COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **22** (4) (2002) 479–496

TRANSITIVE EDGE COLORING OF GRAPHS AND DIMENSION OF LATTICES

ANDRÁS GYÁRFÁS*

Received May 2, 2001

We explore properties of edge colorings of graphs defined by set intersections. An edge coloring of a graph G with vertex set $V=\{1,2,\dots,n\}$ is called *transitive* if one can associate sets $F_1,F_2,\dots,F_n$ to vertices of G so that for any two edges $ij,kl\in E(G)$, the color of ij and kl is the same if and only if $F_i\cap F_j=F_k\cap F_l$. The term *transitive* refers to a natural partial order on the color set of these colorings.

We prove a canonical Ramsey type result for transitive colorings of complete graphs which is equivalent to a stronger form of a conjecture of A. Sali on hypergraphs. This—through the reduction of Sali—shows that the dimension of n-element lattices is $o(n)$ as conjectured by Füredi and Kahn.

The proof relies on concepts and results which seem to have independent interest. One of them is a generalization of the induced matching lemma of Ruzsa and Szemerédi for transitive colorings.

1. Introduction

A. Sali proposed a conjecture ([15], 1996, implicitly also in [14], 1986) on hypergraphs and proved that it would imply $\dim(L)=o(|L|)$ for finite lattices L, suggested by Füredi and Kahn ([7], 1988) as a property distinguishing lattices from posets, where $\dim(P)=\frac{1}{2}|P|$ is possible. The monograph of W. T. Trotter ([17]) is recommended for combinatorial problems related to dimension of partially ordered sets, in particular Section 10.6, which gives a brief survey on the lattice dimension problem.

Mathematics Subject Classification (2000): 05C15

* Research supported in part by OTKA Grant T029074.

0209–9683/102/$6.00

In this paper we prove Sali's conjecture in a stronger form (Theorem 1.3) which, through the reduction of Sali ([15]), proves the conjecture of Füredi and Kahn. The actual lower bound obtained for $\lambda(n)$, the minimum size of a lattice of dimension n, is $\Omega(n\log(\log^{**}(n)))$ (Theorem 7.1), where $\log^{**}$ is the functional inverse of the iterated tower function. This is hopelessly far from $\lambda(n)=\Theta(n^2)$, the "reckless" conjecture of Füredi and Kahn ([7]).

The proof of Theorem 1.3 relies on the notion of *transitive colorings,* formulated in Definition 1.4 as a special type of edge coloring of graphs where vertices are sets and the edge colors are defined by set-intersections. In fact, Theorem 1.3 is, in disguise, a canonical Ramsey type theorem for transitive colorings of complete graphs (Theorem 2.2). The proof of Theorem 2.2 is in Section 6, based on results summarized in the next paragraph.

Section 3 shows that transitive colorings can be equivalently defined through the relation *span* on the color set (Propositions 3.1, 3.2). This allows to define the *color poset* of a transitive coloring which becomes a powerful tool in subsequent sections. Section 4 gives bounds on sizes of subgraphs colored by chains and antichains of the color poset of transitive colorings with bounded degree (Lemmas 4.1, 4.4). Section 5 connects these lemmas through Dilworth theorem and leads to Theorem 5.1, a generalization of the induced matching lemma of Ruzsa and Szemerédi ([13], 1978, Lemma 1) for transitive colorings. For a discussion of this lemma (with its role in the celebrated (6,3)-theorem), see the survey paper of Komlós and Simonovits ([9], 1996).

To formulate Sali's conjecture, we need a few definitions.

Definition 1.1. (Sali, [15], 1996) A set system $F_1,F_2,\dots,F_t$ is a t-configuration if there exist sets A_i, B_i such that $F_i = A_i \cup B_i$, $1 \le i \le t$ where the sets A_i form a chain $A_1 \subseteq \dots \subseteq A_t$ and the sets B_i are pairwise disjoint and also disjoint from A_t.

It is worth noting that t-configurations generalize the well studied and widely applied notion of Delta systems (obtained when $A_1=A_2=\dots=A_t$) and chains (when all B_i-s are empty). It seems that this generalization is unexplored, I know only an unpublished result of Füredi ([5]) which determines the minimum number of edges of a k-uniform hypergraph ensuring a t-configuration.

Assume that $\mathcal{F}=\{F_1,F_2,\dots,F_n\}$ is a system of (not necessarily distinct) finite sets on an arbitrary ground set. Let $\mathcal{M}(\mathcal{F})$ denote the set of intersections of members of $\mathcal{F}$,

$$\mathcal{M}(\mathcal{F}) = \{\cap_{i\in I} F_i : I \subseteq [n]\}.$$

Conjecture 1.2. (Sali, [15], 1996, implicitly also in [14], 1986) For every fixed $c>0$ and fixed positive integer t, there exists $n_0=n_0(c,t)$ such that the following is true for $n\geq n_0$: every system $\mathcal{F}=\{F_1,F_2,\dots,F_n\}$ which satisfies $|\mathcal{M}(\mathcal{F})|\leq cn$ contains a t-configuration.

We prove Conjecture 1.2 in a stronger form, replacing $\mathcal{M}(\mathcal{F})$ by $\mathcal{M}_2(\mathcal{F})=\{F_i\cap F_j : 1\leq i<j\leq n\}$.

Theorem 1.3. *For every fixed $c>0$ and fixed positive integer t there exists $n_0=n_0(c,t)$ such that the following is true for $n\geq n_0$: every system $\mathcal{F}=\{F_1,F_2,\dots,F_n\}$ which satisfies $|\mathcal{M}_2(\mathcal{F})|\leq cn$ contains a t-configuration.*

It is not difficult to show that a cn bound on $\mathcal{M}_2$ does not necessarily imply a $c'n$ bound on $\mathcal{M}$ (not even on triple intersections). Therefore Theorem 1.3 is really stronger than the claim of Conjecture 1.2.

