On Graphs With Strongly Independent Color-Classes

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Abstract: We prove that for every k there is a k-chromatic graph with a k-coloring where the neighbors of each color-class form an independent set. This answers a question raised by N. J. A. Harvey and U. S. R. Murty [4]. In fact we find the smallest graph G_k with the required property for

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every k. The graph G_k exhibits remarkable similarity to Kneser graphs. The proof that G_k is k-chromatic relies on Lovász's theorem about the chromatic number of graphs with highly connected neighborhood complexes.

Keywords: *chromatic number; Kneser graphs; Mycielski's construction; neighborhood complex*

1. INTRODUCTION

Our starting point is the following problem raised by N. J. A. Harvey and U. S. R. Murty [4].

For a given positive integer k, is there a k-chromatic graph G with a k-coloring $c = \{X_1, \ldots, X_k\}$ such that $N_G(X_i)$ is independent[1] for each color-class $X_i, 1 \leq i \leq k$.

By a *k-coloring* of a graph G, we mean a partition $c = \{X_1, \ldots, X_k\}$ of the vertex set of G into k independent sets of G (empty sets are allowed). If x is a vertex of G with $x \in X_i$, then we also write $c(x) = i$ and say that x is colored with color i. The *chromatic number* $\chi(G)$ of G is the smallest number k such that G has a k-coloring. If $\chi(G) = k$, then we briefly say that G is *k-chromatic*. A graph G is called *k-critical* if G is k-chromatic but every proper subgraph of G has a $(k-1)$-coloring. The notation $N_G(X)$ is used for the set of vertices of G adjacent to some vertex of X where $X \subseteq V(G)$. Furthermore, $G[X]$ denotes the subgraph of G induced by X. A k-coloring $c = \{X_1, \ldots, X_k\}$ of a graph G is called a *strong k-coloring* if $N_G(X_i)$ is independent for each color-class $X_i, 1 \leq i \leq k$. Clearly, every graph that has a strong k-coloring must be triangle-free.

We shall give a positive answer to the above question by constructing k-chromatic graphs G_k and H_k (for every integer $k \geq 2$). The graphs G_k are defined non-recursively, they have some resemblance to Kneser graphs. Let $k \geq 2$ be an integer. We let $[k]$ denote the set $\{1, \ldots, k\}$. The graph G_k is defined as the graph whose vertices are the pairs (i, A) that satisfy $i \in [k]$, $A \subseteq [k]$, $i \notin A$ and $A \neq \emptyset$, and whose edges are those tuples $(i, A)(j, B)$ that satisfy $i \in B, j \in A$ and $A \cap B = \emptyset$. Then G_k has $k(2^{k-1} - 1)$ vertices. The smallest of these graphs are $G_2 = K_2$ and $G_3 = C_9$. Figure 1 shows the graph G_4.

Let $\mathbf{S}_k$ denote the class of all graphs that have a strong k-coloring. We prove that G_k is homomorphism universal in $\mathbf{S}_k$, that is, $G \in \mathbf{S}_k$ if and only if G has a homomorphism to G_k (Theorem 2.1). We also show that G_k is k-chromatic (Theorem 2.2) and even k-critical (Theorem 2.3) for all $k \geq 2$. These results give the somewhat surprising corollary that G_k is the unique smallest k-chromatic graph in $\mathbf{S}_k$ (Corollary 2.1).

[1]The problem in its original form asks only if the chromatic number of $G[N_G(X_i)]$ can be made strictly less thank $k - 1$. We will consider and solve only the strongest possible version of this question, i.e., whether this chromatic number can be one, which was suggested by B. Toft.

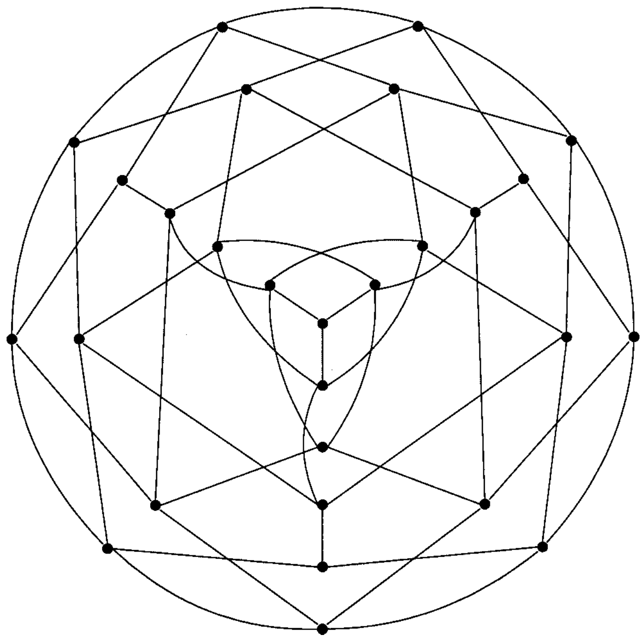

FIGURE 1. The graph G_4.

