ELSEVIER

Discrete Mathematics 257 (2002) 415–421

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

A finite basis characterization of α-split colorings

András Gyárfás[a], André E. Kézdy[b,*], Jenő Lehel[b,1]

[a]Computer and Automation Institute, Hungarian Academy of Sciences, 1111 Budapest, Hungary
[b]Department of Mathematics, University of Louisville, Louisville, KY 40292, USA

Received 17 November 1999; accepted 20 October 2000

Abstract

Fix $t > 1$, a positive integer, and $\boldsymbol{a} = (a_1, \ldots, a_t)$ a vector of nonnegative integers. A t-coloring of the edges of a complete graph is called $\boldsymbol{a}$-split if there exists a partition of the vertices into t sets $V_1, \ldots, V_t$ such that every set of $a_i + 1$ vertices in V_i contains an edge of color i, for $i = 1, \ldots, t$. We combine a theorem of Deza with Ramsey's theorem to prove that, for any fixed $\boldsymbol{a}$, the family of $\boldsymbol{a}$-split colorings is characterized by a finite list of forbidden induced subcolorings. A similar hypergraph version follows from our proofs. These results generalize previous work by Kézdy et al. (J. Combin. Theory Ser. A 73(2) (1996) 353) and Gyárfás (J. Combin. Theory Ser. A 81(2) (1998) 255). We also consider other notions of splitting.

Keywords: Ramsey theory; Coloring; Finite basis

1. Introduction

Fix $t > 1$, a positive integer, and $\mathbf{a} = (a_1, \ldots, a_t)$ a vector of nonnegative integers. A t-coloring of the edges of a complete graph is called $\mathbf{a}$-split if there exists a partition of the vertices into t sets $V_1, \ldots, V_t$ such that every set of $a_i + 1$ vertices in V_i contains an edge of color i, for $i = 1, \ldots, t$. This generalizes the well known family of split graphs: a graph is a *split* graph if there exists a partition of the vertices into two sets so that one set induces a clique and the other set induces an independent set. So a graph is a split graph if and only if the 2-coloring in which its edges are colored red and its nonedges are colored blue defines a $(1, 1)$-split 2-coloring of a complete graph.

* Corresponding author.
E-mail address: kezdy@louisville.edu (A.E. Kézdy).

[1] On leave from the Automation and Computing Research Institute of the Hungarian Academy of Sciences.

0012-365X/02/$ - see front matter
PII: S0012-365X(02)00440-5

A coloring of the edges of a complete graph is **a**-*critical* (or simply, *critical* when **a** is understood) if it is not **a**-split, but becomes **a**-split after the removal of any vertex. In the next section, we show that there is a finite forbidden subcoloring characterization of **a**-split colorings by proving that for any fixed **a**, there are only a finite number of **a**-critical colorings. Our proof combines a theorem of Deza with Ramsey's theorem. This generalizes previous work by Kézdy et al. [7] and Gyárfás [6]. In the final section, we mention related notions of splitting.

2. Main result

In this section we prove the main theorem and introduce a related extremal problem.

Suppose that A and B are finite sets. Let $A \triangle B$ denote the *symmetric difference* of A and B. Clearly, $|A \triangle B| = |A| + |B| - 2|A \cap B|$, so if $|A| + |B|$ is even, then so is $|A \triangle B|$. Given finite sets $F_1, \dots, F_m$, the *degree* of an element $x \in \bigcup_{i=1}^m F_i$ is $d(x) = |\{j\colon\ x \in F_j\}|$. We shall use the following result due to Deza [1] which is Problem 13.17 of the book by Lovász [8] (cf. Section 23.1.2 of the book by Deza and Laurent [2]).