Our approach to prove Theorem 1.3 is to transform it to a problem on edge colorings of graphs through the following definition.

Definition 1.4. An edge coloring of a graph G with vertex set $V=\{1,2,\dots,n\}$ is called transitive if one can associate sets $F_1,F_2,\dots,F_n$ to vertices of G so that for any two edges $ij,kl\in E(G)$, the color of ij and kl is the same if and only if $F_i\cap F_j=F_k\cap F_l$.

We also need a simple but important observation about t-configurations.

Proposition 1.5. *A system of t sets is a t-configuration if and only if its members can be indexed as $F_1,F_2,\dots,F_t$ so that for every $1\leq i<j<k\leq t$, $F_i\cap F_j=F_i\cap F_k$.*

Proof. One direction is clear, Definition 1.1 provides the required indexing. Assume that we have sets F_i, $1\leq i\leq t$, indexed according to Proposition 1.5 on a ground set F_{t+1}. Define $A_i=F_i\cap F_{i+1}$, $B_i=F_i\setminus A_i$ for $1\leq i\leq t$.

Assume that $1\leq i<j\leq t$. Then

$$A_i=A_i\cap A_i=(F_i\cap F_j)\cap(F_i\cap F_{j+1})=F_i\cap(F_j\cap F_{j+1})=F_i\cap A_j\subseteq A_j$$

thus $A_i\subseteq A_j$. Moreover

$$\begin{aligned}B_i\cap B_j&=(F_i\setminus A_i)\cap(F_j\setminus A_j)=(F_i\cap F_j)\setminus(A_i\cup A_j)\\&=(F_i\cap F_{i+1})\setminus(A_i\cup A_j)=A_i\setminus(A_i\cup A_j)=\emptyset\end{aligned}$$

thus $B_i\cap B_j=\emptyset$. Finally, for every $1\leq i\leq t$,

$$B_i\cap A_t=(F_i\setminus A_i)\cap A_t\subseteq F_i\cap A_t=F_i\cap(F_t\cap F_{t+1})=F_i\cap F_t=F_i\cap F_{i+1}=A_i$$

and

$$B_i\cap A_t\subseteq F_i\setminus A_i\subseteq\overline{A}_i$$

showing that $B_i \cap A_t = \emptyset$. Thus the sets $F_i = A_i \cup B_i$, $1 \le i \le t$, satisfy Definition 1.1. ∎

After these preparations Theorem 1.3 is reformulated as an equivalent Ramsey-type theorem.

2. Canonical Ramsey theorem for transitive colorings

Definition 2.1. An edge coloring of a complete graph K_t is *canonical* if and only if its vertices can be labelled by $1,2,\dots,t$ so that for every $1 \le i < j < k \le t$, the color of edge ij is the same as the color of edge ik.

Theorem 2.2. *For every fixed $c > 0$ and fixed positive integer t, there exists $n_0 = n_0(c,t)$ such that the following is true for $n \ge n_0$: in every transitive coloring of the edges of K_n with at most cn colors there is a canonically colored K_t.*

Now we show that Theorem 2.2 and Theorem 1.3 are the same, formulated in a different language. View a set system $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ satisfying the assumptions of Theorem 1.3 as the vertex set of a complete graph with color $F_i \cap F_j$ on the edge F_iF_j. This coloring is transitive by Definition 1.4, uses at most cn colors therefore Theorem 2.2 implies that there is a canonical K_t in this coloring. By Definition 2.1 and by the nontrivial part of Proposition 1.5, the sets assigned to the vertices of K_t form a t-configuration.

The reverse argument is similar. If we have a transitive coloring on the edges of K_n with at most cn colors then Definition 1.4 ensures a representation of this coloring by $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ with at most cn intersections of the pairs. Theorem 1.3 gives a t-configuration which, by Definition 2.1 and by the trivial part of Proposition 1.5, corresponds to a canonically colored K_t.

2.1. Remarks

1. The term canonical is borrowed from canonical Ramsey theory initiated by Erdős and Rado ([4]). The reader familiar with the subject may notice that Definition 2.1 incorporates *mono, min, max* colorings (see [8]) but allows other colorings as well.

2. Observe that for $t = 1,2$, K_t is trivially canonical (in the language of section 1, t sets always form a t-configuration), consequently Theorem 2.2 is trivial. However, the case $t = 3$ is already difficult, a canonically colored K_3 is a triangle with two or three edges colored with the same color. In this case

Theorem 2.2 says that if n is large (in terms of c) then K_n has no proper transitive coloring with cn colors (where proper means that each color class is a set of pairwise disjoint edges).

3. The threshold function $n_0(c,t)$ from our proof of Theorem 2.2 is very large, for fixed c, it is essentially a t-times iterated tower function because it comes from iterations of Szemerédi's regularity lemma.

4. Theorem 2.2 easily provides a K_t which is canonical in a stronger sense, having one of three types: *mono, min, max,* eliminating the fourth type, *distinct* of the Erdős–Rado theorem ([4], for best bounds see [11]). (The terms mono, min, max, distinct are from [8]).

Corollary 2.3. *For every fixed $c>0$ and fixed positive integer t, there exists $n_0 = n_0(c,t)$ such that the following is true for $n \geq n_0$: in every transitive coloring of the edges of K_n with at most cn colors, there is a K_t on which the coloring is one of the three types: mono, min, max.*

Proof sketch. Suppose that the vertex set of K_n is $[n]$. Theorem 2.2 provides a $K_{(t-1)^4+1}$ with vertex set A which is canonical in the sense of Definition 2.1. The labelling provided by this definition can be considered as a permutation on A. Select a monotone increasing (1) or decreasing (2) subsequence $B \subset A$ of this permutation with $|B| = (t-1)^2 + 1$. Then select t vertices from B whose "defining colors" are all distinct (3) or all equal (4). This provides the required K_t : (1)(4) and (2)(4) give type *mono,* (1)(3) gives type *min,* (2)(3) gives type *max*. ∎

3. The color poset of transitive colorings

Here we show that transitive colorings of graphs define a natural partial order on the underlying color set (Proposition 3.1). The converse is also true and can serve as an equivalent definition of transitive colorings (Proposition 3.2).

