# Edge List Multicoloring Trees: An Extension of Hall's Theorem

**Mathew Cropper,[1] András Gyárfás,[2] and Jenő Lehel[3]†***

[1]DEPARTMENT OF MATHEMATICS AND STATISTICS
EASTERN KENTUCKY UNIVERSITY
RICHMOND, KENTUCKY 40475

[2]COMPUTER AND AUTOMATION RESEARCH
INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST XI. KENDE U. 13–17, 1111 HUNGARY

[3]DEPARTMENT OF MATHEMATICAL SCIENCES
THE UNIVERSITY OF MEMPHIS
MEMPHIS, TENNESSEE 38152
E-mail: jlehel@memphis.edu

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**Abstract:** We prove a necessary and sufficient condition for the existence of edge list multicoloring of trees. The result extends the Halmos–Vaughan generalization of Hall's theorem on the existence of distinct representatives of sets.  J Graph Theory 42: 246–255, 2003

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*Correspondence to: Jenő Lehel, Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152. E-mail: jlehel@memphis.edu

## 1. INTRODUCTION

A *multicoloring* of the edges of a graph $G$ is a mapping from its edge set into the power set of an underlying set of colors, the set $\mathcal{N}$ of natural numbers. A multicoloring $\varphi: E(G) \longrightarrow \mathcal{P}(\mathcal{N})$ is *proper* provided $\varphi(e) \cap \varphi(f) = \emptyset$, for any two incident edges $e, f \in E(G)$. A *list multicoloring* is defined by specifying a list $L(e)$ of available colors and the cardinality $w(e)$ of color sets at each edge $e \in E(G)$. Given the list and weight assignment $L: E(G) \longrightarrow \mathcal{P}(\mathcal{N})$ and $w: E(G) \longrightarrow \mathcal{N}$ an $(L, w)$ coloring of $G$ is a proper multicoloring $\varphi$ such that $\varphi(e) \subseteq L(e)$, and $|\varphi(e)| = w(e)$, for all $e \in E(G)$.

To state a necessary condition for the existence of a list multicoloring we introduce a few notions. Let $(G, L, w)$ be a graph $G$ with list and weight assignment $L$ and $w$, and let $H \subseteq G$ (i.e., $H$ is an induced subgraph of $G$). For $c \in \mathcal{N}$, the *support* $H_c$ is the subgraph of $H$ with edge set $\{e \in E(H) \mid c \in L(e)\}$. In any $(L, w)$ coloring $\varphi$ of $G$, and for each color $c \in \mathcal{N}$, the set $\{e \mid c \in \varphi(e)\}$ is a matching in $G_c$. Let $\nu(H)$ denote the *matching* number of $H$, defined as the maximum number of pairwise independent edges of $H$. Then we have

$$\sum_{e \in E(H)} |\varphi(e)| = |\{(e, c) \mid c \in \varphi(e)\}| = \sum_{c \in \mathcal{N}} |\{e \mid c \in \varphi(e)\}| \leq \sum_{c \in \mathcal{N}} \nu(H_c).$$

Thus for the existence of an $(L, w)$ coloring of $G$, we obtain the following necessary condition called Generalized Hall's condition: $(G, L, w)$ satisfies *GH* if and only if

$$\sum_{e \in E(H)} w(e) \leq \sum_{c \in \mathcal{N}} \nu(H_c) \qquad (*)$$

holds for every subgraph $H \subseteq G$.

We prove here that GH is also sufficient for the existence of an $(L, w)$ coloring of $G$, provided $G$ is a tree. If $G$ is a star, and $w$ is the unit weight assignment, $w(e) \equiv 1$, then any $(L, w)$ coloring of $G$ corresponds to a family of distinct representatives for the color lists. The first result on this now well studied concept in transversal theory was Hall's classical matching theorem [7]. The idea of extending list colorings from a star to any graph $G$ and from unit weight assignment to any weight $w$ has been introduced in Hilton et al. [4]. The context there was vertex multicolorings, which, of course, includes edge multicolorings by the device of taking line graphs.

