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A METHOD TO GENERATE GRACEFUL TREES

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Résumé. — Un arbre d'ordre n est dit gracieux s'il existe une numérotation des sommets avec les nombres $0, 1, \ldots, n-1$ telle que :

— deux sommets distincts ont des numéros différents,

— la valeur absolue de la différence entre deux nombres correspondant aux extrémités d'une arête est différente pour toute arête.

Nous présentons ici une méthode de numérotation qui permet de construire des arbres gracieux à partir d'arbres gracieux plus petits. Une application simple de la méthode est montrée pour la classe des arbres « symétriques » (arbres ayant une racine et où chaque niveau ne contient que des sommets de même degré).

1. **Introduction.** — A tree T on n vertices is called graceful [1] if its vertices can be numbered with $0, 1, 2, \ldots, n-1$ so that

a) No two vertices of T get the same number.

b) The absolute value of the difference between two numbers belonging to the endpoints of an edge (it is called the weight of the edge) is different on every edge of T.

The numbering of a graceful tree described above is also called graceful. The conjecture that all trees have a graceful numbering is due to Ringel [2].

This paper presents a numbering method (*) to generate graceful trees from smaller ones (Theorem 1 and Theorem 2).

A simple application of the method is shown for the class of « symmetrical » trees (rooted trees in which every level contains vertices of the same degree) in corollary 1.

2. **Notations.** — Let T be a gracefully numbered tree. The number associated with vertex v is denoted by $g(v)$. The vertices of T (T being a bipartite graph) can be split into two distinct sets A_T and B_T so that every edge of T is spanned between A_T and B_T. As a convention we will always assume the vertex v with $g(v) = 0$ to be in A_T and we consider that vertex as the root of T. Let T_1 and T_2 be trees with distinct vertices.

Definition 1. — Planting T_1 with vertex v_1 into a vertex v_2 of T_2 is an operation which yields a new tree T by identifying v_1 and v_2 in the forest $T_1 \cup T_2$:

(*) We were informed during the conference that a related idea was used by R. G. Stanton, C. R. Zarnke [3].

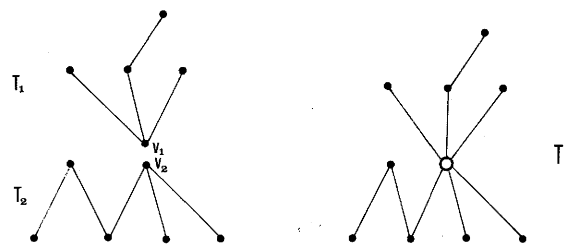

FIG. 1.

Definition 2. — The trees T_1 and T_2 form a beau-pair (denoted by $T_1 \sim T_2$) if

a) T_1 and T_2 have the same number of vertices, they are gracefully numbered (with the numbering functions g_1 and g_2 respectively);

b) A_{T_1} and A_{T_2} (consequently B_{T_1} and B_{T_2}) are numbered with the same numbers;

c) For all edges xy of T_1 and uv of T_2 with

$$x \in A_{T_1}, \quad u \in A_{T_2}, \quad y \in B_{T_1}, \quad v \in B_{T_2}$$

we have

$$g_1(x) - g_1(y) \neq (-1) . (g_2(u) - g_2(v)) .$$

Obviously $\sim$ is an equivalence relation on the graceful trees. The example on figure 2 shows two non isomorphic graceful graphs from an equivalence-class of $\sim$. It can be proved easily that two caterpillars, T_1, T_2 (a caterpillar is a tree which becomes a path after deleting the vertices of degree one together with the incident edges) form a beau-pair if T_1, T_2 and A_{T_1}, A_{T_2} have the same number of vertices.

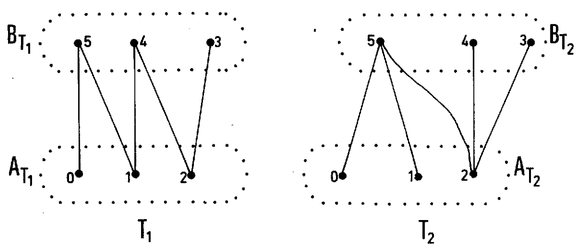

FIG. 2.