Graphs in this paper are finite, loops and multiple edges are excluded. We frequently exclude isolated vertices by considering graphs defined by the span of their edge set. A subgraph of the graph $G=(V,E)$ is defined as (V_1,E_1), where $E_1 \subseteq E$ and V_1 is the set of vertices covered by E_1. A subgraph (V_1,E_1) of $G=(V,E)$ is an *induced subgraph* if $e \in E_1$ holds for every $e \in E$ with both endpoints in V_1. The cardinality of a largest independent set of G is denoted by $\alpha(G)$. A *star* is a set of edges incident to a vertex, the *closed neighborhood of a vertex x* is the set of vertices covered by the edges incident to x.

Assume that $G=(V,E)$ is a graph with vertex set $V=[n]$ with an edge r-coloring $\mathcal{C}$ using colors from the set $[r]$. For convenience, we shall also use the notation $|\mathcal{C}|=r$, without actually referring to the color set. Following the usual convention, colorings with less than r colors are also considered as r-colorings. We shall use the following notations.

For $V_1 \subseteq V$, $col(V_1)$ denotes the set of colors used on edges with both endpoints in V_1, with the convention that $col(V_1)=\emptyset$ if V_1 is an independent set in G. In particular, if $ij \in E(G)$ then $col(\{i,j\})=col(ij)$ is the color of the edge ij.

For $C \subseteq [r]$ let $V(C)$ denote the subset of V covered by edges of G whose color is from C. In particular, $V(\{i\})=V(i)$ is the set of vertices covered by the edges of color i. Similarly, $E(C)$ denotes the subset of edges of G whose color is from C. In particular, $E(\{i\})=E(i)$ is the set of edges of G with color i. Finally, the graph $G(C)$ is defined as $(V(C),E(C))$, in particular, $G(\{i\})=G(i)=(V(i),E(i))$ is the subgraph of G spanned by the edges of color i.

Assume that G is a graph with coloring $\mathcal{C}$ using colors from $[r]$. For $C \subseteq [r]$, $span(C)$, the span of C is defined as $col(V(C))$. (For $C=\{i\}$, $span(\{i\})$ is abbreviated as $span(i)$.) For any positive integer m, $span^m(C)$ is the m times iterated span, which is well defined since $span(C)$ maps subsets of $[r]$ to subsets of $[r]$ (in fact, it is a nondecreasing map).

The binary relation R on $[r]$ is defined as follows: for $i,j \in [r]$, iRj if and only if $j \in span^m(i)$ for some positive integer m. Note that R is a reflexive relation on every colored graph since $i \in span(i)=span^1(i)$.

Proposition 3.1. *If $\mathcal{C}$ is a transitive coloring on a graph G then the relation R is a partial order on $[r]$.*

Proof. From Definition 1.4, $\mathcal{C}$ on G can be represented by assigning a set F_i to vertex i for each $i \in [n]$ and defining $col(ij)=F_i \cap F_j$ for each edge $ij \in E(G)$. It is enough to prove Proposition 3.1 in terms of this representation because it only changes the names of vertices and colors. Thus vertices of G and the colors of the edges are considered as sets. Translating the definition of R, we have to show that $Y \in span^m(X)$ is a partial order. This is immediate from the following claim: if $Y \in span^m(X)$ for some positive integer m then $X \subseteq Y$.

To prove this claim, assume that $a \in X$. This implies that a is an element of every vertex of $V(X)$, which implies that a is an element of every color of $col(V(X))$, i.e. every color of $span(X)$. Repeating this argument, it follows that a is an element of every color of $span^2(X),\dots,span^m(X)$, in particular, a is an element of $Y \in span^m(X)$. ∎

By Proposition 3.1 we can define the *color poset* of a transitive coloring $\mathcal{C}$ as the color set $[r]$ with the partial order $iRj = i \leq_R j$. It is worth proving, although we shall not actually use it, that the converse of Proposition 3.1 is also true.

Proposition 3.2. *If $\mathcal{C}$ is a coloring on G such that $[r]$ is partially ordered under the relation $i \leq_R j = iRj$, then $\mathcal{C}$ is transitive.*

Proof. Let $C(i)$ denote the set of of colors appearing on the edges of G incident to $i \in V(G)$ (the vertex set of G is $[n]$). Let F_i be defined as the set of colors which are smaller than or equal to some color of $C(i)$ under R. Thus F_i is the *down-set* or ideal generated by $C(i)$ in the color poset.

Proposition 3.2 follows by showing that for every $ij, kl \in E(G)$, $F_i \cap F_j = F_k \cap F_l$ if and only if $col(ij) = col(kl)$.

Assume that $F_i \cap F_j = F_k \cap F_l$, $col(ij) = x, col(kl) = y$. Since $x \in F_i \cap F_j$, we get $x \in F_k \cap F_l$. Thus $span^{m_1}(x) \cap C(k) \neq \emptyset$, $span^{m_2}(x) \cap C(l) \neq \emptyset$ which implies, with $m = max(m_1, m_2)$, that $y \in span^{m+1}(x)$, i.e. $x \leq_R y$. The reverse argument gives $y \leq_R x$ which implies $x = y$.