The graphs H_k will be defined recursively, in fact they are obtained by a generalised Mycielski construction. Let G be a graph and let $r \geq 1$ be an integer. We construct a new graph denoted by $M_r(G) = M_r(G, p_1, \ldots, p_r)$ as follows. For $1 \leq i \leq r$, let p_i denote a bijection from the vertex set $V(G)$ to a set X_i, where p_1 is the identity map of $X_1 = V(G)$ and the sets $X_1, \ldots, X_r$ are pairwise disjoint. For $1 \leq i \leq r-1$, let

$$E_i = \{p_i(x)p_{i+1}(y) \,|\, xy \in E(G)\},$$

further, let

$$E_r = \{p_r(x)z \,|\, x \in V(G)\},$$

where z is an additional vertex. Then

$$V(M_r(G)) = \bigcup_{i=1}^{r} X_i \cup \{z\} \quad \text{and} \quad E(M_r(G)) = E(G) \cup \bigcup_{i=1}^{r} E_i.$$

Clearly, if $H = M_r(G)$, then $H[X_1] = G$ and $H - E(G)$ as well as $H - X_1$ are bipartite. The graph $M_1(G)$ is the complete join of G and K_1. For the special case $r = 2$, this construction was invented in 1955 by Mycielski [7] in order to generate a sequence of triangle-free k-chromatic graphs for $k \geq 2$. In 1968, Schäuble [11], see also [6, problem 9.18], proved that $\chi(M_2(G)) = \chi(G) + 1$ and, moreover, that $M_2(G)$ remains critical provided that G is critical.

In 1985, Tuza and Rödl [15] observed that the graph $M_r(K_k)$ is $(k+1)$-critical for all $r \geq 1$; thus they obtained infinitely many $(k+1)$-critical graphs ($k \geq 2$) which can be made bipartite by the deletion of only $\binom{k}{2}$ edges. They also proved in [15] that this bound is best possible. Note that $M_r(K_2) = C_{2r+1}$.

Clearly, $\chi(M_r(G)) \leq \chi(G) + 1$ holds for all $r \geq 1$. Equality holds for $r = 1, 2$ but not in general. For every integer $k \geq 4$, let F_k denote the complete join of the complete graph on $k-4$ vertices with the square of the circuit C_7. Then it is proved in [14] that $\chi(F_k) = \chi(M_3(F_k)) = k$. However, if we repeatedly apply the generalised Mycielski construction to an odd circuit, then in each step the chromatic number increases by one. For $k \geq 3$, let $\mathbf{M}_k$ denote the class of graphs defined recursively as follows:

(1) $\mathbf{M}_3$ consists of all odd circuits, and, for $k \geq 3$,
(2) $\mathbf{M}_{k+1} = \{M_r(G) \mid G \in \mathbf{M}_k \text{ and } r \geq 1\}$.

The following result is due to Stiebitz [13], see also [10].

Theorem 1.1. *Every graph $G \in \mathbf{M}_k$ with $k \geq 3$ is k-chromatic.*

In [13], this theorem is used to generate for every $k \geq 4$ an infinite sequence of k-critical graphs without short odd circuits. Since the proof in [13] is not easily available, it is given in a subsequent section.

The graphs H_k are special members of $\mathbf{M}_k$, defined recursively as $H_{k+1} = M_4(H_k)$ starting with $H_2 = K_2$. Observe that $H_3 = G_3 = C_9$, however H_4 and G_4 are distinct (the former has 37, the latter has 28 vertices). On one hand, we use H_k to prove that G_k is k-chromatic by showing that H_k has a homomorphism to G_k (Theorem 2.2). On the other hand, H_k is a special case ($\ell = 1$) of the graphs H_k^ℓ for which we prove the following property (a generalisation of the Harvey and Murty property).

H_k^ℓ is a k-chromatic graph with a k-coloring $c = \{X_1, \ldots, X_k\}$ such that $N^j_{H_k^l}(X_i)$ is independent for each $i \in \{1, \ldots, k\}$, and for each $j \in \{1, \ldots, \ell\}$, where $N^j_G(X)$ is the set of vertices of G with distance j from X (Theorem 4.1).

Notice that the bound ℓ on j is essential in the generalisation. Without it, only bipartite graphs satisfy this property.

We conclude the Introduction with the following problem. We could prove that G_k is k-chromatic via a homomorphism from H_k which is proved to be k-chromatic by Lovász's theorem [5]. Is it possible to show that G_k is k-chromatic using a more direct proof? Like Bárány's proof for the Kneser graphs [1].

2. PROPERTIES OF G_K

Theorem 2.1. *Let $k \geq 2$ be an integer. A graph G has a strong k-coloring if and only if there exists a homomorphism of G into G_k.*

Proof. For the "if" direction, assume that $\varphi: V(G) \to V(G_k)$ is a homomorphism of G into G_k. For $i \in [k]$, let $Y_i = \{(i, A) \mid \emptyset \neq A \subseteq [k] \backslash \{i\}\}$. Clearly, Y_i is an independent set in G_k. Furthermore, $N_{G_k}(Y_i) = \{(j, B) | i \in B \subseteq [k] \backslash \{j\}\}$ implying that $N_{G_k}(Y_i)$ is also an independent set in G_k. Consequently, $c' = \{Y_1, \ldots, Y_k\}$ is a strong k-coloring of G_k. Since φ is a homomorphism of G into G_k, this implies that the partition $c = \{X_1, \ldots, X_k\}$ with $X_i = \{v \in V(G) | \varphi(v) \in Y_i\}$ for $i \in [k]$ is a strong k-coloring of G.