3. **A method to generate graceful trees.** — The following theorems present constructions to generate graceful trees from smaller ones. In this paragraph we assume that the graceful numbering of the graceful trees T and T_v is defined by the function g and g_v respectively.

Theorem 1. — *Let T be a gracefully numbered tree with k vertices, $T_0, T_1, \dots, T_{k-1}$ be gracefully numbered trees of m vertices satisfying*

$$T_v \sim T_{k-1-v} \quad \text{for} \quad 0 \leqslant v \leqslant k-1 .$$

The tree T^* we get by planting T_v for all v

$$(0 \leqslant v \leqslant k-1)$$

with its root into the vertex v of T satisfying $g(v) = v$ is graceful.

A slightly different result is when we do not plant anything into the vertex w of T with the maximal number :

Theorem 2. — *Let T be a gracefully numbered tree with k vertices $T_0, T_1, \dots, T_{k-2}$ be gracefully numbered trees of m vertices satisfying*

$$T_v \sim T_{k-2-v} \quad \textit{for} \quad 0 \leqslant v \leqslant k-2 .$$

The tree T^* we get by planting T_v for all v

$$(0 \leqslant v \leqslant k-2)$$

with its root into the vertex v of T satisfying $g(v) = v$ is graceful. Moreover T^* has a graceful numbering g^* satisfying $g^*(w) = 0$, where w is the vertex of T for which $g(w) = k - 1$.

The proofs are almost identical for the theorems therefore we give only the proof of theorem 1. The copy of T_i planted into a vertex of T will be refered as T_i, so we can say that every vertex of T^* belongs to exactly one T_i.

Proof. — Let us define g^* on vertices of T^* as follows :

$$g^*(v) = \begin{cases} g_i(v) + im & \text{if} \quad v \in A_{T_i} \\ g_i(v) + (k-1-i)\,m & \text{if} \quad v \in B_{T_i} \end{cases} .$$

We prove that g^* gives a graceful numbering on T^*.

I. $0 \leqslant g^*(v) \leqslant km - 1$ is obvious, we show that g^* is a one-to-one function. If

$$g^*(v_1) - g^*(v_2) = (g_i(v_1) - pm) - (g_j(v_2) - qm) = 0$$

then

$$g_i(v_1) - g_j(v_2) = (p - q)\,m .$$

The left side is divisible by m but its absolute value is less than m which implies $p = q$ and

$$g_i(v_1) = g_j(v_2) . \tag{1}$$

From the definition of g^* $i = j$ or $i = k - 1 - j$ follows which means $T_i \sim T_j$. Using the equality (1) and the property *b*) (Definition 2) only $T_i = T_j$ may be true, and because of g_i is a one-to-one function : $v_1 = v_2$.

II. We show that the weights of the edges of T^* are different in the numbering induced by g^*. From the definition of g^* we can easily derive that

$$g^*(x) - g^*(y) \equiv 0 \pmod m$$

if and only if the edge xy is in T. It is enough, therefore to investigate two cases :

Case A. — xy and uv are edges of T_v and T_μ. Assume that

$$g^*(x) - g^*(y) = g^*(u) - g^*(v) > 0 . \tag{2}$$

We show that the two edges are identical. If

$$g^*(x) = g_v(x) + pm , \qquad g^*(y) = g_v(y) + qm$$
$$g^*(u) = g_\mu(u) + rm , \qquad g^*(v) = g_\mu(v) + sm$$

then from (2)

$$p \geqslant q, \qquad r \geqslant s$$

and

$$(g_v(x) - g_v(y)) + (g_\mu(v) - g_\mu(u)) = \\ = (q - p + r - s)\,m .$$

The absolute value of the left side is less than $2\,m$, so $e = q - p + r - s$, being an even number

$$(q + p = r + s = k - 1) ,$$

must be 0, that is

$$g_v(x) - g_v(y) = g_\mu(u) - g_\mu(v) \tag{3}$$

and $p = q$. From the definition of g^* $v = \mu$ or $v = k - 1 - \mu$ so $T_v \sim T_\mu$ follows. Combining (3) with the property *c*) (Definition 2) we see that $x \in A_{T_v}$ if and only if $u \in A_{T_\mu}$. This means that $T_v = T_\mu$, and because of g_v is a graceful numbering on T_v, (3) implies :