Assume that $col(ij) = col(kl) = x$. If $F_i \cap F_j \neq F_k \cap F_l$ then, by symmetry, we can select $a \in (F_i \cap F_j) \setminus (F_k \cap F_l)$. Since $x \in F_k \cap F_l$, $x \neq a$. From $a \in F_i$, vertex i is covered by $V(span^{m_1}(a))$ for some m_1. From $a \in F_j$, vertex j is covered by $V(span^{m_2}(a))$ for some m_2. Therefore, with $m = max(m_1, m_2)$, $V(span^m(a))$ covers both i and j, which implies $col(ij) = x \in span^{m+1}(a)$, therefore $x >_R a$. However, $x = col(kl) \in C(k) \cap C(l)$ implies that $a <_R x$ is in the down-set of both $C(k)$ and $C(l)$, i.e. $a \in F_k \cap F_l$ which contradicts the choice of a. Therefore $F_i \cap F_j = F_k \cap F_l$. ∎

3.1. Simple facts and examples

Every graph has transitive colorings, one is the monochromatic coloring, when all edges are colored with the same color, here the color poset is trivial. Another is the coloring when each edge is colored with a distinct color, here the color poset is an antichain. Both definitions of transitive colorings imply immediately that if a coloring $\mathcal{C}$ is transitive on $G = (V, E)$ then $\mathcal{C}_{G_1}$, the restriction of $\mathcal{C}$ to a subgraph $G_1 = (V_1, E_1)$ of G is transitive on G_1.

The reader can easily check that among the six nonisomorphic 2-colorings of K_4, three are transitive. The colorings 1212, 1213212 of the paths P_5,P_8 are not transitive: in both cases $1 <_R 2, 2 <_R 1$ (since $1 \in span(2), 2 \in span(1)$). The coloring 131232 of the path P_7 is not transitive either, here $1 <_R 2, 2 <_R 1$ follows from a more subtle reason: $1 \in span^2(2), 2 \in span^2(1)$.

It is an easy exercise to show that a factorization of K_{2n} into $2n-1$ matchings can not be a transitive coloring. However, it is not easy to see that a partition of K_n into at most $2001n$ matchings can not be a transitive coloring if n is large enough, this follows from Theorem 2.2 with $t=3, c=2001$.

A partition of the edge set of a complete graph into complete subgraphs (finite planes, Steiner systems) is transitive, its color poset is an antichain. The converse is also obvious, in a transitive coloring of a complete graph whose color poset is an antichain, the color classes give a partition into complete subgraphs.

Canonical colorings of complete graphs are not necessarily transitive, the smallest example is a K_4 with $V=[4]$, where the edges $23, 24$ are colored with 2, the other edges with 1. Here both colors are in the span of the other color. However, a canonical *min* coloring, where $col(ij)=i$ for all $j>i$, is transitive, its color poset is a chain.

4. Chains and Antichains of Transitive colorings

Let $\Delta(G)$ denote the maximum degree of a graph G. For a graph G with coloring $\mathcal{C}$ using colors from $[r]$, set $\Delta(\mathcal{C})=max\{\Delta(G(i)):i\in[r]\}$. A coloring $\mathcal{C}$ is called *D-bounded* if $\Delta(\mathcal{C})\leq D$, i.e. each color class has maximum degree at most D. We need two lemmas to estimate the number of edges in D-bounded transitive colorings. The first (Chain lemma) gives a bound on the number of edges of chains.

Lemma 4.1. (Chain lemma) *Suppose that $C\subseteq[r]$ is a chain of the color poset in a D-bounded transitive coloring $\mathcal{C}$ of a graph G with n vertices. Then*

(1) $$\sum_{i\in C}|E(i)|\leq Dn.$$

Proof. We may assume, by renaming the colors, that $C=[p]$ (with some $p\in[r]$) and the natural order on $[p]$ agrees with the order $<_R$, i.e $1<_R 2<_R \dots <_R p$.

Claim 4.2. *Assume that $i\in[p]$, and $e\in E(i)$. Then at least one of the two vertices of e is not covered by the set $W(i)=\cup_{j>i}V(j)$.*

Proof of Claim 4.2. Assume that both endvertices of $e=xy$ are in $W(i)$, $x\in V(k)$, $y\in V(l)$ where both k and l are larger than i. Then x is incident to an edge f with $col(f)=k$ and y is incident to an edge g with $col(g)=l$. We may assume by symmetry that $k\leq_R l$. Therefore $l\in span^m(k)$ for some positive integer m from the definition of R. However, $col(e)=i$ and the endvertices of e are covered by f,g of color k,l which implies that $i\in span^{m+1}(k)$. Therefore $i>_R k$ which contradicts the assumption $i<_R k$. ∎

Claim 4.2 implies that one can define the pairwise disjoint sets $A(i)$ for $i=1,2,\dots p$ by selecting a vertex from each edge of $G(i)$ which is not covered by $W(i)=\cup_{j>i}V(j)$. Since each edge of $G(i)$ is incident to $A(i)$ and each vertex of $G(i)$ has degree at most D,

$$\sum_{i\in C}|E(i)|\le\sum_{i\in C}D|A(i)|=D\sum_{i\in C}|A(i)|\le Dn$$

which proves Lemma 4.1. ∎

Antichains of transitive colorings relate to induced subgraphs. Observe that if $C\subseteq[r]$ is an antichain in the color poset of a transitive coloring $\mathcal{C}$ of G then the graphs $\{G(i):i\in C\}$ are induced subgraphs of the graph $G(C)$. To bound $|E(C)|$ in terms of $|C|$ for D-bounded transitive colorings requires the extension of the induced matching lemma of Ruzsa and Szemerédi from matchings to bounded degree subgraphs.

Theorem 4.3. (Induced matching lemma.) *If a graph G with n vertices is the union of n induced matchings, then $|E(G)|=o(n^2)$.*

Theorem 4.3 is essentially Lemma 1 from a celebrated paper of Ruzsa and Szemerédi ([13], 1978). The formulation above is from the survey paper of Komlós and Simonovits ([9], 1996, Theorem 3.2) on Szemerédi's regularity lemma ([16]). The survey also shows the connection of Theorem 4.3 to its better known forms ((6,3)-theorem, $r_3(n)=o(n)$). The proof of the following extension of Theorem 4.3 adapts the approach of Komlós and Simonovits.