For the "only if" direction, assume that $c = \{X_1, \ldots, X_k\}$ is a strong k-coloring of G. We will show that the mapping φ defined as follows is a homomorphism from G to G_k. For every $i \in [k]$ and every $x \in X_i$, let

$$\varphi(x) = \begin{cases} (i, \{j \in [k] \mid x \text{ has a neighbor in } X_j\}) & \text{if } d_G(x) > 0, \text{ and} \\ (i, [k] \backslash \{i\}) & \text{if } d_G(x) = 0. \end{cases}$$

Since no $v \in X_i$ has a neighbor in X_i, φ is clearly a mapping from $V(G)$ to $V(G_k)$.

Let uv be any edge of G, say with $u \in X_i$ and $v \in X_j$. Then $i \neq j$, and φ maps u and v to vertices (i, A) and (j, B) of G_k respectively, with $j \in A$ and $i \in B$. For every $\ell \in [k]$, the fact that c is a strong k-coloring of G implies that at most one of u and v has a neighbor in X_ℓ, so $A \cap B = \emptyset$ follows, by the definition of φ. Hence (i, A) and (j, B) are adjacent in G_k, which proves that φ is a homomorphism. ∎

Theorem 2.1 implies that G_k has a strong k-coloring. Consequently, $\chi(G_k) \leq k$. In order to prove that G_k is a k-critical graph, we first prove the following result.

Proposition 2.1. *There is a homomorphism from $M_4(G_{k-1})$ to G_k for $k \geq 3$.*

Proof. Let $H = M_4(G_{k-1}, p_1, p_2, p_3, p_4)$ and $X = V(G_{k-1})$. For $j = 1, 2, 3, 4$, denote by $x^j = p_j(x)$ the jth copy of the vertex $x \in X$ and, let $X_j = \{x^j \mid x \in X\}$. Then

$$V(H) = X_1 \cup X_2 \cup X_3 \cup X_4 \cup \{z\}$$

and

$$E(H) = E \cup E_1 \cup E_2 \cup E_3 \cup E_4,$$

where

$$E = \{x^1 y^1 \mid xy \in E(G_{k-1})\}, \ E_4 = \{x^4 z \mid x \in X\},$$

and, for $j = 1, 2, 3$,

$$E_j = \{x^j y^{j+1} \mid xy \in E(G_{k-1})\}.$$

For a vertex $x = (i, A) \in X = V(G_{k-1})$, we have $i \in [k-1], A \subseteq [k-1], A \neq \emptyset$, and $i \notin A$. Then, let

$$\varphi(x^j) = \begin{cases} (i, A) & \text{if } \ j = 1, \\ (i, A \cup \{k\}) & \text{if } \ j = 2, \\ (k, A) & \text{if } \ j = 3, \\ (i, \{k\}) & \text{if } \ j = 4. \end{cases}$$

Moreover, let

$$\varphi(z) = (k, \{1, \ldots, k-1\}).$$

Then, clearly, φ is a mapping from $V(H)$ to $V(G_k)$ and it is straightforward to check that if uv is an edge of H, then $\varphi(u)\varphi(v)$ is an edge of G_k. Thus, φ is a homomorphism from $H = M_4(G_{k-1})$ to G_k. ■

Theorem 2.2. *The graph G_k is k-chromatic for every $k \geq 2$.*

Proof. We prove by induction on k that there is a homomorphism from H_k to G_k. For $k = 2$ (in fact for $k = 3$ too), H_k and G_k are isomorphic. Assuming that there is a homomorphism from H_{k-1} to G_{k-1}, one can obviously extend it to a homomorphism of $M_4(H_{k-1}) = H_k$ to $M_4(G_{k-1})$. But Proposition 2.1 ensures that this can be continued to G_k, which results in the required homomorphism. Since $H_k \in \mathbf{M}_k$, we then infer from Theorem 1.1 that $\chi(G_k) \geq \chi(H_k) = k$. Thus, by Theorem 2.1, $\chi(G_k) = k$. ■

Theorem 2.3. *The graph G_k is k-critical for every $k \geq 2$.*

Proof. By Theorem 2.2, we only have to show that every proper subgraph of G_k is colorable with fewer than k colors. To do this, it is sufficient to show that $G_k - e$ has a $(k-1)$-coloring for all edges e of G_k. We prove this by induction on $k \geq 2$. The bottom case is trivial, with $G_2 = K_2$.