If $G$ is a star, then inequality $(*)$ becomes $\sum_{e \in E(H)} w(e) \leq |\bigcup_{e \in E(H)} L(w)|$. By a celebrated result, apparently first noticed by Halmos and Vaughan [2], this inequality, required for all $H \subseteq G$, is sufficient to guarantee a family of disjoint representatives with prescribed size $w(e)$, for each set $L(e)$. This family of disjoint representatives is clearly an $(L, w)$ coloring of the star $G$. A result of Hilton and Johnson Jr. in [3] implies that GH is also sufficient for any tree $G$ with unit weights. A problem proposed in Cropper et al. [1] promptly translates into the conjecture that the claim above is true for trees with arbitrary weight assignment $w$ as well. The conjecture has been supported by resolving several particular cases, paths, double stars, and small trees in Cropper et al. [1]. We prove this for arbitrary trees in Theorem 2.1 below. This solution also settles the full conjecture in Cropper et al. [1] for vertex list multicolorings, in the affirmative (see Theorem 3.2).

The problem of extending partial edge colorings of multigraphs lead Marcotte and Seymour in [6] to a result equivalent to Theorem 2.1 here. Their proof uses polyhedral combinatorics and totally unimodular matrices, meanwhile the proof by induction we shall give in Section 2 is elementary.

In Section 3, we shall determine how far the class of graphs extends when one requires the Generalized Hall's condition to be sufficient for the existence of list multicolorings. First we characterize the class of graphs such that Theorem 2.1 remains true. The answer in Theorem 3.1 is a family of multigraphs rather close to trees.

The vertex list coloring version of Theorem 2.1 is given in Theorem 3.2. An extension of Theorem 3.2 was proposed recently by Johnson Jr. and Wantland in [5]. Their question establishes a clever link between the results concerning the case of unit weights in Hilton and Johnson Jr. [3] and that of arbitrary weights in Theorem 3.2. Our last result, Theorem 3.3, answers this question as a corollary of Theorem 3.2.

## 2. TREE LIST–COLORING THEOREM

**Theorem 2.1.** *Let $T$ be a tree with list and weight assignment $L{:}E(T) \longrightarrow \mathcal{P}(\mathcal{N})$ and $w{:}E(T) \longrightarrow \mathcal{N}$. If $\sum_{e\in E(H)} w(e) \leq \sum_{c\in\mathcal{N}} \nu(H_c)$ holds for every subtree $H \subseteq T$, then $T$ has an $(L,w)$ coloring.*

***Proof.*** The induction proof is based on several reductions of $(T,L,w)$. To keep track of the changes on both sides of inequality $(*)$ in GH, we introduce the following terms. A subtree $H \subseteq T$ is $(L,w)$ *violating* (or $(L,w)$ *tight*) provided $(H,L,w)$ fails $(*)$ (or satisfies $(*)$ with equality). The main property of tightness is: if $H \subseteq T$ is an $(L,w)$ tight subtree, then for every $(L,w)$ coloring $\varphi$ of $H$ and for every color $\gamma$, the color class $\{f \in E(H) | \gamma \in \varphi(f)\}$ is a maximum matching of $H_\gamma$.

We use reductions to define critical counterexamples as follows. Assume that $(T,L,w)$ is a counterexample to the theorem, i.e., it satisfies GH and no $(L,w)$ coloring of $T$ exists. We call $(T,L,w)$ a *critical counterexample* if

(i) the number of edges of $T$ is as small as possible,
(ii) $\sum_{e\in E(T)} w(e)$ is minimum among counterexamples satisfying (i), and
(iii) $\sum_{e\in E(T)} |L(e)|$ is minimum among counterexamples satisfying (i) and (ii).

One of the most used properties of a critical counterexample $(T, L, w)$ is that every proper subtree of $T$ has an $(L, w)$ coloring.

For $e \in E(T)$ and $c \in L(e)$, a subtree $H \subseteq T$ is called *$c$ tight with respect to $e$* if $H$ is $(L, w)$ tight and every maximum matching of $H_c$ contains $e$. A crucial property of a critical counterexample $(T, L, w)$ is the following claim. For every $e \in E(T)$ and $c \in L(e)$, there exists a subtree $H^c(e)$ which is $c$ tight with respect to $e$. Furthermore, if $H^c(e)$ is a proper subtree of $T$, then every $(L, w)$ coloring $\varphi$ of $H^c(e)$ satisfies $c \in \varphi(e)$.