Lemma 4.4. (Antichain lemma) *Suppose that $C\subseteq[r]$ is an antichain of the color poset in a D-bounded transitive coloring $\mathcal{C}$ of a graph G with n vertices. Let $\varepsilon>0$ be arbitrary and $n\ge M(\varepsilon)(1+D\varepsilon)\varepsilon^{-3}(1-\varepsilon)^{-1}$, where $M(\varepsilon)$ is the upper bound for the number of clusters in the regularity lemma. Then*

$$\sum_{i\in C}|E(i)|\le 2.5\varepsilon n^2+\frac{D(D+1)}{2}\varepsilon n|C|.$$

Proof. The regularity lemma is applied to the graph $G(C)$ in the so called "degree form" ([9], Theorem 1.10) with density $d=2\varepsilon$, which means that all pairs of the $k\le M(\varepsilon)$ clusters $X_1,\dots,X_k$ of size $m\le\varepsilon\lceil n\rceil$ are ε-regular, each with a density either 0 or at least $d=2\varepsilon$. The graph G^* denotes the "pure" graph of $G(C)$, i.e. the subgraph of $G(C)$ defined by the edges between the clusters. This notation is extended to all color classes of $G(C)$, $G^*(i)=$

$(V^*(i), E^*(i))$ is the subgraph of $G(i)$ defined by the edges between the clusters. Set $Y(i,j) = V^*(i) \cap X_j$ for $i \in C, j \in [k]$.

Since the first term $2.5\varepsilon n^2$ in the estimate of Lemma 4.4 is the error term $\frac{(d+3\varepsilon)n^2}{2}$ (with $d = 2\varepsilon$) which comes from the application of the degree form of the regularity lemma, it is enough to prove the following claim.

Claim 4.5. *For each $i \in C$, $|E^*(i)| \leq \frac{D(D+1)}{2}\varepsilon n$.*

Proof. Assume that Claim 4.5 is violated by some $i \in C$, i.e.

$$|E^*(i)| > \frac{D(D+1)}{2}\varepsilon n. \tag{2}$$

Let J denote the index set of the clusters which have large intersection with $V^*(i)$:

$$J = \{j : |Y(i,j)| > \varepsilon |X_j| = \varepsilon m\} \tag{3}$$

and partition $V^*(i)$ into two sets, A, B, where $A = \cup_{j \in J} Y(i,j)$, $B = V^*(i) \setminus A$.

Claim 4.6. $|A| > \alpha(G^*(i))$.

Proof. Notice that from the definition of J

$$|B| \leq \sum_{j \in [k] \setminus J} \varepsilon |X_j| \leq \varepsilon n \tag{4}$$

Using (2) and (4), we get

$$|V^*(i)| \left(1 - \frac{D}{D+1}\right) = \frac{|V^*(i)|}{D+1} \geq \frac{2|E^*(i)|}{D(D+1)} > \varepsilon n \geq |B|$$

which implies

$$|A| = |V^*(i)| - |B| > |V^*(i)| \left(\frac{D}{D+1}\right). \tag{5}$$

Observation 4.7. Assume that H is a graph without isolated vertices with maximum degree at most D. Then

$$\alpha(H) \leq |V(H)| \left(\frac{D}{D+1}\right).$$

Proof. Let S be a maximum independent set in H and $T = V(H) \setminus S$. The closed neighborhoods of vertices of T cover every vertex of H and each of them has at most $D+1$ vertices. Thus

$$(|V(H)| - \alpha(H))(D+1) = |T|(D+1) \geq |V(H)|$$

which gives the required inequality. ∎

Applying Observation 4.7 to $G^*(i)$ and using (5) we get

$$\alpha(G^*(i)) \leq |V^*(i)| \left(\frac{D}{D+1}\right) < |A| \tag{6}$$

which proves Claim 4.6. ∎

Claim 4.6 implies that $G^*(i)$ has an edge with both endpoints in $A = \cup_{j\in J} Y(i,j)$, w.l.o.g. between $Y(i,1)$ and $Y(i,2)$. Therefore the density between the clusters X_1 and X_2 is not 0, consequently at least $d = 2\varepsilon$.

The proof of Claim 4.5 will be finished by showing that there is an edge of G^* between $Y(i,1)$ and $Y(i,2)$ with color $l \neq i$. Then $l \in span(i)$ which contradicts the assumption that C is an antichain.

Indeed, by (3), we can select a pair of sets, Z_1, Z_2, of size εm from $Y(i,1)$ and $Y(i,2)$, respectively (integer parts are neglected). There are at most $D\varepsilon m$ edges of color i between Z_1 and Z_2. However, the pair X_1, X_2 is ε-regular with density larger than 2ε therefore the density of the pair Z_1, Z_2 is at least ε. Therefore the number of edges of G^* between Z_1 and Z_2 is at least $\varepsilon^3 m^2$. To reach the required contradiction, we need the inequality

$$\varepsilon^3 m^2 - D\varepsilon m > 1 \tag{7}$$

which, for easier computation, is replaced by $\varepsilon^3 m - D\varepsilon > 1$. Then m is replaced with its lower bound $\frac{n-\varepsilon n}{M(\varepsilon)}$ (εn is subtracted because of the exceptional class). Solving the obtained inequality for n, we get the bound $n \geq M(\varepsilon)(1+D\varepsilon)\varepsilon^{-3}(1-\varepsilon)^{-1}$, stated as the condition of the theorem. This finishes the proof of Claim 4.5 and Lemma 4.4. ∎

5. Extension of Ruzsa–Szemerédi theorem for transitive colorings

The following result is a generalization of the induced matching lemma for transitive colorings. It gives Theorem 4.3 if $D = c = 1$ and the color poset of $\mathcal{C}$ is an antichain. Its key feature is that $\mathcal{C}$ is an *arbitrary transitive coloring* (the extension for D is already in Lemma 4.4 and c is immaterial since Theorem 4.3 can be stated with cn instead of n).