Now, assume that $k \geq 3$. Let xy be an arbitrary edge of G_k. Then $x = (i', A')$ and $y = (j', B')$ where $i', j' \in [k], A', B' \subseteq [k], A' \neq \emptyset, B' \neq \emptyset, i' \notin A'$ and $j' \notin B'$. Since xy is an edge of G_k, we have $i' \in B', j' \in A'$ and $A' \cap B' = \emptyset$. By symmetry and since $k \geq 3$, we may assume that $k \neq i', j'$ and $k \notin A'$. Then we partition the vertex set of G_k into four classes, namely

$$\begin{aligned} X_1 &= \{(i, A) \,|\, i \in [k-1], A \subseteq [k-1], A \neq \emptyset, i \notin A\}, \\ X_2 &= \{(i, A \cup \{k\}) \,|\, (i, A) \in X_1\}, \\ X_3 &= \{(k, A) \,|\, A \subseteq [k-1]\}, \\ X_4 &= \{(i, \{k\}) \,|\, i \in [k-1]\}, \end{aligned}$$

and the additional vertex $z = (k, [k-1])$. Clearly, $x \in X_1$ and $y \in X_1 \cup X_2$. Furthermore, $G_k[X_1] = G_{k-1}$ and if e is an edge of G_k, then e joins two vertices of X_1, or a vertex of X_i with a vertex of X_{i+1} for some $i \in \{1,2,3\}$, or a vertex of X_4 with the vertex z. In particular, X_2, X_3 and X_4 are independent sets in G_k.

If $y \in X_1$, then xy is an edge of $G_{k-1} = G_k[X_1]$, and, by induction, there exists a $(k-2)$-coloring of $G_{k-1} - xy$. This coloring may be extended to a $(k-1)$-coloring of $G_k - xy$ by coloring all vertices of $X_2 \cup X_4$ with the new color $k-1$, and coloring all vertices of $X_3 \cup \{z\}$ with an old color, say color 1.

If $y = (j', B') \in X_2$, then $u = (j', B' - \{k\})$ is a vertex of X_1 that is also adjacent to x, and, by induction, there exists a $(k-2)$-coloring of $G_{k-1} - xu$. This coloring may be extended to a $(k-1)$-coloring of $G_k - xy$ as follows. First, we give each vertex $(i, A \cup \{k\}) \in X_2$ the color of its corresponding vertex $(i, A) \in X_1$. Then we recolor u by the new color $k-1$ and use this color to color all vertices of $X_3 \cup \{z\}$. Finally, we color all vertices of X_4 with an old color.

Therefore, in both cases, we obtain a $(k-1)$-coloring of $G_k - xy$ and the proof is finished. ■

This section is concluded by stating an immediate corollary of Theorems 2.1 and 2.3.

Corollary 2.1. *Assume that G is a graph with the minimum number of vertices among the k-chromatic graphs having a strong k-coloring. Then G is isomorphic to G_k.*

3. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is based on a result of Lovász. We need some concepts from algebraic topology, see [2] or [12] for the details.

An (*abstract*) *simplicial complex* $\mathcal{K} = (V, \Delta)$ is a set $V = V(\mathcal{K})$ (the *vertex set* of $\mathcal{K}$) together with a set $\Delta = \Delta(\mathcal{K})$ of non-empty finite subsets of V (called *simplices* or *faces* of $\mathcal{K}$) such that $\emptyset \neq \sigma \subseteq \tau \in \Delta$ implies $\sigma \in \Delta$. Usually, $V = \bigcup_{\sigma \in \Delta} \sigma$. If $V(\mathcal{K})$ is a finite set, then we briefly say that $\mathcal{K}$ is *finite*.

Let $S^d = \{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\}$ and $B^d = \{x \in \mathbb{R}^d \mid \|x\| \leq 1\}$ denote the *d-dimensional sphere* and *d-dimensional ball*, respectively. A topological space $\mathcal{T}$ is *m-connected* for $m \geq 0$ if every continuous map $f{:}S^d \longrightarrow \mathcal{T}$ for $0 \leq d \leq m$ has a continuous extension over B^{d+1}.

For any graph G, let $\mathcal{N}(G)$ be the simplicial complex, called the *neighborhood complex* of G, whose vertex set is $V(G)$ and whose simplices are those sets of vertices which have a common neighbor in G, i.e., $\sigma \in \Delta(\mathcal{N}(G))$ iff $\emptyset \neq \sigma \subseteq N_G(x)$ for some $x \in V(G)$. In 1978, Lovász [5] proved the following remarkable result.

Theorem 3.1. *For any graph G, if $\mathcal{N}(G)$ is a $(k-3)$-connected topological space where $k \geq 3$, then $\chi(G) \geq k$.*

Let $\mathcal{K}$ be a finite simplicial complex. The *cone* over $\mathcal{K}$, denoted by $C(\mathcal{K})$, is the complex with $V(C(\mathcal{K})) = V(\mathcal{K}) \cup \{a\}$ and $\Delta(C(\mathcal{K})) = \Delta(\mathcal{K}) \cup \{\sigma \cup \{a\} \mid \sigma \in \Delta(\mathcal{K}) \cup \{\emptyset\}$ for an $a \notin V(\mathcal{K})$. The *suspension* of $\mathcal{K}$, denoted by $S(\mathcal{K})$, is the complex with $V(S(\mathcal{K})) = V(\mathcal{K}) \cup \{a, b\}$ and $\Delta(S(\mathcal{K})) = \Delta(\mathcal{K}) \cup \{\sigma \cup \{x\} \mid \sigma \in \Delta(\mathcal{K}) \cup \{\emptyset\}$ and $x \in \{a, b\}\}$, with $a, b \notin V(\mathcal{K})$ and $a \neq b$.