$$x = u \quad \text{and} \quad y = v .$$

Case B. — Let xy be an edge of T, with $g(x) = i$, $g(y) = j$. We know that $g_i(x) = g_j(y) = 0$ and $x \in A_{T_i}$, $y \in A_{T_j}$

$$w^*_{xy} = | g^*(x) - g^*(y) | = | im - jm | = \\ = m \, | i - j | = m \, | g(x) - g(y) |$$

showing that $w^*_{xy} - s$ are different on different xy edges of T.

4. **Applications of the theorems.** — Figure 3 shows a tree on which the graceful numbering was supplied by theorem 1. The $T_i - s(T_0 \sim T_4, T_1 \sim T_3)$ and T can be seen on figure 4.

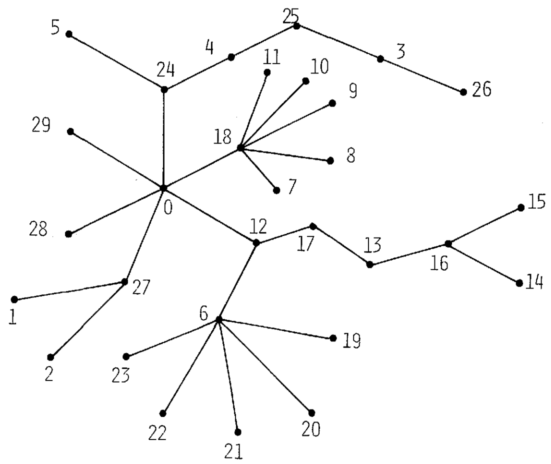

FIG. 3.

Repetitive application of the theorems with special graceful trees in the role of T and $T_v - s$ yields different graceful tree classes. Here we mention one interesting and simple case. Let us call a tree « symmetrical » if it is a rooted tree in which every level contains vertices of the same degree.

Corollary 1. — *Symmetrical trees have a graceful numbering with $g(x) = 0$ at the root x.*

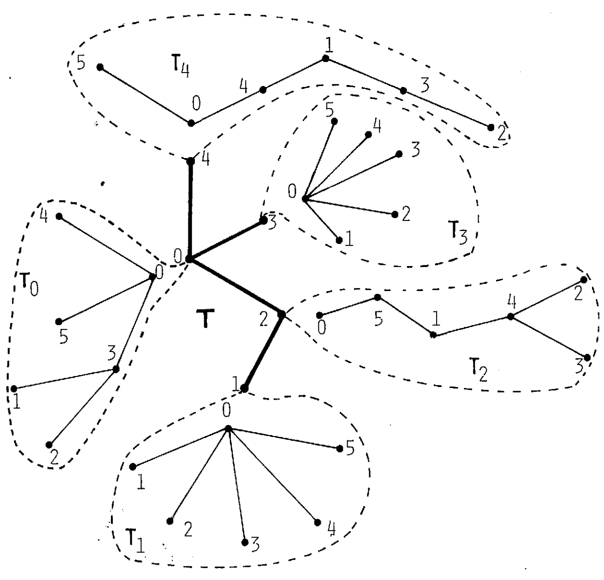

FIG. 4.

Proof. — Induction on the number of levels, denoted by l. The case $l = 2$ is simple. If $l \geqslant 3$, then the symmetrical tree can be derived by planting identical symmetrical trees of level $l - 1$ with their roots into all vertices of a star, except to the centre. Applying theorem 2 the proof is complete.

A complete k-nary tree of level l is a special symmetrical tree, where

$$\text{the degree of } v \begin{cases} k \text{ if } v \text{ is on level 1 (the root)} \\ k + 1 \text{ if } v \text{ is on level } 2, 3, \ldots, l - 1 \\ 1 \text{ if } v \text{ is on level } l. \end{cases}$$

Corollary 2 (cf. [3]). — *Complete k-nary trees are graceful.*

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References

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