Theorem 5.1. *Suppose that $\mathcal{C}$ is a D-bounded transitive coloring with at most cn colors on a graph G with n vertices. Then $|E(G)| = o(n^2)$ for every fixed c, D.*

Proof. Assume indirectly that for some fixed c, D there exists a $\rho > 0$ and a sequence of graphs G_n with n vertices, at least ρn^2 edges with D-bounded transitive colorings using at most cn colors. Delete the edges of all color classes of G_n with less than $\frac{\rho n}{2c}$ edges. This sparser graph sequence has graphs with at least $\frac{\rho n^2}{2}$ edges and (from the restrictions of the colorings of G_n) have transitive D-bounded colorings with at most cn colors. For convenience, this sparser graph sequence is considered as G_n.

For an arbitrary $\varepsilon > 0$ which will be fixed later, consider G_n with $n \geq M(\varepsilon)(1+D\varepsilon)\varepsilon^{-3}(1-\varepsilon)^{-1}$. Let $\mathcal{C}$ denote the coloring of G_n and apply the antichain lemma (Lemma 4.4) for G_n with a maximum antichain C of the color poset of $\mathcal{C}$. We get

$$\frac{\rho n}{2c}|C| \leq \sum_{i \in C} |E(i)| \leq 2.5\varepsilon n^2 + \frac{D(D+1)}{2}\varepsilon n|C|, \tag{8}$$

implying that

$$|C| \leq \varepsilon n \left(\frac{5c}{\rho - cD(D+1)\varepsilon} \right). \tag{9}$$

By Dilworth's theorem, the color poset of $\mathcal{C}$ can be partitioned into $s = |C|$ chains, $C_1, \ldots, C_s$. From the chain lemma (Lemma 4.1),

$$|E(C_j)| = \sum_{i \in C_j} |E(i)| \leq Dn \tag{10}$$

for every $j \in [s]$. Combining (9) with (10),

$$\frac{\rho n^2}{2} \leq |E(G_n)| = \sum_{j=1}^{s} |E(C_j)| \leq |C|Dn \leq \varepsilon n^2 \frac{5cD}{\rho - cD(D+1)\varepsilon} \tag{11}$$

which implies

$$\frac{\rho^2}{10cD + \rho cD(D+1)} \leq \varepsilon. \tag{12}$$

Since ρ, c, D are all fixed, we get a contradiction by selecting an ε which is smaller than the left-hand side of (12). ∎

The following corollary is a quantitative version of Theorem 5.1 which will be applied in the proof of Theorem 2.2.

Corollary 5.2. *Let $\mathcal{G}(\rho, n)$ denote the family of graphs with n vertices, and at least ρn^2 edges. For a suitable function $f(\rho, c, n)$, which tends to infinity with n for every fixed ρ, c, the following holds: if $\mathcal{C}$ is a transitive coloring of $G \in \mathcal{G}(\rho, n)$ using at most cn colors then $\Delta(\mathcal{C}) \geq f(\rho, c, n)$, i.e. there is a monochromatic star with at least $f(\rho, c, n)$ edges. A suitable f is essentially $\log^*(n)$ if the other two parameters are fixed.*

Proof. Follow the steps of the proof of Theorem 5.1. To avoid complicated calculations, we deliberately use the word essentially for inequalities which are not (but can be) stated precisely.

Let h be the threshold function of Lemma 4.4:

$$h(\varepsilon, D) = M(\varepsilon)(1 + D\varepsilon)\varepsilon^{-3}(1 - \varepsilon)^{-1}. \tag{13}$$

Since $M(\varepsilon)$ is essentially a tower of height ε^{-5} (see [2], [9]), we can replace $h(\varepsilon, D)$ with $tower(\varepsilon^{-5})D$. Thus Lemma 4.4 can be applied if $n \geq tower(\varepsilon^{-5})D$, i.e.

$$\varepsilon \geq \left(\log^*\left(\frac{n}{D}\right)\right)^{-\frac{1}{5}}. \tag{14}$$

Let $G \in \mathcal{G}(\rho, n)$ with a transitive coloring $\mathcal{C}$ using at most cn colors. Set $D = \Delta(\mathcal{C})$. Like in the proof of Theorem 5.1, we may assume that each color class of $\mathcal{C}$ has at least $\frac{\rho n}{2c}$ edges. From this point the argument of the proof of Theorem 5.1 can be applied which leads to contradiction if an ε can be selected which is smaller than the left-hand side of (12) and satisfies (14) (to ensure that Lemma 4.4 is applicable) i.e. if

$$\frac{\rho^2}{10cD + \rho cD(D+1)} > \varepsilon \geq \left(\log^*\left(\frac{n}{D}\right)\right)^{-\frac{1}{5}}. \tag{15}$$

We conclude that no ε can satisfy (15) therefore

$$\log^*\left(\frac{n}{D}\right) \leq \left(\frac{10cD + \rho cD(D+1)}{\rho^2}\right)^5$$

which is essentially

$$\log^*(n) \leq \rho^{-10}c^5D^{10}. \tag{16}$$

Therefore

$$f(\rho, c, n) = \rho c^{-\frac{1}{2}}(\log^*(n))^{\frac{1}{10}} \leq D, \tag{17}$$

giving the claimed (essentially) $\log^*(n)$ lower bound on D. ∎

6. Proof of canonical Ramsey theorem

Here we repeat Theorem 2.2 for convenience, including its threshold function.

Theorem. *For every fixed $c > 0$ and fixed positive integer t, there exists $n_0 = n_0(c, t)$ such that the following is true for $n \geq n_0$: in every transitive coloring of the edges of K_n with at most cn colors there is a canonically colored K_t.*

A suitable $n_0(c,t)$ is a t-times iterated tower function with $8^{10}4^{5t}c^5$ in the innermost tower.

