$ of a perfect graph G?

Recall that a graph is *perfect* if, for every induced subgraph G', $\chi(G') = \omega(G')$; that is, the chromatic number of G' is equal to its maximum clique size.

Duffus et al. [4] observe that the answer to Question 1 is positive for two subclasses of perfect graphs: the clique-chromatic number of comparability graphs is at most 2, and that of cocomparability graphs is at most 3 by a result of Duffus, Kierstead, and Trotter [3]. In this paper we show that the answer to Question 1 is yes in some other cases, and again with $C = 2$ or $C = 3$. We do not have any example of a perfect graph, and not even of an odd-hole-free graph, with clique-chromatic number greater than 3.

Let us finally introduce some more notation and terminology. For $U \subseteq V$ we will use the notation $N(U) := \{v \in V : v \notin U$, and there exists $u \in U$ such that $uv \in E\}$, $N[U] := N(U) \cup U$. Instead of $\{x\}$ we will often write x. The *border* $B(U)$ of U is $N(U) \cup N(V \setminus U)$; that is, $B(U)$ is the set of vertices of U or $V \setminus U$ that has a neighbor in $V \setminus U$ or U, respectively. ($B(U) = B(V \setminus U)$). We will say that $u \in U$ is a *border-guard* of U if $N[u] \supseteq B(U)$. Borders and border-guards will be useful for clique-colorations because of the simple fact that any $Q \in \mathcal{E}(\mathcal{H}(G))$ is either entirely contained in U, in $V \setminus U$, or in $B(U)$; in the latter case Q contains all the border-guards of U.

Given $U \subseteq V$ and $u \in U$ it is easy to test whether u is a border-guard of U. This is to be appreciated, because it is not as easy to exhibit a "reasonable" clique-coloration as it is a coloration; the main difficulty is that it is NP-hard already to check whether a given mapping is a clique-coloration! The mentioned properties of border-guards are helpful for achieving these tasks whenever border-guards exist.

In section 2, we analyze various aspects of the complexity of clique-coloring. In section 3, we show some simple but general (greedy) methods to clique-color graphs. In section 4, we exhibit connections between $\kappa(G)$ and other parameters of the graph G. In section 5, we prove that some classes of clique-hypergraphs are 2- or 3-colorable. Finally, in section 6, we show that almost all perfect graphs are 3-clique-colorable.

2. The complexity of clique-coloring. In this section, we study several aspects of the complexity of clique-coloring.

It is already coNP-complete to check whether a given function c defined on the vertices of a graph is a clique-coloration. More precisely, the following problem is shown to be NP-complete.

MAXIMAL CLIQUE CONTAINMENT.

INPUT: Graph $G = (V, E)$ and $T \subseteq V$.

QUESTION: Is there a maximal clique K of G such that $K \subseteq T$?

Therefore deciding whether a k-clique-coloration exists is not clearly in NP nor clearly in coNP.

THEOREM 1. MAXIMAL CLIQUE CONTAINMENT *is NP-complete and remains NP-complete if the complement of the input graph G is restricted to be $K_{1,4}$-free.*

Proof. The 3-DM (that is, three-dimensional matching; see [5]) can be very simply reduced to this problem (a similar proof of [1] can be shortcut for this simpler situation): let $(X, Y, Z, \mathcal{T})$ be an instance of 3-DM; that is, X, Y, Z are finite sets, $|X| = |Y| = |Z|$, and $\mathcal{T} \subseteq X \cup Y \cup Z$ so that for all $T \in \mathcal{T}$, $|T \cap X| = |T \cap Y| = |T \cap Z| = 1$. Let $\mathcal{E} := \mathcal{T} \cup \{\{y\} : y \in Y\}$.

We let G be the intersection graph of the hypergraph $(X \cup Y \cup Z, \mathcal{E})$, that is, the vertex-set of G is $\mathcal{E}$, and we join two vertices if they intersect. The following statements can be easily checked: $\mathcal{T}$ contains a maximal stable set of G if and only if the 3-DM problem has a solution, that is, if the family $\mathcal{T}$ contains a partition of $X \cup Y \cup Z$; since the cardinality of every set in $\mathcal{E}$ is at most three, G is $K_{1,4}$-free.

Thus the 3-DM problem for $(X, Y, Z, \mathcal{T})$ is reduced to the existence of a maximal clique of $\bar{G}$ contained in $\mathcal{T}$, where $\bar{G}$ is $K_{1,4}$-free. ∎

If the maximal cliques of a graph are given, it can of course be checked in polynomial time if a coloration is a clique-coloration. So, for general algorithmic considerations it is reasonable to consider the problem in a setting where $\mathcal{H}(G)$ is given as part of the input.

We will in fact consider the following seemingly more general problem.

k-CLIQUE-COLORING.

INPUT: A family $\mathcal{H}$ of maximal cliques of G, and $k \in \mathbb{N}$.

QUESTION: Can $\mathcal{H}$ be k-colored?

The problem of coloring $\mathcal{H}$ is not really more general than that of coloring $\mathcal{H}(G)$. Indeed, adding to G a vertex v_K for every clique $K \in \mathcal{H}(G) \setminus \mathcal{H}$, and joining v_K exactly to the vertices of K, we obtain a graph G' with the property that $\mathcal{H}$ is k-colorable if and only if $\mathcal{H}(G')$ is k-colorable ($k \geq 2$).

This does not mean that $\mathcal{H}$ arises as the hypergraph of *all* the maximal cliques of some graph: let G be the graph consisting of a circuit on 6 vertices and 3 chords forming a triangle T; then $\mathcal{H}(G) \setminus \{T\}$ does not arise as the set of all maximal cliques of a graph.

Notice also that the problem of coloring clique-hypergraphs is more restrictive than that of general hypergraph coloring: the hypergraph $\{1,2\}$, $\{2,3\}$, $\{3,1\}$ does not arise as a clique-hypergraph.

Since the computation of the chromatic number is NP-hard for triangle-free graphs [12], it is also *NP-hard to compute the clique-chromatic number* of triangle-free graphs, even if all the cliques are given explicitly as part of the input.