To prove the first part of the claim, define $L'(e) = L(e) \setminus \{c\}$ and let $L'(f) = L(f)$, for $f \neq e$. Because $(T, L', w)$ is not a counterexample, there is an $(L', w)$ violating subtree $H \subseteq T$. Of course $H$ is not $(L, w)$ violating which implies that $H$ is $(L, w)$ tight. Also $e$ must be in every maximum matching of $H_c$. Thus the existence of the required $H^c(e)$ follows. The second part of the claim is immediate from the main property of tight subtrees.

Starting with a critical counterexample $(T, L, w)$, the proof of the theorem proceeds in four claims. Notice first that $T$ must have at least two edges. Consider a longest path $(u, v, x, \ldots)$ in $T$. Let $e$ be the edge $vx$, and let $S$ be the set of all edges of $T$ incident with $v$ and different from $e$.

**Claim 1.** $w(f) = 1$, for every $f \in S$.

Assume that $w(f) > 1$. Define $w'(f) = w(f) - 1$ and $w'(g) = w(g)$, for all $g \in E(T - f)$. Let $\varphi$ be an $(L, w')$ coloring of $T$ and select $c \in \varphi(f)$. Define $L'$ by removing $c$ from the lists of all edges of $T$ incident with $v$, i.e., those in $S \cup \{e\}$. We verify that $(T, L', w')$ satisfies GH. Obviously, $\varphi$ is an $(L', w')$ coloring of the tree $T - f$. Consequently, inequality $(*)$ holds for every subtree of $T - f$ with $L'$ and $w'$. For any subtree $H \subseteq T$ containing $f$ inequality $(*)$ must hold as well, otherwise the $(L', w')$ violating $H$ would be also $(L, w)$ violating subtree of $T$. Because $(T, L', w')$ satisfies GH and $(T, L, w)$ is critical, $T$ has an $(L', w')$ coloring. Adding $c$ to the color set of $f$ results in an $(L, w)$ coloring of $T$, contradiction.

**Claim 2.** $L(f) \cap L(g) = \emptyset$, for any two distinct edges $f, g \in S$.

Assume that $f_0 \in S$, and $f_1, f_2 \in S \cup \{e\}$ are edges such that $f_0 \notin \{f_1, f_2\}$ and let $c \in L(f_0) \cap L(f_1) \cap L(f_2)$. For $i = 1, 2$, let $H^c(f_i)$ be a $c$ tight subtree with respect to $f_i$. Note that each $H^c(f_i)$ is a proper subtree of $T$, because $f_0 \notin H^c(f_i)$. Consider an $(L, w)$ coloring $\varphi$ of $T - f_0$. Clearly, $\varphi$ is an $(L, w)$ coloring for both $H^c(f_1)$ and $H^c(f_2)$. Because $\varphi$ uses color $c$ at most once at vertex $v$, there is only one edge among $f_1$ and $f_2$ which could be colored with $c$. Therefore $f_1 = f_2$ and $|(S \cup \{e\}) \cap T_c| \leq 2$ follows.

Now assume that $f_0$ and $f_1$ are two distinct edges of $S \cap E(T_c)$ for some $c$. Define $(T - f_0,\, L', w')$ by modifying $L$ and $w$ only for $f_1$ as follows: $L'(f_1) = L(f_0) \cup L(f_1)$ and $w'(f_1) = w(f_0) + w(f_1) = 2$. By the previous paragraph, among the $L'$ lists of $(S \cup \{e\}) \setminus \{f_0\}$, the only list containing $c$ is $L'(f_1)$. Because $(T - f_0,\, L', w')$ satisfies GH and $T$ is critical, $T - f_0$ has an $(L', w')$ coloring $\varphi$. Let $c_0 \in \varphi(f_1)$. If $c_0 \notin L(f_0)$, then color $f_0$ with $c$ and $f_1$ with $c_0$. If

$c_0 \in L(f_0)$, then color $f_0$ with $c_0$ and $f_1$ with $c$. In either case keep $\varphi$ unchanged. Thus we would obtain an $(L,w)$ coloring of $T$, contradiction.

**Claim 3.** $\bigcup_{f\in S} L(f) \subseteq L(e)$, and $\sum_{f\in S}(|L(f)|-1) < w(e)$.

Notice that $f \in S$ and $c \in L(f)$ imply that $c \in L(e)$. Otherwise, by Claims 1 and 2, an $(L,w)$ coloring of $T-f$ is extendable to an $(L,w)$ coloring of $T$. To verify the second part of the claim, select an edge $f \in S$, and for every $c \in L(f)$, consider a subtree $H^c(e) \subset T$ which is $c$ tight with respect to $e$. Note that $f$ is not an edge of $H^c(e)$.