Furthermore, for $r \geq 1$, we construct a new complex denoted by $M_r(\mathcal{K}) = M_r(\mathcal{K}, q_1, \ldots, q_r)$ as follows. For $1 \leq i \leq r$, let q_i denote a bijection from the set $V(\mathcal{K})$ to a set Y_i, where the sets $Y_1, \ldots, Y_r$ are pairwise disjoint. For $1 \leq i \leq r-1$, let

$$\Delta_i = \{\sigma \mid \emptyset \neq \sigma \subseteq q_i(\tau) \cup q_{i+1}(\tau) \ \ \& \ \ \tau \in \Delta(\mathcal{K})\},$$

further, let

$$\Delta_0 = \{\sigma \mid \emptyset \neq \sigma \subseteq Y_1\},$$

and,

$$\Delta_r = \{\sigma \mid \emptyset \neq \sigma \subseteq q_r(\tau) \cup \{z\} \ \ \& \ \ \tau \in \Delta(\mathcal{K})\},$$

where z is a new vertex. Then

$$V(M_r(\mathcal{K})) = \bigcup_{i=1}^{r} Y_i \cup \{z\} \quad \text{and} \quad \Delta(M_r(\mathcal{K})) = \bigcup_{i=0}^{r} \Delta_i.$$

Note that, for $1 \leq i \leq r$, the complex $\mathcal{K}_i = (Y_i, \{q_i(\tau) \mid \tau \in \Delta(\mathcal{K})\})$ is an isomorphic copy of $\mathcal{K}$. Clearly, $(Y_r \cup \{z\}, \Delta_r) = C(\mathcal{K}_r)$ and (Y_1, Δ_0) is a full simplex and, therefore, a contractible space. Furthermore, for $1 \leq i \leq r-1$, the complex $(Y_i \cup Y_{i+1}, \Delta_i)$ is homotopy equivalent to $\mathcal{K} \times I$ where I is the unit interval. Then the complex $(\bigcup_{i=1}^{r} Y_i, \bigcup_{i=1}^{r-1} \Delta_i)$ is homotopy equivalent to $\mathcal{K} \times [0, r-1]$. Therefore, the complex $M_1(\mathcal{K})$ is homotopy equivalent to a space obtained from the cone over $\mathcal{K}$ by identifying the base of the cone and, for $r \geq 2$ the complex $M_r(\mathcal{K})$ is homotopy equivalent to a space that is the quotient with respect to $\mathcal{K}$ of a full simplex (Y_1, Δ_0), a homotopy $\mathcal{K} \times [0, r-1]$ and a cone $C(\mathcal{K})$. Consequently, we have the following result.

Lemma 3.1. *For every finite simplicial complex $\mathcal{K}$ and every $r \geq 1$, the complex $M_r(\mathcal{K})$ is homotopy equivalent to the suspension $S(\mathcal{K})$.*

Lemma 3.2. *For every graph G and every $r \geq 1$, $\mathcal{N}(M_r(G)) = M_r(\mathcal{N}(G))$.*

Proof. Let $H = M_r(G, p_1, \ldots, p_r)$ and $\mathcal{K} = \mathcal{N}(G)$. If $r = 2s$, then it is not difficult to check that $\mathcal{N}(H) = M_r(\mathcal{K}, q_1, \ldots, q_r)$ with $q_i = p_{r-2i+2}$ and $q_{s+i} = p_{2i-1}$ for $1 \leq i \leq s$. Similarly, if $r = 2s+1$, then $\mathcal{N}(H) = M_r(\mathcal{K}, q_1, \ldots, q_r)$ with $q_i = p_{r-2i+2}$ for $1 \leq i \leq s+1$ and $q_{s+i+1} = p_{2i}$ for $1 \leq i \leq s$. ∎

Proof of Theorem 1.1. First, we claim that for every integer $k \geq 3$ and every graph $G \in \mathbf{M}_k$, the topological space $\mathcal{N}(G)$ is homotopy equivalent to the $(k-2)$-dimensional sphere S^{k-2} which is a $(k-3)$-connected space.

We prove this claim by induction on k. For $k = 3$ the claim is evident, since every graph in $\mathbf{M}_3$ is an odd circuit. Now, assume that $H \in \mathbf{M}_{k+1}$ where $k \geq 3$. Then $H = M_r(G)$ for some graph $G \in \mathbf{M}_k$ and some integer $r \geq 1$. From the induction hypothesis, it follows that $\mathcal{N}(G)$ is homotopy equivalent to the sphere S^{k-2}. By Lemmas 3.1 and 3.2, $\mathcal{N}(H)$ is homotopy equivalent to the suspension of $\mathcal{N}(G)$ and, therefore, to the sphere $S^{k-1} = S(S^{k-2})$. This proves the claim.