Quite general classes of hypergraphs can be 2-colored. Using the Lovász local lemma, McDiarmid [15] proves that all hypergraphs whose hyperedges are "large" (in a well-defined sense), as compared to the degrees, are 2-colorable. Almost all perfect graphs are 3-clique-colorable (see section 6), but deciding if a perfect graph of maximum clique-size four is 2-clique-colorable is already NP-complete, by Kratochvíl and Tuza [10]. On the other hand, Mohar and Škrekovski [16] have shown that every planar graph is 3-clique-colorable, and Kratochvíl and Tuza [10] proposed a polynomial algorithm to decide if a planar graph is 2-clique-colorable (the set of cliques is given in the input).

The following result is inspired by the methods of [10].

THEOREM 2. *2-clique coloring is NP-complete even if the input graph G is restricted to be of maximum degree* 3.

Proof. We use the not-all-equal satisfiability problem (NAE-SAT), which is known to be NP-complete [21].

NAE-SAT.

INPUT: A set X of Boolean variables and a collection C of clauses (set of literals over U), each clause containing three different literals.

QUESTION: Is there a truth assignment for X such that every clause contains at least one true and at least one false literal?

Given an instance $\mathcal{F}$ of NAE-SAT, we build a graph $G(\mathcal{F})$ as follows.

To the clauses we associate vertex disjoint triangles; each vertex corresponds to one of the literals of the clause. For each variable x, vertex disjoint paths P_x are added to the graph as follows. Let $C_1, \ldots, C_k$ be the clauses in which x or its negation occur, the path P_x is defined with vertices $v_{x_1} \ldots v_{x_{2k}}$ (in this order). The path P_x and the triangles are joined with the following rule: if C_i contains x (resp., $\overline{x}$), we add the edge from the vertex of the triangle representing x to $v_{x_{2i-1}}$ (resp., to $v_{x_{2i}}$). This construction is clearly polynomial in the size of $\mathcal{F}$, and it is easy to verify that $G(\mathcal{F})$ is 2-clique-colorable if and only if $\mathcal{F}$ is not-all-equal satisfiable. Furthermore, $G(\mathcal{F})$ is of maximum degree 3. ∎

Because of the nature of the clique-coloring problem, the NP-completeness of the 2-clique-coloring problem does not immediately imply the NP-completeness of the k-clique-coloring problem (for any fixed $k \geq 2$). Nevertheless it is true; here is a simple reduction.

COROLLARY 1. *For any fixed $k \geq 2$, the k-clique-coloring problem is NP-complete.*

Proof. Let G be an instance of the k-clique-coloring problem. Add a copy of the $(k+2)$-chromatic Mycielski graph G_{k+2}. Remove an edge incident to x_{k+2} (we use the notation given in the introduction), and replace x_{k+2} by $|V(G)|$ copies of x_{k+2}. Pairing these copies of x_{k+2} with the vertices of G, we obtain a new graph G'. Observe now that in any $(k+1)$-coloration of G', all copies of x_{k+2} have the same color. Hence a $(k+1)$-clique-coloration of G' yields a k-clique-coloration of G, which completes the reduction. ∎

3. How to clique-color a graph? It is not difficult to provide clique-coloration of a graph: just color every vertex with a different color; a coloration of the graph is also a proper clique-coloration, etc. However, the clique-chromatic number is typically much smaller than the chromatic number. For instance, for perfect graphs the chromatic number is ω and the clique-chromatic number is conjectured to be a constant, maybe 3!

We need heuristics that may provide better estimates than the chromatic number. Besides the difficulty of coloring with a small number of colors, it is also difficult to realize that a procedure is good, since by Theorem 1 we cannot even check whether a partition of the vertices is a clique-coloration.

However, certain constructions inherently guarantee that the result is a proper coloration, and at the same time the number of occurring colors can be bounded in a helpful way. We present in this section three such frameworks. These are meant to be used more as frameworks than algorithms: in the realizations queues can be broken in various ways, and this arising freedom will be exploited in the particular procedures we will present later.

A neighborhood-coloration is any clique-coloration obtained by the following greedy framework.

NEIGHBORHOOD COLORING.

INPUT: Graph $G = (V, E)$ and $\mathcal{H} \subseteq \mathcal{H}(G)$.

0. In each iteration, the algorithm updates the set D of "considered" vertices and the set L of "colored" vertices, $D \subseteq L$. Initially set $D := \emptyset$, $L := \emptyset$.

While not all the vertices are colored do the following:

1. Choose $v \in V \setminus D$, and consider v.

2. If $v \notin L$, then assign to v a color which does not occur in $N(v)$; $L := L \cup \{v\}$.

3. Let c be a color different from all colors occurring among the neighbors of vertices in $N(v) \setminus L$. Assign to all vertices in $N(v) \setminus L$ the color c.

4. Update: $D := D \cup \{v\}$, $L := L \cup N(v)$.

LEMMA 1. *The coloration found by the algorithm is a clique-coloration of G.*

Remark. At each iteration the set of considered vertices dominates the set of colored vertices, so that the set D obtained at the end of the algorithm is a dominating set of G; that is, $N[D] = V$.

The order in which the vertices are considered, or the free choices for the colors, for instance, for color c, will be replaced by particular rules in more specific coloring procedures.

The next lemma shows that if a graph admits a certain partition of the vertices, then it is k-clique-colorable. A clique-coloration obtained by the way described in the proof of Lemma 2 will be called a *partition coloration*.

LEMMA 2. *Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$, $k \geq 2$.*

If G admits a partition $\{V_1, \ldots, V_p\}$ of V such that

- *$G(V_i)$ is k-clique-colorable, and V_i has a border-guard in G ($i = 1, \ldots, r \leq p$);*
- *$G(V_i)$ ($i = r+1, \ldots, p$) does not contain a maximal clique of G;*
- *the graph H obtained by identifying the vertices of each V_i (denote the new vertices by x_i, $i = 1, \ldots, p$) has $\chi(H) \leq k$;*

then G is k-clique-colorable.