Proof. The proof is induction on t. For $t=2$, $n_0(c,2)=2$ is trivially a good definition for every c. Assuming that $n_0(c,t)$ is already defined for every c, $n_0(c,t+1)$ is defined as follows:

$$n_0(c,t+1) = \min\{n : f(1/4,c,n) > n_0(4c,t)\}. \tag{18}$$

Observe that $n_0(c,t+1)$ is well defined by Corollary 5.2. Using $f(1/4,c,n)$ as defined in (17), the recursion (18) gives $n_0(c,t+1)$ as the smallest n for which a $(t-2)$-times iterated $\log^*$-type function remains over 2. Since the parameter c is multiplied by four at each step of the iteration, $8^{10}4^{5(t-3)}c^5$ appears in the innermost tower. The effect of using $\frac{1}{4}c^{-\frac{1}{2}}(\log^*(n))^{\frac{1}{10}}$ instead of $\log^*$ is negligable, it results only in a $o(t)+o(c)$ increase of the innermost function. This is very generously compensated by iterating t-times and using 4^{5t} in the innermost tower.

To carry out the inductive step, assume that $n \geq n_0(c,t+1)$ and consider a transitive coloring $\mathcal{C}$ on K_n (with vertex set $V=[n]$) such that $|\mathcal{C}| \leq cn$.

Claim 6.1. *K_n contains a canonically colored K_{t+1}.*

Proof of Claim 6.1. Partition the edges of K_n into two parts as follows. Edge ij $(1 \leq i < j \leq n)$ is an A-edge if both i and j have degrees less than $n_0(4c,t)$ in color $col(ij)$, otherwise it is a B-edge. The A-edges and B-edges give a partition of K_n into subgraphs G_A and G_B. Both subgraphs are transitively colored by $\mathcal{C}_A$, $\mathcal{C}_B$, the restrictions of $\mathcal{C}$ to G_A, G_B.

Observe that G_A has less than $\frac{n^2}{4}$ edges. Indeed, otherwise $G_A \in \mathcal{G}(1/4,n)$ and since $\mathcal{C}_A$ is a transitive coloring of G_A with at most cn colors, we can apply Corollary 5.2 which states that G_A has a monochromatic star with at least $f(1/4,c,n) \geq n_0(4c,t)$ edges. However, this contradicts to the definition of the A-edges.

Assume that ij is a B-edge $(1 \leq i < j \leq n)$. From the definition, either i or j has large degree in color $col(ij)$, where large means at least $n_0(4c,t)$. Orient the edge ij into its endpoint of large degree (if both endpoints have large degree, orient arbitrarily). Orienting each B-edge this way, G_B becomes a digraph with more than $\binom{n}{2} - \frac{n^2}{4}$ edges. Select a vertex x with indegree at least $n/4$ in G_B. Without loss of generality, assume that the B-edges oriented into x are colored with colors $1,2,\ldots,k$.

For each $i \in [k]$, define $X_i = \{y \in [n] : col(xy) = i\}$. Since for each $i \in [k]$ there is a B-edge of color i oriented into x,

$$|X_i| \geq n_0(4c,t) \quad \text{for} \quad \text{every } i \in [k] \tag{19}$$

and

$$\sum_{i=1}^{k} |X_i| \geq n/4 \tag{20}$$

We show that for some $i \in [k]$, $\mathcal{C}$ colors $K_n[X_i]$, the complete subgraph of K_n induced by X_i, with at most $4c|X_i|$ colors. This will finish the proof of Claim 6.1 because (19) allows to use the inductive hypothesis on $K_n[X_i]$: there is a canonically colored K_t within X_i which, if extended by x, gives a canonically colored K_{t+1} in K_n.

Assume indirectly that for every $i \in [k]$, $\mathcal{C}$ colors $K_n[X_i]$ with $r_i > 4c|X_i|$ colors. Observe that for every $1 \leq i < j \leq k$, and for every pair of edges e_i, e_j such that e_i is in $K_n[X_i]$, e_j is in $K_n[X_j]$, $col(e_i) \neq col(e_j)$. Indeed, if $col(e_i) = col(e_j)$ then $i \in span^2(j), j \in span^2(i)$ which contradicts to transitivity. Therefore, using (20),

$$|\mathcal{C}| \geq \sum_{i=1}^{k} r_i > \sum_{i=1}^{k} 4c|X_i| \geq 4c(n/4) = cn$$

contradicting to the assumption that $\mathcal{C}$ uses at most cn colors. ∎

7. The size and dimension of lattices

Until this point we followed the traditional view of the Ramsey function: $n \geq n_0(c,t)$ vertices guarantee a canonical K_t in every transitive cn-coloring of K_n. Sali's reduction uses the inverse function $f(c,n)$ with the property: in every transitive cn-coloring of K_n there is a canonical $K_{f(c,n)}$. This amounts to estimate t from $n = n_0(c,t)$. Using the t-times iterated tower threshold of Theorem 2.2 for $n = n_0(c,t)$ and applying its functional inverse, $\log^{**}$, we get

$$\log^{**}(n) = 8^{10} 4^{5t} c^5 \tag{21}$$

which gives

$$t = f(c,n) = \frac{1}{10}\log(\log^{**}(n)) - 3 - \frac{1}{2}\log(c). \tag{22}$$

After this preparation we can apply Sali's reduction inequality ([15], Theorem 3.3) to estimate $\lambda(n)$, the minimum size of a lattice of order dimension n. Sali's reduction (apart from a constant factor about 1/2) gives the following lower bound on $\lambda(n)$ in terms of $f(c,n)$.

$$\lambda(n) \geq \min\left(cn, \sum_{\substack{i \leq n \\ i \equiv n \pmod 3}} f(c,i) \right) \tag{23}$$

Disregarding indices under $\frac{n}{2}$ in the sum of (23) we get

$$\lambda(n) \geq \min\left(cn, \frac{n}{6} f\left(c, \frac{n}{2}\right)\right) \tag{24}$$

from which $\lambda(n) \geq ng(n)$ follows by solving

$$g(n) = \frac{1}{6} f\left(g(n), \frac{n}{2}\right). \tag{25}$$

Using (22),

$$6g(n) + \frac{1}{2}\log(g(n)) + 3 = \frac{1}{10}\log(\log^{**}(n/2)) \tag{26}$$

implying the following estimate for $\lambda(n)$.

