BOLYAI SOCIETY
MATHEMATICAL STUDIES, 11

Paul Erdős and his Mathematics. II,
Budapest, 2002, pp. 367–373.

Erdős Problems on Irregularities of Line Sizes and Point Degrees

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Problems and results are given concerning some variants on the irregularities of line sizes and point degrees. All questions are initiated by Paul Erdős.

A *linear space* (Pairwaise balanced design) is a set of *lines* (edges, blocks) on n *points* (vertices, treatments) with the property that each pair of distinct points is covered by a unique line. The *size* of a line is the number of points on it and the *degree* $d(p)$ of a point p is the number of lines containing it. A linear space is *planar* if its points can be placed in R^2 so that the lines are the maximal collinear subsets. A well-known example of a nonplanar linear space is the Fano plane. We shall assume that line sizes are between 2 and $n-1$, i.e. singleton lines and trivial spaces are excluded.

This note addresses irregularity problems asked by Paul Erdős: what is the maximum number of distinct line-sizes (point degrees) and what is the minimum number of repetitions for them. Here the maximum and minimum is taken over all linear spaces of n points. The four questions becomes eight since each has a planar variant. Similar questions of Erdős involving distances are not treated here, see [6] and its references. For problems and results concerning linear spaces in general see [7].

* Partly supported by grant OTKA T029074.

1. Maximum Number of Distinct Line Sizes: $f_1(n)$

One can state a surprisingly sharp result here. Erdős, Duke, Fowler and Phelps ([3]) prove that $f_1(n) = 2\lfloor n^{1/2}\rfloor - 2$. There is a slight error in the proof, in fact $f_1(n)$ is one larger for certain values of n. The correct value is stated in the next theorem.

Theorem 1. *For $k^2 \le n < k^2 + k$, $f_1(n) = 2k - 2$ and for $k^2 + k \le n < (k+1)^2$, $f_1(n) = (2k-1)$.*

Proof. The upper bound comes from the following

Claim. *Assume that $e_1, e_2, \ldots, e_t$ are lines in a linear space and $|e_1| < |e_2| < \ldots < |e_t|$. Then*

$$|\cup_{i=1}^{t} e_i| \ge |e_1|t.$$

Proof of Claim. For each i $(1 \le i \le t-1)$, e_{i+1} intersects $\cup_{j=1}^{i} e_j$ in at most i points and $|e_{i+1}| \ge |e_1| + i$.

Case 1, $n < k^2 + k$. Assume that $k \le |e_1| < \ldots |e_t|$. Then, using the claim, $k^2 + k > n \ge kt$ which gives $k + 1 > t$, i.e there are at most k distinct line sizes not smaller than k. Since at most $k-2$ line sizes are in the interval $[2, k-1]$, $f_1(n) \le k + k - 2 = 2k - 2$.

Case 2, $n < (k+1)^2$. Assume that $k + 1 \le |e_1| < \ldots |e_t|$. Then, using the claim, $(k+1)^2 > n \ge (k+1)t$ which gives $k + 1 > t$, i.e there are at most k distinct line sizes not smaller than $k + 1$. Since at most $k - 1$ line sizes are in the interval $[2, k]$, $f_1(n) \le k + k - 1 = 2k - 1$.

The lower bound of the theorem comes from the following construction.

Construction 1. Consider lines $e_1, \ldots, e_k$ in general position and parallel segments $f_1, \ldots, f_k$. One can easily arrange these lines so that f_i intersects precisely $e_1, \ldots, e_i$ and these intersection points are all distinct from the $\binom{k}{2}$ intersection points of the e_i-s. We have k^2 points with line sizes $2, 3, \ldots, 2k-1$, i.e. $2k-2$ line sizes. For $1 \le t \le k-1$ it is trivial to extend the example with t new points not collinear to any lines e_i, f_i. Therefore $f_1(n) \ge 2k - 2$ for $k^2 \le n < k^2 + k$.