Proof of Lemma 2. Consider a k-coloration $c_H : V(H) = \{x_1, \ldots, x_p\} \longrightarrow \{1, \ldots, k\}$ of H and also a k-clique-coloration $c_i : V_i \longrightarrow \{1, \ldots, k\}$ of $G(V_i)$ ($i = 1, \ldots, r$).

By assumption V_i has a border-guard v_i in G ($i = 1, \ldots, r$). We can suppose that $c_i(v_i) = c_H(x_i)$ (otherwise we interchange two colors in the coloration of $G(V_i)$). Furthermore, for $i = r+1, \ldots, p$ we define $c_i(v) = c_H(x_i)$ for all $v \in V_i$. Define for $v \in V(G)$ $c(v) := c_i(v)$ if $v \in V_i$.

Now let Q be a maximal clique of G. If Q is contained in some V_i, then by the assumption $i \leq r$ and $c(q) = c_i(q)$ for all $q \in Q$. Therefore, at least two colors occur in Q. If Q is not contained in some V_i, then say $Q \cap V_i \neq \emptyset \neq Q \cap V_j$. Let $v_i \in Q \cap V_i$ (resp., $v_j \in Q \cap V_j$) be an arbitrary vertex in $Q \cap V_i$ (resp., $Q \cap V_j$) for $i \geq r$ (resp., $j \geq r$).

Clearly, $v_i, v_j \in Q$. Since $c(v_i) = c_H(x_i) \neq c_H(x_j) = c(v_j)$ because of $x_i x_j \in E(H)$, two different colors do occur in Q. ◻

A third simple but useful method is presented in the following lemma. A pair (d, D) is called a *dominating pair* if $d \in V$, $D \subseteq N(d)$, and any maximal clique K of G containing d satisfies $K \cap D \neq \emptyset$. The following lemma shows that such a pair can be useful for our coloring problem.

LEMMA 3 (dominating pair lemma). *Let (d, D) be a dominating pair, and let k be a nonnegative integer with $|D| < k$. If $\mathcal{H}(G - d)$ is k-colorable, then so is $\mathcal{H}(G)$.*

Proof. Let c be a k-coloration of $\mathcal{H}(G-d)$. Since $k > |D|$, there exists a color i that does not occur in D. Let $c' : V \to \{1, 2, \ldots, k\}$, with $c'(v) = c(v)$ for all $v \in G - d$ and $c'(d) = i$. Since c is a k-coloration of $\mathcal{H}(G - d)$, it is sufficient to check that any maximal clique K which contains d is not monocolored by c'. By definition of a dominating pair, there exists a vertex $v \in K \cap D$. By the choice of i, we have $c'(d) = i \neq c(v) = c'(v)$. Thus c' is a k-coloration of $\mathcal{H}(G)$. ◻

Let G be a graph with the property that every induced subgraph contains a vertex u whose neighborhood has at most k connected components, each of which is a clique. A direct consequence of the dominating pair lemma is that G is $k+1$-clique-colorable.

4. Rough general bounds. In this section we estimate the clique-chromatic number with some other graph parameters.

Recall that a *dominating set* D is a subset of V such that $N[D] = V$. The *domination number* $\gamma(G)$ of a graph G is the smallest cardinality of such a set. Note that $\gamma(G)$ is always smaller than or equal to the stability number $\alpha(G)$.

We assume G to be connected, leaving to the reader the trivial extension of the following theorem to graphs with several connected components.

THEOREM 3. *If $G = (V, E)$ is a connected graph, then $\kappa(G) \leq \gamma(G) + 1$, and if $\kappa(G) = \gamma(G) + 1$, then every dominating set D of minimum size is a stable set, and one of the following holds:*

- *$|D| < \alpha(G)$,*
- *D is a set of two nonadjacent vertices of $G = C_5$,*
- *$|D| = 1$ and $G = K_n$, $n \geq 2$.*

Proof of Theorem 3. Let $D = \{x_1, \ldots, x_k\}$ be a dominating set of G, and $n := |V(G)|$. If there exists $a, b \in D$, $ab \in E(G)$, suppose $x_k = b$. Apply a neighborhood coloring with the following specifications: the order of considering the vertices is $x_1, \ldots, x_k$; in the ith iteration ($i = 1, \ldots, k$), if x_i is not yet colored, color it with color 1; moreover, for $i = 1, \ldots, k-1$, color the not yet colored vertices of $N(x_i)$ with color $i+1$; if $c(x_k) \neq 1$, then color $N(x_k) \setminus \cup_{j=1}^{k-1} N[x_j]$ with color 1, otherwise with color $k+1$. It can be checked immediately that the defined colors are allowed, and the number of colors is $k+1$ only if D is a stable set. More exactly, we have the following claims.

Claim 1. If $\kappa(G) = k+1$, then D is a maximal stable set of minimum size.

Indeed, if there exists a maximal stable set D' of smaller size $k' := |D'| < k$, then it is also a dominating set. Hence $\kappa(G) \leq k' + 1 \leq k$, as required.

Assume now that $k = \alpha(G)$.

Claim 2. If $\kappa(G) = k + 1$, $k = \alpha(G) \leq 2$, then either $G = C_5$, or $G = K_n$, $n \geq 2$.

Indeed, if $k = \alpha = 1$, then $G = K_n$. Let now $k = \alpha = 2$. We prove by induction on the number of vertices that $\kappa(G) = 2$, unless $G = C_5$.

Let a and b be two nonadjacent vertices; then because of $\alpha < 3$, $N[a] \cup N[b] = V(G)$.