Assume that for some $g \in S\setminus\{f\}$, $g$ is not an edge of $\cup\{H^c(e) \mid c \in L(f)\}$. Clearly any $(L,w)$ coloring of $T-g$ is an $(L,w)$ coloring of the graph $\cup\{H^c(e) \mid c \in L(f)\}+f$. Because $c$ tightness of $H^c(e)$ with respect to $e$ implies that $c \in \varphi(e)$ holds, for every $c \in L(f)$, we obtain that $\varphi(f)\cap\varphi(e) \neq \emptyset$. This contradiction shows that $g$ must be an edge of $\cup\{H^c(e) \mid c \in L(f)\}$, for every $g \in S\setminus\{f\}$.

Now consider an $(L,w)$ coloring $\varphi$ of $T-f$. Because each edge $g \in S\setminus\{f\}$ belongs to a tight subtree $H^c(e)$, for some $c \in L(f)$, every color $\gamma \in L(g)$ is used in a maximum matching of $H^c_\gamma(e)$. Hence either $\gamma \in \varphi(g)$ or $\gamma \in \varphi(e)$. By Claim 1, $|\varphi(g)| = w(g) = 1$, hence at least $|L(g)|-1$ colors of $L(g)$ are used in $\varphi(e)$, for every $g \in S\setminus\{f\}$. For distinct edges $g \neq h$ in $S\setminus\{f\}$, we have $L(g)\cap L(h) = \emptyset$, by Claim 2. Furthermore, $L(f) \subset \varphi(e)$ is a set of further $|L(f)|$ colors in $\varphi(e)$ and different from those mentioned earlier. Thus $w(e) > \sum_{f\in S}(|L(f)|-1)$ follows.

To prepare the last step of the proof, we will modify $(T,L,w)$ and create an associated tree $(T^*,L^*,w^*)$. Let $S=\{f_1,\ldots,f_p\}$, define $\ell_i = |L(f_i)|$, $i=1,\ldots,p$, and set $s=\sum_{i=1}^p \ell_i$. By Claims 2 and 3, $L(e) = (\cup_{i=1}^p L(f_i)) \cup K$, where $L(f_1),\ldots,L(f_p)$, and $K$ are pairwise disjoint sets. By Claim 3, $w(e)=s+t$, for some integer $t>-p$. Define $T^*$ as a tree obtained from $T$ by removing all edges $f_1,\ldots,f_p$ incident with $v$ and adding new pendant edges $e_1,\ldots,e_p$ incident with the other vertex $x$ of $e$. Define $L^*(e)=K, w^*(e)=p+t, L^*(e_i)=L(f_i)\cup K, w^*(e_i)=\ell_i-1$, for $1\le i\le p$, and $L^*(g)=L(g), w^*(g)=w(g)$ for $g\in T-(S\cup\{e\})$.

**Claim 4.** $(T^*,L^*,w^*)$ satisfies GH.

Let $H^*$ be an $(L^*,w^*)$ violating subtree of $T^*$, i.e., inequality $(*)$ is not true:

$$\sum_{f\in E(H^*)} w^*(f) > \sum_{c\in\mathcal{N}} \nu(H^*_c)\ .$$

Observe that $H^*$ must contain at least one edge among $e_1,\ldots,e_p$ and $e$, thus every color from $K$ contributes to the right hand side of the inequality above. Note also that, by Claim 3, $w^*(e)=p+t>0$. Thus we obtain that the subtree $H^*+e$ is $(L^*,w^*)$ violating as well. So we assume that $e\in E(H^*)$. We also assume w.l.o.g. that, for some $q\ge 0$, $e_i\in E(H^*)$ if and only if $1\le i\le q$.