Lemma 1 (Deza [1]). *If $m > k^2 + k + 2$ and $F_1, \dots, F_m$ are finite sets satisfying $|F_i \triangle F_j| = 2k$, for $1 \leqslant i < j \leqslant m$, then every element in $\bigcup_{i=1}^m F_i$ has degree 1, $m-1$, or m.*

Recall that the *Ramsey number* $R_j(i)$ is the smallest positive integer p such that, any coloring of the edges of K_p with j colors contains a monochromatic K_i. The existence of $R_j(i)$ is guaranteed by Ramsey's theorem; in particular, this number is finite. Let $\|\mathbf{a}\| = \|\mathbf{a}\|_\infty = \max\{a_1, \dots, a_t\}$. Let $\mathbf{Z}$ denote the set of integers.

Theorem 2. *For any fixed integer $t > 1$ and $\mathbf{0} \leqslant \mathbf{a} \in \mathbf{Z}^t$, the number of **a**-critical colorings is finite.*

Proof. Fix $t > 1$ and $\mathbf{0} \leqslant \mathbf{a} = (a_1, \dots, a_t) \in \mathbf{Z}^t$. To prove the theorem, it suffices to prove that if there is a critical coloring of K_n, then n is bounded by a function of t and $\|\mathbf{a}\|$.

Set $M = (t-1)R_t(\|\mathbf{a}\| + 1) + 1$ and recursively define the function

$$N(i) = \begin{cases} 2R_M((M-1)^2 + (M-1) + 3) & \text{if } i = 1, \\ 2R_M(N(i-1)) & \text{if } i > 1. \end{cases}$$

We shall prove that there is no critical coloring of K_n, if $n \geqslant N(t)$.

Suppose, on the contrary, that there is a critical coloring of $G = K_n$, for some $n \geqslant N(t)$. For each vertex $v \in V(G)$, choose and fix an **a**-splitting of $V(G) - \{v\}$ into t sets $V_1^v, \dots, V_t^v$.

Claim 1. *For all $u, v \in V(G)$ and $i = 1, \dots, t$, $|V_i^u \triangle V_i^v| < 2M$.*

To see this, observe that

$$V_i^u \triangle V_i^v = \bigcup_{j \neq i} [(V_j^u \cap V_i^v) \cup (V_j^v \cap V_i^u)] \cup \{u, v\}.$$

Thus, because of symmetry, it is enough to prove that $|(V_j^u \cap V_i^v)| < R_t(\|\mathbf{a}\| + 1)$. This latter statement follows from the observation that the largest monochromatic clique in the graph induced by $V_j^u \cap V_i^v$ $(i \neq j)$ has at most $\max\{a_i, a_j\}$ vertices since V_j^u induces a graph with independence number at most a_j in color j, and V_i^v induces a graph with independence number at most a_i in color i.

Claim 2. *There is a set of vertices $S \subset V(G)$, and nonnegative integers $k_1, \ldots, k_t \leqslant (M-1)$, such that $|S| \geqslant (M-1)^2 + (M-1) + 3$, and for all $i = 1, \ldots, t$, $|V_i^u \cap V_i^v| = 2k_i$ for all $u, v \in S$.*

We show the existence of such a set of vertices by iterating a certain procedure that we now describe. Set $S_0 = V(G)$, so $|S_0| = n$. Because $n \geqslant N(t) = 2R_M(N(t-1))$, there is a set S_0^* of at least $n/2 \geqslant R_M(N(t-1))$ of the vertices in S_0 with the property that $|V_1^u| \equiv |V_1^v| \bmod 2$, for all $u, v \in S_0^*$. In particular, $|V_1^u \triangle V_1^v|$ is even, for all $u, v \in S_0^*$. Claim 1 guarantees that $|V_1^u \triangle V_1^v| < 2M$, for all $u, v \in S_0^*$. So, by considering a complete graph whose vertices correspond to the sets V_1^v $(v \in S_0^*)$ and whose edges are colored with the colors from $\{0, 1, \ldots, M-1\}$ according to the rule that assigns color $|V_1^u \triangle V_1^v|/2$ to the edge $V_1^u V_1^v$, Ramsey's theorem implies that there exists a nonnegative integer $k_1 < M$ and a set $S_1 \subset S_0^*$ satisfying $|S_1| \geqslant N(t-1)$ and $|V_1^u \triangle V_1^v| = 2k_1$, for all $u, v \in S_1$.