If we can 2-clique-color the subgraph N_{ab} induced by $N(a) \cap N(b)$, then we extend this coloration to all G: define $c(v) := 1$ if $v \in \{a\} \cup N(b) \setminus N(a)$, and $c(v) := 2$ if $v \in \{b\} \cup N(a) \setminus N(b)$. If Q is a maximal clique of G and, say, $c(q) = 1$ for all $q \in Q$, then all vertices of $Q \setminus a$ are adjacent to b. Since $c(b) = 2$ it follows that $a \in Q$. But then $Q \setminus a$ is a maximal clique of N_{ab}, and, since c is a 2-clique-coloration of N_{ab}, $Q \setminus a$ is a single vertex, v. If $\{b, v\}$ is not a maximal clique, then by giving color 2 to v we get a 2-clique-coloration of G. Else v is adjacent only to a and b, and so, since $\alpha = 2$, $V(G) \setminus \{a, b, v\}$ is a clique. We may assume that $N_{ab} \setminus \{v\}$ is empty and that $N(a) \setminus N(b)$ and $N(b) \setminus N(a)$ are nonempty, since, else, there exists a dominating edge in G and hence, by Claim 1, a 2-clique-coloration of G. In case a or b has at least two neighbors distinct from v, then let w be one of those, give color 1 to a, b, and w, and give color 2 to all the other vertices: this a 2-clique-coloration of G. The only remaining case is when $|N(a) \setminus N(b)| = |N(b) \setminus N(a)| = 1$; then $G = C_5$.

We now assume that N_{ab} has no 2-clique-coloration. Thus by induction hypothesis, at least one connected component of N_{ab} induces a C_5. Since $\alpha = 2$, we have $N_{ab} = C_5$. Label $v_1, \ldots, v_5$ its vertices in the cyclic order. If $N(a) \setminus N(b) = N(b) \setminus N(a) = \emptyset$, then G is 2-clique-colorable; else fix a vertex v in, say, $N(a) \setminus N(b)$. Since $\alpha(G) = 2$, v is adjacent either to v_1 or to v_3, say v_1, and v is adjacent either to v_2 or to v_5, say v_2. Now give color 1 to a, v_1, v_2, v_4, and all the vertices in $N(b) \setminus N(a)$, and give color 2 to all the other vertices: this a 2-clique-coloration of G.

The claim is now proved.

To finish the proof of Theorem 3, suppose that $k \geq 3$ and that D is a stable set of cardinality $k = \alpha(G)$. In the above constructed neighborhood coloring, let x_{k-2}, x_{k-1}, x_k be the three pairwise nonadjacent vertices colored last. The neighborhood coloring assigns colors $c(x_{k-2}) = c(x_{k-1}) = c(x_k) = 1$ and new colors $k-1$, k, $k+1$ to the set of their not-yet-colored neighbors.

Claim 3. The graph induced by vertices of color $k-1$, k, $k+1$ and x_{k-2}, x_{k-1}, x_k can be 3-clique-colored.

The claim finishes the proof of the theorem. Indeed, choose the three colors to be 1, $k-1$, and k to get a k-clique-coloration of G. (The colors $k-1$ and k do not occur previously, and all previously colored vertices of color 1 are nonadjacent to the vertices that are present in the claim.)

To prove Claim 3, we can suppose $k = 3$; then the notation is simplified, and we only have to prove $\kappa(G) \leq 3$.

If $G - N[v]$ is not a C_5 for some $v \in V(G)$, then by Claim 2 it can be colored with colors 1 and 2; completing this coloration with $c(v) := 1$ and $c(x) := 3$ if $x \in N(v)$, the statement is proved.

Suppose now that $G - N[v]$ is a C_5 for all $v \in V(G)$. Then G is $n-6$-regular. If there is no triangle in G, then $N(v)$ is a stable set for all $v \in V(G)$, and therefore $n - 6 \leq 3$. The equality holds here, because if G is 2-regular, then $G - N[v]$ cannot be a C_5 for all $v \in V(G)$. But if the equality holds, then the number of edges with exactly one endpoint in $N[v]$ is, on one hand, $2|N(v)| = 6$ and, on the other hand, 5 (because there is exactly one such edge for every vertex of $G - N[v]$).

So G has a triangle. Let $ab \in E(G)$ be one of its edges. If $\{a, b\}$ is a dominating set, then we can 2-clique-color G by Claim 1. Let us suppose that v is adjacent neither to a nor to b. Since $G - N[v]$ is a C_5 containing the edge ab, where ab is contained in a triangle of G, the following coloration is correct: $c(v) := c(a) := c(b) := 1$, $c(x) := 2$ if $x \in N(v)$, and the remaining three vertices forming a path in the C_5 can be colored $3, 1, 3$. □

Remark that for any integer k, a path P_{3k} on $3k$ vertices has a dominating number equal to k and $\kappa(P_{3k}) = 2$.

On the other hand, Mycielski's graphs provide an infinite class of triangle-free graphs G_k for which $\kappa(G_k) = \chi(G_k) = \gamma(G_k) + 1 = k$ (for $k \geq 4$ the first case of the theorem holds, for $k = 3$ the second, and for $k = 2$ the third). Let $D_2 = \{v\}$, where v is either vertex of G_2, and define $D_k := D_{k-1} \cup \{x_k\}$ (we use the notation given in the introduction). By construction, D_k is a dominating set of G_k and $|D_k| = k - 1$. By the theorem, and since $\kappa(G_k) = \chi(G_k) = k$, we have that $\gamma(G_k) = k - 1$, and it follows that D_k is a maximal stable set of minimum size (and not maximum as soon as $k \geq 4$).

Corollary 2. *For any graph $G \neq C_5$ with $\alpha(G) \geq 2$, we have $\kappa(G) \leq \alpha(G)$.* □

This first corollary sharpens Theorem 2 in [7]. Indeed, it is stated there that $\kappa(G) \leq \alpha(G) + 1$ and the strict inequality holds for C_5-free noncomplete graphs.

Corollary 3. *For any graph G of order n, we have $\kappa(G) \leq 2\lceil \sqrt{n}\, \rceil$.*

Proof. Let $D = \{v_1, \ldots, v_k\}$ be a subset of k vertices with the following properties:
- $|N(v_1)| \geq \sqrt{n}$,
- $|N(v_i) - (\cup_{j<i} N[v_j])| \geq \sqrt{n}$ for $i = 2, \ldots, k$,
- any vertex $v \in V(G)$ satisfies $|N(v) - N[D]| < \sqrt{n}$.