Removing edges $e_1, \ldots, e_q$ from $H^*$ and adding the edges $f_{q+1}, \ldots, f_p$ we get a subtree $H \subseteq T$. Concerning inequality $(*)$ for $(H, L, w)$ we have

$$\sum_{f\in E(H)} w(f) - \sum_{f\in E(H^*)} w^*(f) = \left[w(e) + \sum_{i=q+1}^{p} w(f_i)\right] - \left[w^*(e) + \sum_{i=1}^{q} w^*(e_i)\right]$$

$$= (s+t+p-q) - \left(p+t+\sum_{i=1}^{q}(l_i-1)\right)$$

$$= s - \sum_{i=1}^{q} l_i = \sum_{i=q+1}^{p} l_i.$$

To see that

$$\sum_{c\in\mathcal{N}} \nu(H_c) - \sum_{c\in\mathcal{N}} \nu(H_c^*) \le \sum_{i=q+1}^{p} l_i,$$

note that when passing from $H^*$ to $H$ the matching number cannot increase for any color of the set $(\cup_{i=1}^{q} L(e_i)) \cup K$. Thus we obtain that

$$\sum_{f\in E(H)} w(f) > \sum_{c\in\mathcal{N}} \nu(H_c).$$

Hence $H$ is $(L, w)$ violating, contradiction.

Now the proof of the theorem is concluded as follows. Note that

$$\sum_{i=1}^{p} w^*(e_i) + w^*(e) = \sum_{i=1}^{p}(\ell_i - 1) + p + t = s + t\ ,$$

and

$$w(e) + \sum_{i=1}^{p} w(f_i) = s + t + p.$$

We have $w^*(g) = w(g)$, for every other edge $g$, hence

$$\sum_{g\in E(T^*)} w^*(g) < \sum_{g\in E(T)} w(g)\ .$$

Because $(T, L, w)$ is a critical counterexample, Claim 4 ensures that $T^*$ has an $(L^*, w^*)$ coloring $\varphi^*$. Define $\varphi(e) = (\cup_{i=1}^{p} \varphi^*(e_i)) \cup \varphi^*(e)$, and for $i = 1, \ldots, p$, let $\varphi(f_i) = c_i$, where $c_i \in L(f_i) \setminus \varphi^*(e_i)$. Note that $|\varphi(e)| = \sum_{i=1}^{p}(\ell_i - 1) + p + t = s + t = w(e)$, and $L(f_i) \setminus \varphi^*(e_i) \ne \emptyset$, because $|L(f_i)| = \ell_i$ and $|\varphi^*(e_i)| = \ell_i - 1$. Setting $\varphi(g) = \varphi^*(g)$ for every $g \in T - (S \cup \{e\})$, we obtain that $\varphi$ is an $(L, w)$ coloring of $T$, contradiction. ■

## 3. EXTENSIONS

We start with remarks on the proof of Theorem 2.1. Claims 1 and 2 of the proof are merely reproductions of Lemmas 5 and 7 in Cropper et al. [1]. It is worth noting that the proof does not use Hall's theorem on distinct representatives of sets. Actually, when $T$ is a star with $w \equiv 1$, Claim 2 does prove it. When $T$ is a star with arbitrary $w$, Claims 1 and 2 together (applied twice for different $e$) prove the Halmos, Vaughan generalization of Hall's theorem in [2]. It is natural to ask whether Theorem 2.1 extends from trees to more general multigraphs. The possibilities are rather restricted as the counterexamples below indicate (see [3]).

Figure 1 shows $(G, L, w)$ satisfying GH and admitting no $(L, w)$ coloring for $G$. In each case $w \equiv 1$ is the unit weight, the graphs indicated together with the corresponding list assignments in Figure 1 are: (a) triangle with a pendant edge, (b) even cycles, (c) odd cycles, and (d) 4 path with a double edge in the middle.

Assume that $G$ is connected and for every list and weight assignment $L$ and $w$, GH is sufficient for $G$ to posess an $(L, w)$ coloring. One learns from Figure 1a that if $G$ has a triangle, then it has no more vertices. If $G$ has more than three vertices, then Figures 1b,c show that $G$ has no cycles of length more than two. Further-