Assume $1 \leqslant p < t-1$ and that we have defined $S_p \subset V(G)$ and nonnegative integers $k_1, \ldots, k_p$ satisfying $|S_p| \geqslant N(t-p)$ and, for all $i = 1, \ldots, p$, we have $|V_i^v \triangle V_i^u| = 2k_i$, for all $u, v \in S_p$. Now we describe how to define k_{p+1} and S_{p+1}. Because $|S_p| \geqslant N(t-p) = 2R_M(N(t-p-1))$, there is a set S_p^* of at least $n/2 \geqslant R_M(N(t-p-1))$ vertices in S_p with the property that $|V_{p+1}^v \triangle V_{p+1}^u|$ is even, for all $u, v \in S_p^*$. As in the previous paragraph, Ramsey's theorem implies that there exists a nonnegative integer $k_{p+1} < M$ and a set $S_{p+1} \subset S_p^*$ satisfying $|S_{p+1}| \geqslant N(t-p-1)$ and $|V_{p+1}^v \triangle V_{p+1}^u| = 2k_{p+1}$, for all $u, v \in S_{p+1}$.

In the final step, $S_{t-1} \geqslant N(1) = 2R_M((M-1)^2 + (M-1) + 3)$, so the procedure yields a nonnegative integer k_t and a set $S_t \subset S_{t-1}$ satisfying $|S_t| \geqslant (M-1)^2 + (M-1) + 3$ and, for all $i = 1, \ldots, t$,

$$|V_i^v \triangle V_i^u| = 2k_i, \quad \text{for all } u, v \in S_t.$$

So the set $S = S_t$ contains the desired vertices. This concludes Claim 2.

Using the set S from Claim 2, define for all $u \in V(G)$ and $i = 1, \ldots, t$, the degrees

$$d_i(u) = |\{v \in S: \ u \in V_i^v\}|.$$

Because $\{V_i^v\}_{i=1}^t$ is a partition of $V(G)-\{v\}$, we obtain easily for all $u\in V(G)$,

$$\sum_{i=1}^{t} d_i(u) = \begin{cases} |S|-1 & \text{if } u\in S, \\ |S| & \text{if } u\notin S. \end{cases} \qquad (*)$$

Observe that, by choice of S, we have $|S|\geqslant(M-1)^2+(M-1)+3$, $k_i\leqslant M-1$, and $|V_i^v \triangle V_i^u|=2k_i$, for all $u,v\in S$. Applying Lemma 1 to $\{V_i^u\colon\ v\in S\}$ we obtain $d_i(u)\in\{0,1,|S|-1,|S|\}$, for all $u\in V(G)$ and for all $i=1,\dots,t$.

To complete the proof of the theorem, we obtain the following contradiction.

Claim 3. *The sets* $B_i=\{u\in V(G)\colon\ d_i(u)\geqslant|S|-1\}$ $(i=1,\dots,t)$ *constitute an* $\mathbf{a}$*-splitting of* G.

By $(*)$, $\bigcup_{i=1}^t B_i=V(G)$ follows because $|S|$ is much larger than t. If $u\in B_i\cap B_j$ $(i\neq j)$, then $d_i(u)+d_j(u)\geqslant 2(|S|-1)$, contradicting $(*)$. Hence the B_i's partition $V(G)$.

Now consider a set $X\subset B_i$ of cardinality a_i+1. By definition, $d_i(x)\geqslant|S|-1$, for all $x\in X$. Hence each element of X is missing from at most one of the sets $\{V_i^v\colon\ v\in S\}$. Because $|S|>a_i+1$, there exists some $v\in S$ such that $X\subset V_i^v$. In particular, X cannot be an independent set in color i. Therefore B_i induces a subgraph with independence number at most a_i in color i. It follows that the B_i's constitute an $\mathbf{a}$-splitting of G. □