Note that D can be empty. Since D is a dominating set of $N[D]$, and $|D| < \sqrt{n}$, by Theorem 3, we can clique-color the subgraph induced by $N[D]$ with $\lceil \sqrt{n}\, \rceil$ colors, say $\{1, \ldots, \lceil \sqrt{n}\, \rceil\}$.

On the other hand, in the subgraph induced by $V \setminus N[D]$ the degree of every vertex is strictly smaller than $\sqrt{n}$, so we can color this subgraph with $\lceil \sqrt{n}\, \rceil$ colors, say $\{\lceil \sqrt{n}\, \rceil + 1, \ldots, 2\lceil \sqrt{n}\, \rceil\}$, by a sequential algorithm. This coloration is a clique-coloration too. □

This bound is not best possible: Kotlov [9] proved that $\kappa(n) \leq \lfloor \sqrt{2n} \rfloor$. We do not even know whether the maximum of the clique-chromatic number for graphs on n vertices divided by $\sqrt{2n}$ is a constant or tends to 0.

Theorem 4. *Let $G = (V, E)$ be a graph and q be an integer, $q > 1$. Then the hypergraph $\mathcal{H}_q := \{K \in \mathcal{H}(G) : |K| \geq q\}$ is $\lceil \frac{\chi(G)}{q-1} \rceil$-colorable.*

Proof. Let $k := \lceil \chi(G)/(q-1) \rceil$. Let $S_1, \ldots, S_{\chi(G)}$ be the color classes of a $\chi(G)$-coloration of G. For $i = 1, \ldots, k$, we consider the union of $q - 1$ color-classes: $C_i = \bigcup_{j=(i-1)(q-1)+1}^{i(q-1)} S_j$ if $i = 1, \ldots, k-1$, and $C_k = \bigcup_{j=(k-1)(q-1)+1}^{\chi(G)} S_j$.

Observe that $\omega(C_i) < q$ for every $i = 1, \ldots, k$. Thus, the coloration c, defined by $c(x) = i$ if $x \in C_i$, is a k-coloration of $\mathcal{H}_q$. □

Corollary 4. *If G is an arbitrary graph, then $(\kappa - 1)(\bar{\kappa} - 1) \leq 2\min\{\chi, \bar{\chi}\} - 2$.*

Proof. Let k be the size of a smallest maximal stable set of G. Since a maximal stable set of G is a dominating set of G, by Theorem 3, we have that $\kappa(G) - 2 \leq k - 1$.

By the choice of k, we have that any maximal clique of $\overline{G}$ has size at least k. If $k > 1$, by Theorem 4, we obtain $\kappa(\overline{G}) - 1 \leq \frac{\chi(\overline{G})-1}{k-1}$. Multiplying the two inequalities, we obtain $(\kappa - 2)(\bar{\kappa} - 1) \leq \bar{\chi} - 1$.

If $k = 1$, then $\kappa = 2$ and trivially $(\kappa - 2)(\bar{\kappa} - 1) \leq \bar{\chi} - 1$.

In both cases we get $(\kappa - 1)(\bar{\kappa} - 1) \leq \bar{\chi} + \bar{\kappa} - 2 \leq 2(\bar{\chi} - 1)$.

Applying this again after interchanging the role of G and $\bar{G}$, we get the claim. □

This bound can be sharpened under various assumptions. For instance, if κ or $\bar{\kappa}$ are close to χ or $\bar{\chi}$, like for Mycielski graphs (see section 1), if $\kappa = \chi$, then $\bar{\kappa} \leq 3$. (In fact, for Mycielski graphs the statement "$\bar{\kappa} = 2$ except for $G_3 = C_5$" is easy to prove directly.) The bound can also be refined using other parameters: as Kotlov [9] noticed, $(\kappa - 1)(\bar{\kappa} - 1) \leq \frac{k}{k-1}(\bar{\chi} - 1)$ if $k > 1$.

5. Claw-free and perfect graphs. In this section we study $\kappa(G)$ and $\kappa(\bar{G})$ when G is a claw-free or a perfect graph or both.

If G is a perfect graph, then we have $\kappa(G) \leq \chi(G) = \omega(G)$. Applying also Corollary 2, if G is not a complete graph, then we have $\kappa(G) \leq \min\{\alpha(G), \omega(G)\}$. (This is better than the bound of Corollary 4 only if $\bar{\kappa} = 2$.) Moreover, when G is perfect, $\alpha(G)$ and $\omega(G)$ can be computed in polynomial time [6].

Furthermore, it seems that in perfect graphs not only the maximum cliques but also the maximal cliques behave well from the viewpoint of clique-colorations. A consequence could be that there exists a constant C such that $\mathcal{H}(G)$ is C-colorable for a perfect graph G; that is, Question 1 has a positive answer. We prove that such a C exists for some classes of perfect graphs.

For example, the hypergraph of maximal cliques of a strongly perfect graph G (defined by the property that every induced subgraph of G contains a stable set intersecting all maximal cliques) is obviously 2-colorable: indeed, color a stable set intersecting all maximal cliques of G with one color and the rest of the vertices with another color.

Note that $\kappa(G)$ can be greater than 2, even for a perfect graph G (see Figure 5.1).

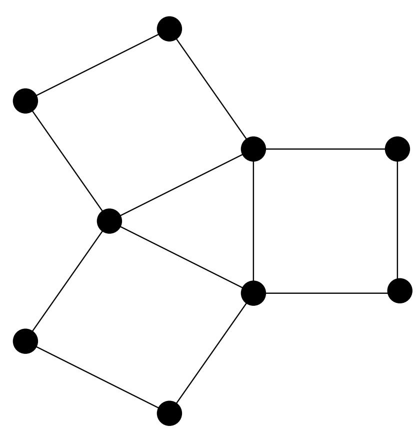

FIG. 5.1. *The clique-hypergraph of this perfect graph is clearly not* 2*-colorable since it contains edges of* C_9 *as hyperedges.*

We saw in the introduction that the clique-chromatic number of claw-free graphs or even of line-graphs is not bounded. The following theorem shows that triangles are the only source of difficulty.