To see that $f_1(n) \ge 2k - 1$ for $n = k^2 + k$, add k new points to the previous construction, one to each e_i, so that the new points are not collinear to any f_j. Each e_i is extended by one point so we have line sizes

$k+1, k+2, \ldots, 2k$ and the f_j-s retain their sizes $2, 3, \ldots, k$ thus we have $2k-1$ distinct line sizes. As before, it is trivial to extend this with t new points for $1 \le t \le k-1$.

This finishes the proof of Theorem 1. ∎

Since the equality in Theorem 1 is shown by a planar example, we get the following corollary.

Corollary 1. *n points of the plane determine at most $2k-2$ $(2k-1)$ distinct line sizes if $k^2 \le n < k^2+k$ (if $k^2+k \le n < (k+1)^2$). This is best possible for every n.*

2. Minimum Number of Line Size Repetitions: $f_2(n)$

Erdős and Purdy introduced $f_2(n)$ in 1978 ([4]) as the minimum r such that every linear space on n points has a line size repeated at least r times.

Conjecture 1 (Erdős–Purdy, [4]). *$f_2(n) \le cn^{1/2}$ where c is an absolute constant.*

In [4] an outline of a probabilistic argument is given to show that $f_2(n) \le cn^{3/4}$. Their example is a linear space obtained from a projective plane by the intersections of its lines with a random subset of points. In fact, as pointed out by Blokhuis, Szőnyi and Wilbrink ([1]), one can not improve the upper bound this way because among the line sizes defined by any set of at least $c_1 q^2$ points of a projective plane of order q there must be a line size repeated at least $c_2 q^{3/4}$ times.

It is also a remark in [4] that the upper bound $f_2(n) \ge cn^{1/2}$ follows easily from De Bruijn–Erdős theorem ([2]). Indeed, since there are less than $2\lfloor n^{1/2} \rfloor$ line sizes (Theorem 1) and at least n lines ([2]), some line size must be repeated at least $\frac{n^{1/2}}{2}$ times.

Corollary 2. $f_2(n) > \frac{n^{1/2}}{2}$.

Conjecture 1 does not hold for the planar version of the problem. The reason is that a planar set of n points must determine at least $\frac{3n}{7}$ Gallai lines (lines which contain precisely two points). In fact, Erdős and Purdy proved ([4]) more which gives that $f_2(n) = n-1$ for planar sets:

Theorem 2 ([4]). *Assume that $n \geq 25$ points of the plane are not collinear. Then there are at least $n-1$ Gallai lines or at least $n-1$ three-point lines.*

3. Maximum Number of Distinct Point Degrees: $f_3(n)$

Problem 1 (Erdős and Makai, 1992). *Estimate $f_3(n)$, the maximum number of distinct point degrees in a linear space of n points. How does $f_3(n)$ change if restricted to planar linear spaces?*

Construction 2. The following construction shows that $f_3(n) \geq n/3$ for planar sets. Let a, b, c denote three parallel lines such that the distance of a, b and the distance of b, c are the same. Select k points $B_1, B_2, \ldots, B_k$ on b so that the distances B_iB_{i+1} are the same. Select point C_1 on line c. The points A_i are defined as the intersection of a and the line C_1B_i and the points C_i are defined as the intersection of c and the line A_1B_i $(1 \leq i \leq k)$. The set of point degrees is equal to $\{k+1, k+2, \ldots, 2k\}$ on lines a, b, c.

Construction 2a. If line c in Construction 2 is replaced by $\binom{k}{2}$ 2-point lines, one gets a nonplanar example showing that $f_3(n) \geq 2n/3$.

The upper bound is much weaker than the lower bounds provided by the above constructions:

Theorem 3 (Erdős–Gyárfás unpublished).

$$f_3(n) \leq n - (n/2)^{1/2}\bigl(1 + o(1)\bigr).$$

Proof. Let x be a point of minimum degree t with lines $l_1, l_2, \ldots l_t$ through x, set $|l_i| = n_i$, $n_1 \geq n_2 \geq \ldots \geq n_t$. The set of n points is denoted by V.

Claim. *For $y \in l_i, y \neq x$,*

$$\frac{n - n_i}{t-1} < d(y) \leq n - n_i + 1.$$

Proof of Claim. The lines through y different from l_i intersect $V \setminus l_i$ in at most $t-1$ points. This gives the lower bound and the upper bound is obvious.