Theorem 2 proves the existence, for any vector $\mathbf{a}$, of *split numbers*, $S(\mathbf{a})$ defined as the maximum order of an $\mathbf{a}$-critical coloring. Actually the proof we give also works for hypergraphs, so $S_r(\mathbf{a})$ exists for r-uniform hypergraphs. Recall that Ramsey's theorem guarantees, for any vector $\mathbf{a}\in\mathbf{Z}^t$, a number $R_t(\mathbf{a})$ such that, for any t-coloring of the 2-sets of an n-set satisfying $n\geqslant R_t(\mathbf{a})$, there exists $1\leqslant i\leqslant t$ such that the t-coloring contains an a_i-set whose 2-sets all have color i (i.e. a monochromatic clique of order a_i). For $\mathbf{a}\in\mathbf{Z}^t$, a *Ramsey* $\mathbf{a}$*-coloring* is a t-coloring of the edges of a complete graph on $R_t(\mathbf{a})-1$ vertices such that there is no monochromatic clique of order a_i for all $1\leqslant i\leqslant t$. An explicit description of $\mathbf{a}$-critical colorings seems very difficult, even when $r=t=2$, since Gyárfás [6] has shown that among the (a_1,a_2)-critical colorings are the Ramsey (a_1+2,a_2+2)-colorings. For $t>2$, the argument does not generalize, though accidentally $(3,3,3)$-Ramsey colorings are $(2,2,2)$-critical.

The next proposition shows that $S(\mathbf{a})$ grows at least exponentially in $\|\mathbf{a}\|$.

Proposition 3. *For any* $a,\ t\in\mathbf{Z}^+$, *if* $(a,a,1,\dots,1)=\mathbf{a}\in\mathbf{Z}^{t+2}$, *then*

$$S(\mathbf{a})\geqslant R_2(a+2,a+2)+t(a+1)-1.$$

Proof. To prove the bound we must demonstrate an $\mathbf{a}$-critical $(t+2)$-coloring of a complete graph on $R_2(a+2,a+2)+t(a+1)-1$ vertices. To this end, consider the $(t+2)$-colored complete graph $G(a)$ defined as follows. The graph has $t+1$ pieces. The first piece, R, is a Ramsey graph on $R_2(a+2,a+2)-1$ vertices that does not contain a monochromatic complete graph K_{a+2} in colors 1 and 2. The other t pieces of $G(a)$, $S_1,\dots,S_t$ are complete graphs with $a+1$ vertices whose edges are all colored 1. Edges between the $t+1$ pieces are colored 2. Thus, $G(a)$ is a graph with

$R_2(a+2, a+2) + t(a+1) - 1$ vertices whose edges are colored using only colors 1 and 2. The following two claims establish that $G(a)$ is **a**-critical.

Claim 1. *For $v \in V(G(a))$, the graph $G(a) - v$ has an* **a**-*splitting.*

If v is a vertex from R, then $R - v$ has a natural partition into two sets $A = \{w \in R - v\colon\ vw$ has color $2\}$ and $B = \{w \in R - v\colon\ vw$ has color $1\}$ in which A induces a graph in which every set of $a+1$ vertices contains an edge of color 1, and B induces a graph in which every set of $a+1$ vertices contains an edge of color 2. Extend B to B^* by adding a vertices from each S_i $(i = 1, \dots, t)$. The remaining vertices are the singleton sets in the $(a, a, 1, \dots, 1)$-splitting of $G(a) - v$.

On the other hand, if $v \in S_i$, for some $1 \leqslant i \leqslant t$, then the t singleton sets in the $(a, a, 1, \dots, 1)$-splitting of $G(a) - v$ will consist of one vertex y from R together with a single vertex y_j from each S_j $(1 \leqslant j \neq i \leqslant t)$. Now define the sets $A = \{w \in R - y\colon\ yw$ has color $2\}$ and $B = \{w \in R - y\colon\ yw$ has color $1\}$, and extend B to B^* by adding $S_i - v$ and $S_j - y_j$ for each S_j $(1 \leqslant j \neq i \leqslant t)$.