Now, we conclude from Theorem 3.1 that $\chi(G) \geq k$ for every graph $G \in \mathbf{M}_k$. On the other hand, $\chi(G) \leq k$ for every graph $G \in \mathbf{M}_k$. Thus Theorem 1.1 is proved. ∎

In [8] as well as in [14], a purely combinatorial proof for Theorem 1.1 in case of $k = 4$ is given. However, such a proof is not known for $k \geq 5$.

Lemma 3.3. *Let G be a k-critical graph $(k \geq 2)$ and let $H = M_r(G)$ for some integer $r \geq 1$. Then H is $(k+1)$-critical provided that $\chi(H) = k + 1$.*

Proof. Assume that $H = M_r(G, p_1, \ldots, p_r)$ and $X_i = p_i(V(G))$ for $i = 1, \ldots, r$. Note that $V(H) = X_1 \cup \cdots \cup X_r \cup \{z\}$, $X_1 = V(G)$ and $G = H[X_1]$. For the proof of Lemma 3.3 it is sufficient to show that $H - e$ has a k-coloring for all edges e of H. We distinguish three cases.

Case 1. $e \in E(G)$. Then there is a $(k-1)$-coloring of $G - e$ and, since $H - X_1$ is bipartite, this coloring can be extended to some k-coloring of $H - e$.

Case 2. $e = p_i(a)p_{i+1}(b)$ with $1 \leq i \leq r - 1$. Then $e' = ab \in E(G)$ and, therefore, there is a $(k-1)$-coloring f of $G - e'$. Since G is not $(k-1)$-colorable, $f(a) = f(b)$. Now, we define a map g as follows. For $1 \leq j \leq i$ and $x \in V(G)$, let

$$g(p_j(x)) = \begin{cases} f(x) & \text{if } x \neq b, \\ k & \text{if } x = b. \end{cases}$$

Furthermore, let $g(p_{i+1}(x)) = f(x)$ for all $x \in V(G)$. Then it is easy to check that g is a k-coloring of the subgraph $H[X_1 \cup \cdots \cup X_{i+1}] - e$. Clearly, g can be extended to some k-coloring of $H - e$.

Case 3. $e = p_r(a)z$. Since G is k-critical, there is a k-coloring f of $G = H[X_1]$ such that $f(x) = k$ only for $x = a$. Now, color each vertex $p_i(x)$ with $f(x)$ and color the vertex z with the new color k. This results in a k-coloring of $H - e$. ∎

As a consequence of Lemma 3.3 and Theorem 1.1, we obtain the following result.

Theorem 3.2. *Every graph $G \in \mathbf{M}_k$ with $k \geq 3$ is k-critical.*

4. GRAPHS H_K^ℓ

For fixed integer $\ell \geq 1$, we recursively define a sequence of graphs H_k^ℓ for $k = 2, 3, \dots$ as follows. For $k = 2$, we let $H_2^\ell = K_2$ (independently of ℓ). For $k \geq 3$, we define $H_k^\ell = M_{3\ell+1}(H_{k-1}^\ell)$.

For $X \subseteq V(G)$ and an integer $j \geq 0$, we will let $N_G^j(X)$ denote the jth distance class from X in G, i.e., the set of those $v \in V(G)$ such that G contains some path of length j from v to a vertex in X, but no shorter path of this type. Note that $N_G^0(X) = X$.

Let k, ℓ be integers with $k \geq 2$ and $\ell \geq 1$. Let $\mathbf{S}_k^\ell$ denote the class of all graphs G that have a k-coloring $c = \{X_1, \dots, X_k\}$ such that $N_G^j(X_i)$ is independent in G for each $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, \ell\}$.

Theorem 4.1. *For every $k \geq 2$ and $\ell \geq 1$, the graph H_k^ℓ is k-chromatic and belongs to $\mathbf{S}_k^\ell$.*

Proof. It follows directly from the construction and using Theorem 1.1 that H_k^ℓ is k-chromatic. Note that $H_2^\ell = K_2$ and $H_3^\ell = C_{6\ell+3}$.

It remains to prove the existence of a k-coloring $c = \{X_1, X_2, \dots, X_k\}$ of H_k^ℓ with the required property. For this, we use induction on $k \geq 2$. Since $H_2^\ell = K_2$, the statement is trivial when $k = 2$.

Now, assume that $k \geq 3$. Then $H_k^\ell = M_{3\ell+1}(H_{k-1}^\ell)$ and, by induction, there exists a $(k-1)$-coloring $c = \{X_1', X_2', \dots, X_{k-1}'\}$ of H_{k-1}^ℓ such that $N_{H_{k-1}^\ell}^j(X_i')$ is independent in H_{k-1}^ℓ for all i,j with $1 \leq i \leq k-1$ and $1 \leq j \leq \ell$.