Theorem 7.1. $\lambda(n) = \Omega(n\log(\log^{**}(n)))$.

8. Concluding remarks

1. Complete graphs can not be replaced by dense graphs in Theorem 2.2 as the following example shows. Consider a complete $(n/\nu)^{1/2}$ -partite graph with parts of size $(n\nu)^{1/2}$. Color the edges between each pair of partite classes with a distinct color. This graph has n vertices, $\frac{n^2}{2} - o(n^2)$ edges and transitively colored with $\frac{n}{2\nu}$ colors, in fact its color poset is an antichain. However, since all triangles are colored with three distinct colors, it contains no canonically colored K_3. By iterating this construction, one can obtain a transitive coloring of K_n with at most cn colors ($c = 1/\nu$) in which the size of the largest canonical complete subgraph is about $\log\log(n) - \log\log(c)$. This indicates that the method of the paper can not give better bound for $\lambda(n)$ (in the lattice dimension problem) than $\Omega(n\log\log(n))$.

2. Transitive colorings of graphs exclude many color patterns on various subgraphs. Perhaps the simplest excluded color pattern is an alternating 2-coloring of a path or cycle with four edges. Colorings without these patterns emerged from a Ramsey type problem and can be called *abab*-free colorings. It was asked in [3] (Problem 1) whether it is possible to find proper *abab*-free colorings of K_n with cn colors. Some results have been obtained by Axenovich ([1]) and Rosta ([12]). In [3] it was predicted that the answer is negative and might come from Szemerédi's regularity lemma. For the special case of transitive colorings the prediction was right, Theorem 2.2 gives this for $t = 3$ (using the regularity lemma).

3. The notion of transitive colorings can be naturally extended to hypergraphs by generalizing Definition 1.4 as follows.

An edge coloring of a hypergraph H with vertex set $V=\{1,2,\dots,n\}$ is called transitive if one can associate sets $F_1,F_2,\dots,F_n$ to vertices of H so that for any two edges $e,f\in E(H)$, the color of e and f are the same if and only if

$$\cap_{i\in e}F_i=\cap_{i\in f}F_i.$$

It is easy to see that the notions (span relation, color poset) and propositions of Section 3 remain valid in this general setting.

4. Z. Füredi noticed ([6]) that Lemma 4.4 can be proved (with a loss of a constant factor) by a direct reduction to the induced matching lemma. Füredi's argument is that the edge set of any color class $G(i)$ of an antichain C in a D-bounded transitive coloring can be recolored with at most $2D^2$ new colors, each forming an induced matching in $G(i)$. This gives a recoloring of the edges of $G(C)$ with $2D^2|C|$ new colors, each forming an induced matching in $G(C)$. Then the induced matching lemma (Theorem 4.3) can be applied to get a bound similar to the one stated in Lemma 4.4. The author thinks that the price is too high for this shortcut because in the present form Lemma 4.4 is an extension of the induced matching lemma for bounded degree subgraphs with a self-contained proof (to a reader familiar with the regularity lemma).

Acknowledgment. I am indebted to my friend Jenő Lehel for stimulating conversations on the subject and for his remarks on preliminary versions of this paper ([10]). I am also grateful to Zoltán Füredi ([6]) for valuable comments which led to improvements of the presentation.

References

[1] M. Axenovich: A generalized Ramsey problem, *Discrete Mathematics,* **222** (2000), 247–249.

[2] B. Bollobás: *Modern Graph Theory,* Graduate Texts in Mathematics, 184, Springer–Verlag, 1998.

[3] P. Erdős, A. Gyárfás: A variant of the classical Ramsey problem, *Combinatorica,* **17** (1997), 459–467.

[4] P. Erdős, R. Rado: A Combinatorial Theorem, *J. London Math. Soc.,* **25** (1950), 249–255.

[5] Z. Füredi: unpublished, personal communication, 2001.

[6] Z. Füredi: personal communication, 2001.

[7] Z. Füredi, J. Kahn: Dimension Versus Size, *Order,* **5** (1988), 17–20.

[8] R. Graham, B. Rothchild, J. Spencer: *Ramsey Theory,* 2nd. ed., Wiley, New York (1990).

[9] J. Komlós, M. Simonovits: Szemerédi's Regularity Lemma and its Applications in Graph Theory, in: *Combinatorics, Paul Erdős is Eighty, Vol. 2.,* (D. Miklós, V. T. Sós, T. Szőnyi, eds.) Bolyai Society Mathematical Studies, 2. (1996), 295–352.

[10] J. Lehel: personal communication, 2001.

[11] H. Lefmann, V. Rödl: On Erdős–Rado numbers, *Combinatorica,* **15** (1995), 85–104.

[12] V. Rosta: personal communication, 1998.

[13] I. Z. Ruzsa, E. Szemerédi: Triple systems with no six points carrying three triangles, in: *Combinatorics, Vol. II.,* Colloq. Math. Soc. J. Bolyai **18** (1978), 939–945.

[14] A. Sali: Extremal Theorems for Finite Partially Ordered Sets, Ph.D. Thesis, Ohio State University, 1986.

[15] A. Sali: On the Problem of Dimension Versus Size for Lattices, *European Journal of Combinatorics,* **17** (1996), 421–425.

[16] E. Szemerédi: Regular partitions of graphs, in: *Problémes Combinatoires et Théorie des Graphes,* Colloques Internationaux C.N.R.S. No.260, Orsay (1976), 399–401.

[17] W. T. Trotter: *Combinatorics and Partially Ordered Sets, Dimension Theory,* The Johns Hopkins University Press, Baltimore and London, 1992.

András Gyárfás

Computer and Automation Research Institute
of the Hungarian Academy of Sciences,
Budapest, P. O. Box 63, Hungary-1518.
gyarfas@sztaki.hu