Claim 2. *The graph $G(a)$ has no* **a**-*splitting.*

Suppose on the contrary that A^*, B^*, plus t singleton sets determine a partition of $V(G(a))$ constituting an **a**-splitting of $G(a)$, where A^* induces a graph in which every set of $a+1$ vertices contains an edge of color 1 (i.e. $\alpha_1(A^*) \leqslant a$), and B^* induces a graph in which every set of $a+1$ vertices contains an edge of color 2 (i.e. $\alpha_2(B^*) \leqslant a$). Define $A = R \cap A^*$, $B = R \cap B^*$, $U = R - (A \cup B)$. Since $|B^* \cap S_i| \leqslant a$, for $i = 1, \dots, t$, it follows that every S_i intersects either A^* or the singleton sets. Set $s = |\{i\colon\ A^* \cap S_i = \emptyset\}|$. Clearly $u = |U| \leqslant t - s$, since the singletons must intersect every S_i satisfying $S_i \cap A^* = \emptyset$. Also note that

$$\alpha_1(A^*) = \alpha_1(A) + t - s \quad \text{and} \quad \alpha_2(B) \leqslant a.$$

Create a new graph R' with vertex set $A \cup B \cup U'$, where $|U'| = t - s + 1 > |U|$. The edges within U' and between U' and A receive color 1 and those between U' and B receive color 2. Now

$$\alpha_1(R') = \max\{\alpha_1(A \cup B), \alpha_1(A) + |U'|\} \leqslant a + 1$$

and

$$\alpha_2(R') = \max\{\alpha_2(A \cup B), \alpha_2(B) + 1\} \leqslant a + 1,$$

contradicting that R was Ramsey. This proves the claim. □

3. Variations

In this section we mention observations concerning some of the myriad variations on the theme of "split" partitions. The original split graph recognition problem could be stated this way: Given a 2-coloring (using colors 1 and 2) of the edges of a complete graph, does there exist a partition of the vertices into two sets, V_1 and V_2 so that V_i induces a subgraph with no edge with color i ($i=1,2$)? There are many ways one could generalize this notion. One way to generalize is to bound independence number in each part as we have done earlier in this paper.

Another way to generalize is to bound the clique number in each part, as follows. Given a 3-coloring (using colors 1, 2 and 3) of the edges of a complete graph, an (r,s)-*fracture* is a partition of the vertices into two sets, V_1, and V_2 so that there exist $1 \leqslant i \neq j \leqslant 3$, such that V_1 induces a subgraph with clique number at most r in color i, and V_2 induces a subgraph with clique number at most s in color j. We say that a 3-edge coloring of a complete graph is (r,s)-*fracturable* if it admits an (r,s)-fracture.

We do not know the complexity of recognizing (r,s)-fracturable 3-edge colorings for general fixed values of r and s. However, for the special case $r=s=1$, one can reduce the problem quite easily to the 2-colors graph partition problem (2-CGP) introduced by Gavril [5].

2-CGP. *Instance*: A graph $G(V,E)$ and two sets $X,Y \subseteq E$ such that $E \subseteq X \cup Y$.

Question: Does there exist a partition of V into two sets A and B so that the graph induced by A has no edges from X, and the graph induced by B has no edges from Y?

Note that the sets X and Y need not be disjoint. Indeed an interesting case is when $X=Y=E$, in which case a partition exists if and only if G is bipartite. Gavril [5] shows that a linear time sequential algorithm for 2-CGP can be derived from the well known linear time sequential algorithms for 2-CNF satisfiability.

Proposition 4. *The* $(1,1)$-*fracturable* 3-*edge colorings of complete graphs can be recognized in linear time*.