From the construction of $M_{3\ell+1}(H_{k-1}^\ell)$, there is, in particular, a partition $V^0 \cup V^1 \cup \dots V^{3\ell-1} \cup V^{3\ell} \cup \{z\}$ of $V(H_k^\ell)$ with $V^0 = V(H_{k-1}^\ell)$, and where $V^t = \{v^t = p_t(v) | v \in V(H_{k-1}^\ell)\}$ are independent sets, $t = 1, 2, \dots, 3\ell$ (and p_t is the bijection of V^0 to V^t). We define for $i = 1, 2, \dots, k-1$,

$$X_i = \{v^t \mid v \in X_i', \ t = 0, 1, \dots, \ell-1, \ell, \ell+2, \ell+4, \dots, 3\ell-2, 3\ell\}.$$

That each of these sets is independent in H_k^ℓ follows from the construction of $M_r(G)$, where we use that each X_i' is independent in H_{k-1}^ℓ. The final class X_k is given by

$$X_k = V^{\ell+1} \cup V^{\ell+3} \cup \dots \cup V^{3\ell-3} \cup V^{3\ell-1} \cup \{z\}.$$

This is again an independent set in H_k^ℓ, since it is, by the construction of $M_r(G)$, a union of independent sets no two of which are joined by an edge. Thus $c = \{X_1, X_2, \dots, X_k\}$ indeed defines a k-coloring of H_k^ℓ.

For $j = 1, \dots, \ell$, we need to show that the jth distance class from each X_i is an independent set in H_k^ℓ, $i = 1, 2, \dots, k$. Suppose that this fails for some $i \leq k-1$. Then there are vertices $u^s, v^t \in X_i$ and a walk W of odd length at most $2\ell+1$ from u^s to v^t in H_k^ℓ. Then W does not use the vertex z, since every walk in H_k^ℓ using z and having length at most $2\ell+1$ must use vertices from X_k alternately, but W has odd length and no end in X_k. So by projecting W down to H^ℓ_{k-1}, there exists a walk W' of odd length at most $2\ell+1$ in H^ℓ_{k-1} between vertices $u, v \in X'_i$, which is a contradiction to the induction hypothesis (possibly for a different color class than X'_i in H^ℓ_{k-1} in the case when W' contains an odd circuit).

Finally, consider a walk in H^ℓ_k which starts and ends in X_k. If it uses any vertex from V^0, then its length is at least $2\ell+2$, since the distance from V^0 to X_k is $\ell+1$ in H^ℓ_k. Since $H^\ell_k - V_0$ is bipartite, this implies that the length of the walk is at least $2\ell+3$ if it is odd. Hence the jth distance classes from X_k are independent, $j = 1, \ldots, \ell$. ■

As pointed out by one of the referees, there is also a homomorphism universal graph G^ℓ_k in $\mathbf{S}^\ell_k$. The vertices of G^ℓ_k are the couples of the form (i, A) where $i \in [k]$ and A is a subset of $[k]^\ell$ consisting of strings $(x_1, \ldots, x_\ell)$ where $x_1 \neq i$ and $x_j \neq x_{j-1}$ for $j = 2, \ldots, \ell$. The edges of G^ℓ_k are the pairs $(i, A), (j, B)$ such that for every

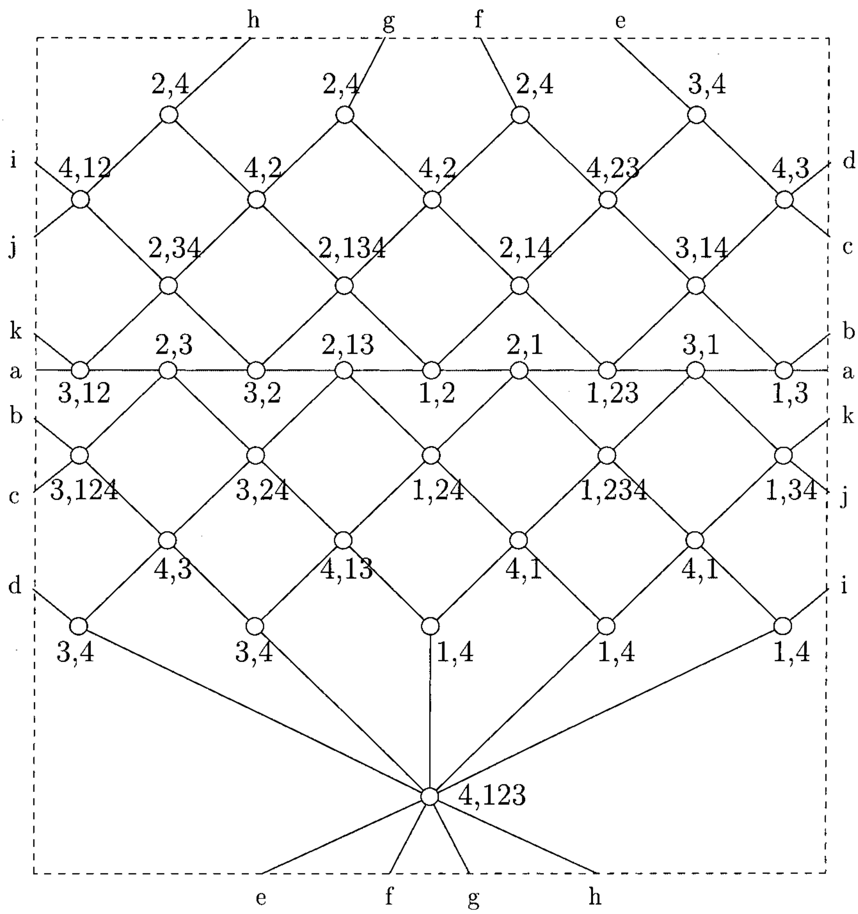

FIGURE 2. An embedding of H_4 quadrangulating the real projective plane.