Using the claim we conclude that no point (except possibly x) has degree less than $\frac{n-n_1}{t-1}$. If $n_1 \leq (1/2+\varepsilon)n$ for some $\varepsilon > 0$ then no point has degree less than

$$\frac{n(1/2-\varepsilon)}{t-1} \leq \frac{n-n_1}{t-1}.$$

On the other hand, by the definition of t, no point has degree less than t. Therefore

$$\max\left\{t-1, \frac{n(1/2-\varepsilon)}{t-1} - 1\right\}$$

'small' degrees are missing which is clearly at least $\left(n(1/2-\varepsilon)\right)^{1/2}$.

If $n_1 \geq (1/2+\varepsilon)n$ then (from the claim) each point of l_1 (except x) has degree at most $n - n_1 + 1 \leq n(1/2-\varepsilon)+1$. On the other hand, each point $y \neq x$ of any l_i for $i \neq 1$ has degree at least $n_1 \geq n(1/2+\varepsilon)$ because y and the points of l_1 are on distinct lines. Therefore each degree (with the possible exception of $d(x)$) is missing from the interval $\left[n(1/2-\varepsilon)+1, n(1/2+\varepsilon)\right]$ and the proof is finished. ∎

The following two problems had been formulated by Paul Erdős and the author.

Problem 2. *Is it true that* $f_3(n) \geq n - cn^{1/2}$*?*

Problem 3. *Is it true that for planar sets* $f_3(n) \leq (1-\varepsilon)n$ *for some positive* ε*?*

4. Minimum Number of Degree Repetitions: $f_4(n)$

Construction 2 gives a planar linear space where each point degree is repeated at most three times, i.e. $f_4(n) \leq 3$.

Problem 4. *Is* $f_4(n) \geq 3$ *for planar linear spaces or there exists a planar point set where each point degree is repeated at most twice?*

Notice that, like for degrees of a graph, some degrees must be repeated. It is well known that there are graphs in which each degree is repeated at most twice. The next construction shows that if planarity is not required then (like in the case of graphs) there are linear spaces in which each point degree is repeated at most twice.

Construction 3. We start with Construction 2 assuming that k is odd and $k \geq 7$. After some modifications we arrive to L^*, a linear space on $3k+1$ points, $2k-2$ distinct point degrees, each repeated at most twice.

Line a is replaced by a near pencil with center A_{k-1}. Line b is replaced by a near pencil with center B_1. Line c is replaced by $\binom{k}{2}$ 2-point lines. The resulting linear space is denoted by L.

Next the following partition $P = \left\{P_1, P_2, \ldots P_{\frac{k+9}{2}}\right\}$ is defined on the points of L. Set $P_1 = a \setminus A_{k-1}$, $P_2 = \{A_{k-1}, B_1\}$, $P_3 = b \setminus B_1$, $P_i = C_{i-3}$ for $i = 4, 5, 6$ and for $7 \leq i \leq \frac{k+9}{2}$ set $P_i = \{C_{2i-10}, C_{2i-9}\}$.

The linear space L^* is formed by adding a new point D to L which form a line with each P_i. The blocks P_i which formed lines in L (all but P_4, P_5, P_6) are removed.

The degree of D is $\frac{k+9}{2}$ while the degrees of points in L do not change, except at C_1, C_2, C_3 which are increased by one. The degree sequence can be divided into four intervals I_j as follows. $I_1 = \left[\frac{k+9}{2}\right]$ repeated once; $I_2 = [k+2, 2k+2]$ repeated twice; $I_3 = [2k+3, 3k-4]$ repeated once; $I_4 = [3k-3, 3k-2]$ repeated twice. To ensure separation of these intervals, we need the conditions $\frac{k+9}{2} < k+2$ and $2k+2 < 3k-3$ thus the construction works for odd $k > 5$.

It is easy to check that adding one or two new points to Construction 3 with 2-point lines preserves the property that each degree is repeated at most twice.

Corollary 3. *For every $n \geq 22$, $f_4(n) = 2$.*

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