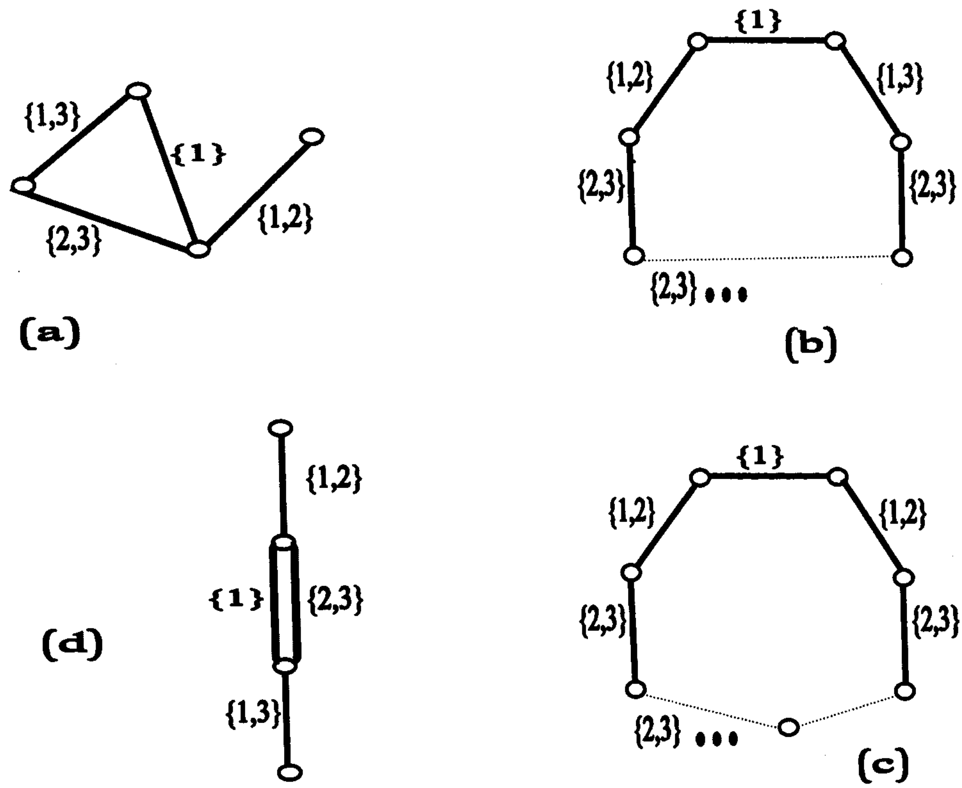


FIGURE 1. Graphs with GH and no list coloring.

more, as Figure 1d indicates, $G$ is obtained from a tree by replacing any number of pendant edges with multiple edges.

Since triangle components can be replaced with stars, and multiple pendant edges of a tree can be "pulled apart" into simple endstars, we obtain easily the following slight extension of the tree list–coloring theorem.

**Theorem 3.1.** *The Generalized Hall's condition is sufficient (and necessary) for the existence of an $(L, w)$ coloring of G, for every list and weight assignment L and w, if and only if each connected component of G is a triangle with possible multiple edges or a tree with possible multiple pendant edges.* ■

Next we show how the tree list-coloring theorem solves the list vertex multicoloring problem investigated in Cropper et al. [1] and Hilton et al. [4]. Given a simple graph $G$ with vertex list and weight assignment $L: V(G) \longrightarrow \mathcal{P}(\mathcal{N})$ and $w: V(G) \longrightarrow \mathcal{N}$ a vertex $(L, w)$ coloring of $G$ is a vertex multicoloring $\varphi$ such that $\varphi(v) \subseteq L(v)$, $|\varphi(v)| = w(v)$, for all $v \in V(G)$, and $\varphi(v) \cap \varphi(u) = \emptyset$ provided $uv \in E(G)$. The vertex version of the Generalized Hall's condition uses the inequality

$$\sum_{v \in V(H)} w(v) \leq \sum_{c \in \mathcal{N}} \alpha(H_c) \qquad (**)$$

where $\alpha$ is the independence number. A graph $G$ with vertex list and weight assignment $L$ and $w$ satisfies Vertex GH (vertex version of the Generalized Hall's condition) if and only if inequality $(**)$ holds for every induced subgraph $H \subseteq G$.

Clearly, any proper edge list multicoloring of a graph $G$ is a proper vertex list multicoloring of its line graph. Furthermore, the Generalized Hall's condition on $G$ translates into Vertex GH on its line graph. This means that the line graph of each counterexample in Figure 1 is a counterexample in the vertex multicoloring version. These line graphs are the induced cycles of length at least four and the diamond (clique $K_4$ minus one edge). A counterexample which is not derived as a line graph is the claw itself (see [1]). Consider a 3 star (claw) with leaves $a, b, c$ and center $d$, and define $L(a) = \{1, 2\}$, $L(b) = \{1, 3\}$, $L(c) = \{2, 3\}$, $L(d) = \{1, 2, 3\}$, $w(a) = w(b) = w(c) = 1$, and $w(d) = 2$. The claw with these list and weight assignment satisfies Vertex GH and has no vertex list multicoloring.