$(x_1,\ldots,x_\ell)\in A,\ (y_1,\ldots,y_\ell)\in B$ we have $(j,y_1,\ldots,y_{\ell-1})\in A,\ (i,x_1,\ldots,x_{\ell-1})\in B$ and $x_t\neq y_t$ for $t=1,\ldots,\ell$.

5. CONCLUDING REMARKS

We will indicate a separate argument for proving that H_4 and G_4 are 4-chromatic graphs. It is straightforward to show that H_4 may be drawn as a quadrangulation of the real projective plane, an embedding such that every face is bounded by a 4-circuit, see Figure 2. We observe that the subgraph $H_3=G_3$ of H_4 corresponds precisely to the horizontally drawn edges of Figure 2. Each vertex of H_4 is marked in this figure with the name of the vertex of G_4 to which it is mapped by the homomorphism φ from H_4 to G_4, the existence of which is proved in Theorem 2.2. Since φ is a composition of homomorphisms, each of which maps two opposite corners of a face to a single vertex and fixes all the remaining

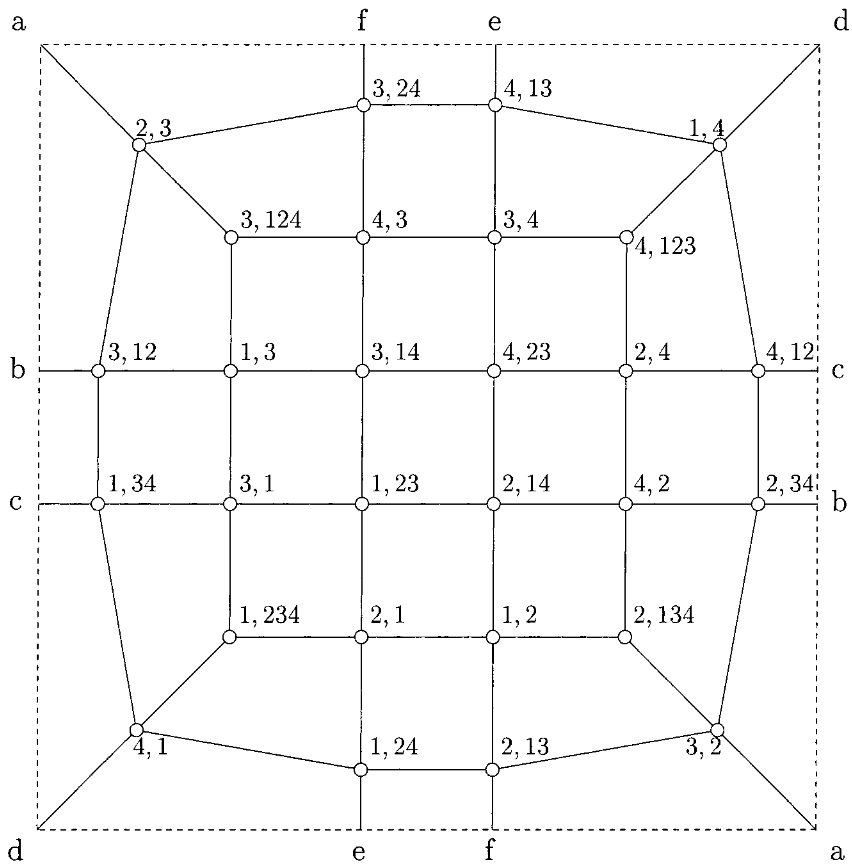

FIGURE 3. An embedding of G_4 quadrangulating the real projective plane.

vertices, it follows that also G_4 may be drawn as a quadrangulation of the real projective plane. Figure 3 shows such a drawing of G_4 in a more symmetric way than obtained directly from Figure 2 by applying φ. Arguments due to Gallai [3], later developed in a more explicit form by Youngs [16], show that any graph which quadrangulates the projective plane is either bipartite or 4-chromatic. Hence G_4 is 4-chromatic, since G_4 contains an odd circuit, namely $G_3 = C_9$. By the same reason also, H_4 is 4-chromatic. This argument can be also used to show that all graphs of the form $M_r(C_{2s+1})$ $(r, s \geq 2)$ are 4-chromatic.

For every $r \geq 1$, the graph $L_r = M_r(C_{2r+1})$ is a 4-critical graph with $n = r(2r+1)+1$ vertices such that every odd circuit of L_r has length $\geq 2r+1 \geq \sqrt{2n}$. This fact was first proved by Stiebitz [13] (see also [10]) and later rediscovered, independently, by Ngoc and Tuza [8] and by Youngs [16]. That all graphs of the family $M_r(C_{2s+1})$ are 4-chromatic was also proved by Payan [9] and, recently, by Tardif [14].

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