We do not know the complexity of clique-coloring line-graphs of graphs optimally. Observe that in the case of line-graphs, it is easy to check whether a given coloration is correct since all maximal cliques of a line-graph $L(G)$ are either stars or triangles of G, and therefore the number of maximal cliques is small (bounded by a polynomial of the number of vertices).

A multigraph is a graph that may contain an arbitrary number of parallel edges.

THEOREM 5. *Let G be a multigraph, $\mathcal{H} = (V, \mathcal{E})$ where $V := E(G)$, and $\mathcal{E}$ is the collection of stars of G. Then $\chi(\mathcal{H}) \leq 3$. Moreover, $\chi(\mathcal{H}) = 3$ if and only if G has a component which is an odd circuit.*

Proof. Without loss of generality, assume that G is connected. Let G' be obtained from G by adding to it a perfect matching M of its odd-degree vertices, if any. Let T be an Eulerian tour of G'. Color the edges of T alternatively black and white, starting at a vertex of degree at least four (if any) or with an edge of M (if any). If there is such a vertex or such an edge, then this coloring induces a proper 2-coloration of $\mathcal{H}$. Else, G is a cycle, and this 2-coloration of $\mathcal{H}$ is not proper if and only if G is an odd cycle. □

We are highly indebted to Kotlov [9] for short-cutting most of our original proof.

For complements of claw-free graphs, the following simple bound holds.

THEOREM 6. *Let $2 \leq k \leq \alpha(G)$. If G is $K_{1,k}$-free, then $\kappa(\bar{G}) \leq k$.*

Proof. Since $k \leq \alpha(G)$, there exists a stable set $S \subseteq V(G)$, $|S| = k$. Since G is $K_{1,k}$-free, S induces a dominating clique (not necessarily maximal) of $\overline{G}$. We achieve the proof of Theorem 6 by applying Theorem 3. □

Notice that the complements of Mycielski's graphs are $K_{1,3}$-free, showing that the condition $k \leq \alpha(G)$ in the preceding theorem is necessary.

We have now arrived at the most difficult result of this paper: we determine the clique-chromatic number of claw-free perfect graphs.

THEOREM 7. *If G is a claw-free perfect graph, then $\mathcal{H}(G)$ is 2-colorable.*

By Theorem 6 any graph which is the complement of a claw-free graph of stability number at least 3 is 3-clique-colorable even if it is not perfect. On the other hand, we saw that line-graphs (which are, of course, claw-free) may have arbitrary large clique-chromatic number, unless they arise from triangle-free graphs.

In [7] it is proved that the hypergraph of *maximum* cliques of a claw-free graph is 2-colorable if and only if it does not contain an odd hole. A common feature of the proof of [7] and our proof below is the use of Ben Rebea's lemma (as cited in [2]); however, an essential difference is that the main part of our proof is the perfect case.

COROLLARY 5. *If G is a claw-free graph without an odd hole, then $\kappa(G) \leq 2$.*

Proof of Corollary 5. Let G be claw-free without an odd hole. If $\alpha(G) \leq 2$, then by Corollary 2, $\kappa(G) \leq 2$.

If $\alpha(G) \geq 3$, then G is perfect because of the following: by Ben Rebea in [2] a connected claw-free graph G with $\alpha(G) \geq 3$ containing an odd antihole also contains an odd hole; Parthasaraty and Ravindra [18] proved that a claw-free graph with neither an odd hole nor an odd antihole is perfect.

Since G is perfect, Theorem 7 can now be applied. □

In order to prove Theorem 7, we use the structural property of claw-free graphs explored by Chvátal and Sbihi [2] and Maffray and Reed [14].

Chvátal and Sbihi [2] defined two special classes of claw-free perfect graphs: the *elementary graphs* and *peculiar graphs*. A graph is called elementary if its edges can be colored with two colors such that every induced P_3 (chordless path on three vertices) has its two edges colored differently. Clearly elementary graphs are claw-free, but not vice versa, as C_5 shows. A graph is called peculiar if it can be obtained as follows: take three pairwise vertex-disjoint cobipartite graphs; call them $(A_1, B_2), (A_2, B_3), (A_3, B_1)$, such that each of them has at least one pair of non-adjacent vertices; add all edges between every two of these cobipartite graphs; then add three cliques Q_1, Q_2, Q_3 that are pairwise disjoint and disjoint from the A_i's and B_i's; add all the edges between Q_i and $A_j \cup B_j$ for $j \neq i$; there is no other edge in the graph. Chvátal and Sbihi [2] proved that every claw-free perfect graph can be decomposed via clique-cutsets into indecomposable graphs that are either peculiar or elementary.

THEOREM 8 (see [2]). *If G is a claw-free perfect graph without a clique cutset, then G is either elementary or peculiar.*

The structure of elementary graphs was determined by Maffray and Reed in [14] as follows. An edge is called *flat* if it does not lie in a triangle. Let xy be a flat edge of a graph G and $(X, Y; F)$ be a cobipartite graph disjoint from G and containing at least one edge with one extremity in X and the other in Y. We obtain a new graph from $G - \{x, y\}$ and $(X, Y; F)$ by making the union of their sets of vertices and edges and adding all possible edges between X and $N_G(x) \setminus \{y\}$ and between Y and $N_G(y) \setminus \{x\}$. This is called *augmenting* the flat edge xy with the cobipartite graph $(X, Y; F)$. The result of augmenting a set of pairwise independent (nonincident) flat edges $e_1, \ldots, e_h$ successively is called an *augmentation* of G.

THEOREM 9 (see [14]). *A graph G is elementary if and only if it is an augmentation of the line-graph of a bipartite multigraph.*

Proof of Theorem 7. We now prove Theorem 7 through several lemmas.