Because every block (i.e., maximal 2 connected component or a single edge) of a diamond free chordal graph is a clique, as an immediate corollary of Theorem 2.1 we obtain the following theorem.

**Theorem 3.2.** *The vertex version of the Generalized Hall's condition is sufficient (and necessary) for the existence of a vertex $(L, w)$ coloring of G, for every L and w, if and only if G is claw free and every block of G is a clique.* ■

As mentioned in the Introduction, Theorem 3.2 was motivated by the following result.

**Theorem** (Hilton and Johnson Jr. [3]). *The vertex version of the Generalized Hall's condition is sufficient (and necessary) for the existence of a vertex* $(L, w \equiv 1)$ *coloring of G for every L if and only if every block of G is a clique.* ■

We conclude the paper with the common extension of this result and Theorem 3.2 as conjectured very recently by Johnson Jr. and Wantland in [5].

**Theorem 3.3.** *The vertex version of the Generalized Hall's condition is sufficient (and necessary) for the existence of a vertex* $(L, w)$ *coloring of G, for every L and w such that* $w(x) = 1$ *whenever x is the center vertex of a claw in G, if and only if every block of G is a clique.*

***Proof.*** To show the sufficiency of Vertex GH let $(G, L, w)$ be a minimal counterexample complying the required properties and having no $(L, w)$ coloring. We claim that $G$ is claw free. Then we are done, because no claw free counterexample exists by Theorem 3.2.

Suppose on the contrary that $x$ is a center vertex of some claw in $G$. By minimality, $G$ is connected, and because each block of $G$ is a clique, $x$ is a cut vertex of $G$. Hence $x$ belongs to $t \geq 3$ blocks of $G$. Consider the $t$ connected components of $G - x$ and extend each component with $x$. Let $G_i, i = 1, \ldots, t$, be these subgraphs of $G$. Let $L_i$ and $w_i$ be the restriction of $L$ and $w$ to $V(G_i)$, respectively.

Because $G$ has no $(L, w)$ coloring, for each color $c \in L(x)$, there exists at least one graph $G_k$ ($k$ depends on $c$) such that $G_k$ has no $(L_k, w_k)$ coloring with $x$ colored $c$. Without the loss of generality, we will assume that $L(x)$ is partitioned as $\Gamma_1 \cup \cdots \cup \Gamma_s$ with some $s$ $(1 \leq s \leq t)$, where $c \in \Gamma_k$ implies that $G_k$ has no $(L_k, w_k)$ coloring with $x$ colored $c$, for every $c \in L(x)$ and $1 \leq k \leq s$.

For $k = 1, \ldots, s$, define $L'_k(x) = \Gamma_k$ and $L'_k(v) = L_k(v)$ provided $v \neq x$. Observe that, by definition, $G_k$ has no $(L'_k, w_k)$ coloring. Hence, by the minimality of $(G, L, w)$, there is a subgraph $H'_k$ in $(G_k, L'_k, w_k)$ violating $(**)$:

$$\sum_{v \in V(H'_k)} w_k(v) > \sum_{c \in \mathcal{N}} \alpha((H'_k)_c) \quad k = 1, \ldots, s.$$

Let $H$ be the union of the subgraphs $H'_k$, $k = 1, \ldots, s$. Notice that $x \in V(H'_k)$, and $w_k(x) = 1$, for every $k = 1, \ldots, s$. Because $L'_1 \cup \cdots \cup L'_s$ is a partition of $L(x)$, we obtain

$$\sum_{c \in \mathcal{N}} \alpha(H_c) = \sum_{k=1}^{s} \sum_{c \in \mathcal{N}} \alpha((H'_k)_c)$$

$$\leq \sum_{k=1}^{s} \left[ \left( \sum_{v \in V(H'_k)} w_k(v) \right) - 1 \right] = \sum_{v \in V(H) \setminus \{x\}} w(v) < \sum_{v \in V(H)} w(v).$$

Thus the subgraph $H$ in $(G, L, w)$ violates $(**)$, a contradiction. This implies that $G$ has no claw, and concludes the proof. ■

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