Proof. Let $G=K_n$ be a complete graph whose edges are colored with colors 1, 2, and 3. Let G_i denote the graph with vertex set $V(G)$ and edge set consisting of those edges of G of color different from i. Let C_i denote the edges of color i in G. Observe that G has a $(1,1)$-fracture if and only if at least one of $(G_1, X=C_2, Y=C_3)$, $(G_2, X=C_1, Y=C_3)$, or $(G_3, X=C_1, Y=C_2)$ admits a partition such as the one in the 2-CGP problem. □

It is worth mentioning that the family of $(1,1)$-fracturable 3-edge colorings is not characterized by a finite list of forbidden subcolorings. Indeed, the following describes an infinite family of critical 3-edge colorings. Color the edges of two vertex disjoint odd cycles of the same size, say size $2k+1$, with color 1, and color the edges of a perfect matching between them with color 2; color the remaining edges with color 3. One can check that any set of vertices that avoids edges of color 1 (respectively 2) has cardinality at most $2k$ (respectively, $2k+1$), whereas any set of three vertices induces an edge of color 3. Since there are $4k+2$ vertices, it follows that there is no $(1,1)$-fracture of this coloring. It is straightforward to show that the removal of any vertex admits a $(1,1)$-fracture. So this is an infinite family of critical 3-edge colorings.

Finally, we mention briefly one more variation on the split theme. Given a 3-coloring (using colors 1–3) of the edges of a complete graph, does there exist a partition of the vertices into three sets V_1, V_2, and V_3 so that V_i induces a subgraph with no edge of color i $(i = 1–3)$? We do not know whether there is a polynomial algorithm to recognize 3-edge colorings that can be split in this way. This variation on the split theme has been studied by Erdős and Gyárfás [3]. They were concerned with estimating the minimum number of vertices in a critical coloring of this sort. Here we describe an infinite list of critical colorings.

Connect the three vertices of a triangle of color 1 to the three vertices of a triangle of color 2 using three pairwise vertex disjoint alternating 2-1 paths; we assume that the alternating paths contain an even number of edges and each end edge has a color different from that of the incident triangle. Call this 2-colored graph an *even stretcher*. Observe that an even stretcher must have an odd number, $2k+1$, of vertices ($k \geqslant 4$). For every $i = 1, 2$, the $2k+1$ vertices of an even stretcher H_k are partitioned into $k-1$ edges and one triangle of color i. Therefore, the vertex set of H_k has no partition $V_1 \cup V_2$ (where V_i induces a subgraph with no edge of color i, $i = 1, 2$). Note that any even stretcher is also vertex-minimal with respect to this property, i.e., $V(H_k - v)$ has a split partition $V_1 \cup V_2$, for each vertex v.

Take two disjoint copies of an arbitrary even stretcher H_k and color all uncolored edges induced in their union with color 3. Based on the properties above of an even stretcher, it is straightforward to verify that in the obtained 3-coloring there is no split partition $V_1 \cup V_2 \cup V_3$, furthermore, this coloring is critical. □

References

[1] M. Deza, Solution d'un problème de Erdős-Lovász, J. Combin. Theory Ser. B 16 (1974) 166–167.

[2] M. Deza, M. Laurent, Geometry of Cuts and Metrics, Springer, Berlin, 1997.

[3] P. Erdős, A. Gyárfás, Split and balanced colorings of complete graphs, Discrete Math. 200(1–3) (1999) 79 – 86.

[4] S. Földes, P.L. Hammer, Split graphs, in: F. Hoffman et al. (Eds.), Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory, and Computing, Louisiana State Univ., Baton Rouge, Louisiana, 1977, pp. 311– 315. Congr. Numer. No. XIX, Utilitas Math., Winnipeg, Man., 1977.

[5] F. Gavril, An efficiently solvable graph partition problem to which many problems are reducible, Info. Proc. Lett. 45 (1993) 285–290.

[6] A. Gyárfás, Generalized split graphs and Ramsey numbers, J. Combin. Theory Ser. A 81(2) (1998) 255 –261.

[7] A.E. Kézdy, H.S. Snevily, C. Wang, Partitioning permutations into increasing and decreasing subsequences, J. Combin. Theory Ser. A 73(2) (1996) 353 – 359.

[8] L. Lovász, Combinatorial Problems and Exercises, North-Holland, New York, 1979.