LEMMA 4. *If G is an elementary graph, then $\mathcal{H}(G)$ is 2-colorable.*

Proof of Lemma 4. For line-graphs of bipartite multigraphs the statement follows from Theorem 5. Furthermore, if G has a 2-clique coloration, the graph obtained by augmenting a flat edge xy with $B = (X, Y; F)$ still has a 2-clique-coloration: keep the same color for all vertices of $G - \{x, y\}$; choose an edge ab of B with $a \in X$ and $b \in Y$; and give color 1 to a and to all vertices in $Y \setminus \{b\}$ and color 2 to b and to all vertices in $X \setminus \{a\}$. $\square$

Using previous results, it is also not difficult to check the following.

LEMMA 5. *If G is a peculiar graph, then $\mathcal{H}(G)$ is 2-colorable.*

Proof of Lemma 5. Let $G = (V, E)$ be a peculiar graph composed of (A_1, B_2), (A_2, B_3), (A_3, B_1), Q_1, Q_2, Q_3 as in the definition of a peculiar graph. Let $a \in A_1$ and let $b \in B_3$ (by definition all the A_i's, B_i's are nonempty). It is easy to verify that the edge ab is dominant, and hence by Theorem 3 we obtain that $\mathcal{H}(G)$ is 2-colorable. $\square$

LEMMA 6. *If G is a claw-free graph and Q is a clique which is a minimal cutset, then $G - Q$ has two components; denote their set of vertices V_1 and V_2, and at least one of the following holds:*

(a) *Either for $i = 1$ or for $i = 2$ both V_i and $V \setminus V_i$ have a border-guard.*

(b) *Both V_1 and V_2 have a border-guard.*

(c) *Both $V_1 \cup Q$ and $V_2 \cup Q$ have two border-guards.*

Proof of Lemma 6. Since Q is a minimal cutset, every $q \in Q$ has a neighbor in all the components. Since G is claw-free, $G - Q$ has two components, and $N(q) \cap V_i$ is a clique for all $i = 1, 2$ and all $q \in Q$.

Claim 1. For all $a, b \in Q$, either $N(a) \cap V_1 \subseteq N(b) \cap V_1$ or $N(a) \cap V_2 \subseteq N(b) \cap V_2$.

Indeed, if not, let $a_i \in N(a) \cap V_i \setminus (N(b) \cap V_i)$ $(i = 1, 2)$. Clearly, a, b, a_1, a_2 induce a claw, a contradiction.

Claim 2. Either there exists a border-guard in V_1, or there exist two distinct border-guards in $V_1 \cup Q$.

Indeed, suppose the first possibility does not hold. Then there are $a \neq b \in Q$ so that $N[a] \cap V_1$ and $N[b] \cap V_1$ are not equal, and they are both inclusionwise minimal among $N[q] \cap V_1$ ($q \in Q$). (If there were a unique inclusionwise minimal $N[q] \cap V_1$ ($q \in Q$), then any $v_1 \in N[q] \cap V_1$ would be a border-guard of V_1.)

Since neither $N[a] \cap V_1$ nor $N[b] \cap V_1$ contains the other, by Claim 1 both $N[a] \cap V_2 \subseteq N[b] \cap V_2$ and $N[b] \cap V_2 \subseteq N[a] \cap V_2$ hold; that is, $N[a] \cap V_2 = N[b] \cap V_2 =: N_2$.

Now by the minimal choice of $N[a] \cap V_1$ and of $N[b] \cap V_1$, $N[q] \cap V_1$ for any $q \in Q$ cannot be a subset of both. So by Claim 1, $N[q] \cap V_2 \subseteq N_2$ for all $q \in Q$. Since $B(V_1 \cup Q) = Q \cup N_2$, we proved that both a and b are border-guards of $V_1 \cup Q$ and the claim is proved.

To finish the proof of Lemma 6, note that by symmetry, Claim 2 also holds if we replace 1 by 2. From these two variants of Claim 2 we get that one of the following cases holds:

– Both V_1 and V_2 have a border-guard, and then each of these is adjacent with every vertex in Q. So Q is not a maximal clique, and "b" of the lemma holds.

– Both $V_1 \cup Q$ and $V_2 \cup Q$ have two border-guards, and then we have "c."

– V_1 and $V_2 \cup Q$ have border-guards or V_2 and $V_1 \cup Q$ have border-guards. This is just "a." □

The proof of Theorem 7 works by induction on $|V|$. Let $G = (V, E)$ be a claw-free perfect graph. If G has one, two, or three vertices, then clearly $\mathcal{H}(G)$ is 2-colorable. Suppose now that G has n vertices and that the theorem has been proved for any claw-free perfect graph with less than n vertices. If G is either elementary or peculiar, then, by Lemmas 4 and 5, $\mathcal{H}(G)$ is 2-colorable. So by Theorem 8, we may assume that G has a clique cutset.

We can now finish the proof of Theorem 7 by applying the idea of Lemma 2 in a very simple special case.

If Lemma 6(a) holds for say $i = 1$, by the induction hypothesis, we can 2-clique-color $G(V_1)$ and $G(V \setminus V_1)$. Without loss of generality, we may assume that the border-guard of V_1 has a different color from that of $V \setminus V_1$. Every maximal clique of G is contained either in V_1 or in $V \setminus V_1$, or contains both border-guards. In any case, both colors occur in it.

If Lemma 6(b) holds, then by the induction hypothesis, we can 2-clique-color $G(V_1)$ and $G(V_2)$. Without loss of generality, we may assume that both their border-guards have color 1. Color all vertices of Q with color 2. Since every maximal clique of G is contained in V_1 or V_2 or contains a border-guard and a vertex of Q, we defined a 2-clique-coloration.

Finally, if Lemma 6(c) holds, then color Q so that the two border-guards of $V_1 \cup Q$, and also those of $V_2 \cup Q$, have different colors, and otherwise arbitrarily. We complete this coloration by a 2-clique-coloration of $G(V_1)$ and $G(V_2)$. Now every maximal clique of G is contained in V_1 or in V_2, or for some $i \in \{1, 2\}$ it contains both border-guards of $V_i \cup Q$. □.

Note that the proof of Theorem 5 is algorithmic; moreover, either it reduces the clique-coloration of G into the clique-coloration of two smaller graphs or the graph itself is easy to color.

Using the following ingredients, the proof provides a way of 2-clique-coloring an arbitrary claw-free perfect graph G in polynomial time:

– Whitesides's algorithm [23] that finds a clique cutset;

- Chvátal and Sbihi's Theorem 8 [2];
- Maffray and Reed's canonical decomposition algorithm of an elementary graph into a line-graph of a bipartite graph and some augmentations [14];
- checking for border-guards is polynomial (obvious);
- the number of graphs occurring through the decomposition can be bounded by a polynomial of the number of vertices of the input graph. (These graphs are not the same as in Chvátal and Sbihi's algorithm for recognizing claw-free perfect graphs, since the clique-cutset is not left in both two decomposing graphs.)

Furthermore, this algorithm uses only the graph G and not a list of its maximal cliques.

Diamond-free perfect graphs constitute another interesting class of perfect graphs (a diamond is a K_4 minus an edge). It is known [22, 19] that a diamond-free graph is perfect if and only if it does not contain an odd hole. Unfortunately we cannot prove $\kappa \leq 3$ for this class. This is somewhat frustrating, because Tucker [22] proved that a diamond-free perfect graph has a vertex which is contained in at most two maximal cliques of size at least 3, which implies the following.

PROPOSITION 1. *The hypergraph of maximal cliques of size at least* 3 *of a diamond-free perfect graph is* 3*-colorable. In particular, if G is a diamond-free perfect graph without flat edges, then $\kappa(G) \leq 3$.*

The conjecture $\kappa \leq 3$ for diamond-free perfect graphs (equivalently diamond- and odd-hole-free graphs) could contain many of the difficulties of coping with odd-hole-free graphs in general. We wonder whether the clique-chromatic number of odd-hole-free graphs could be bounded as well: we also do not know of any odd-hole-free graph with clique-chromatic number greater than three.

6. Generalized split graphs. A graph G is a *generalized split graph* if either G or the complement of G has a vertex partitioned into sets A, B_i $(1 \leq i \leq k)$ so that A and all B_i's span complete graphs and there are no edges between B_i and B_j if $i \neq j$. Generalized split graphs are perfect and have been introduced in the paper of Prömel and Steger [20]; this class plays a crucial role in their proof of the asymptotic version of the strong perfect graph conjecture: almost all Berge graphs are perfect. In fact, they proved in [20] that almost all C_5-free graphs are generalized split graphs. ("Almost all" means here that the ratio of the number of labelled n-vertex C_5-free graphs to the number of n-vertex generalized split graphs tends to one if n tends to infinity.) Therefore any property of generalized split graphs holds for almost all perfect graphs. In our case the property in question is the chromatic number of the clique hypergraph.

THEOREM 10. *The clique-hypergraph of a generalized split graph is* 3*-colorable.*

Proof. Assume that G is a generalized split graph. If the complement of G has the required partition into A, B_i's, then a proper coloration for the maximal cliques of G is trivial: the vertices of A are colored with color 1, the vertices of B_1 are colored with color 2, and the vertices in all other B_i's (if there are any) are colored with color 3.

If G has the required partition, then two cases are considered. If $|A| \leq 1$, then we color the B_i's with colors 1 and 2 so that each of them with at least two vertices gets both color 1 and color 2, and if A is nonempty, we color it with color 3. Finally, if $|A| > 1$, a fixed vertex $x \in A$ is colored by color 2, all other vertices of A are colored with color 3, the sets B_i with one vertex are colored with color 1, and any set B_i with at least two vertices is colored using the same rule: if x is adjacent to all vertices of

B_i, then color all vertices of B_i with color 1; otherwise, a fixed vertex of B_i which is not adjacent to x is colored with color 2 and all other vertices of B_i are colored with color 1. It is straightforward to check that under this coloration every maximal clique of G gets at least two colors. □

It is worth noting that the theorem is sharp in the sense that there are generalized split graphs with 3-chromatic clique-hypergraphs, for instance, the graph in Figure 5.1.

The result of Prömel and Steger [20] mentioned above yields the following corollary, which is an asymptotic answer to Question 1.

COROLLARY 6. *Almost all perfect graphs are* 3*-clique-colorable.*

7. Open problems. In Theorem 1, we proved that MAXIMAL CLIQUE CONTAINMENT is NP-complete for the complements of $K_{1,4}$-free graphs. It is therefore natural to first ask the following question.

Question 2. Is MAXIMAL CLIQUE CONTAINMENT polynomially solvable for the complements of $K_{1,3}$-free graphs?

Since it is NP-complete to compute the chromatic number of a triangle-free graph [12], it is NP-complete to compute the clique-chromatic number of a complement of a $K_{1,3}$-free graph. Nevertheless, we know by Theorem 6 that $\chi(\mathcal{H}(\bar{G})) \leq 3$ when G is $K_{1,3}$-free and $\alpha(G) \geq 3$. Hence we should ask the next question.

Question 3. Is it NP-complete to determine whether $\bar{G}$ is 2-clique colorable when G is $K_{1,3}$-free?

We saw that it is NP-complete to determine whether a graph of maximum degree 3 is 2-clique-colorable. Moreover, Corollary 5 gives that any $K_{1,3}$-free graph with no odd hole is 2-clique colorable.

Question 4. Is it NP-complete to determine whether G is 2-clique colorable when G is $K_{1,3}$-free?

Most of our results concern classes of graphs defined by forbidden configurations. Thus it would be interesting to study hereditary properties of the clique-chromatic number of a graph. Hoàng and McDiarmid in [7] studied such questions. Concerning the complexity aspect, we ask the following.

Question 5. What is the complexity of deciding whether a graph and all its induced subgraphs can be 2-clique-colored?

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