*Colloques internationaux C.N.R.S.*
N° 260 - Problèmes combinatoires et théorie des graphes

# DIRECTED GRAPHS AND COMPUTER PROGRAMS

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**Résumé.** — La notion du graphe de flot (flow graph) joue un rôle important dans la théorie mathématique de la programmation. Un graphe orienté sans boucle ni arc multiple est appelé graphe de flot s'il a un sommet $S$ (la racine) tel qu'il part une chaîne de $S$ vers tout autre sommet. Nous considérons ici deux classes des graphes de flot :

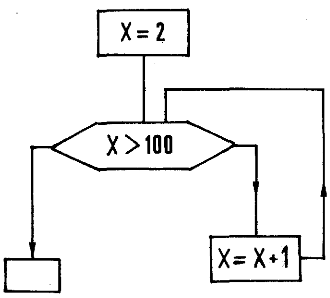
— les graphes de flot dans lesquels le demi-degré extérieur de tout sommet est au plus 2 (graphes de flot 2-bornés) ;

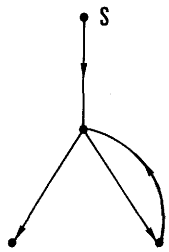
— les graphes de flot ou tout cycle élémentaire admet un seul nœud d'entrée (le sommet $v_C$ du cycle $C$ est appelé nœud d'entrée unique de $C$ si toute chaîne allant de la racine à $C$ traverse $v_C$).

Pour la première classe le problème suivant sera discuté : quelle est la condition nécessaire et suffisante pour qu'un graphe puisse être orienté de telle façon que le graphe obtenu soit un graphe de flot 2-borné ?

Pour les graphes de la deuxième classe nous donnons un théorème minimax qui montre que le système des cycles élémentaires d'un tel graphe forme un hypergraphe normal.

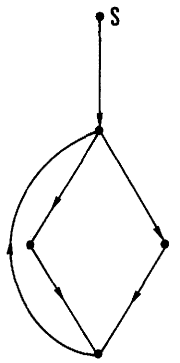
1. **Introduction.** — The notion of flow-graphs plays an important role in the mathematical theory of programming. *A flow-graph* is a directed graph without multiple edges and loops with a vertex $S$ (the root) from which any other vertex can be reached by a directed path. We remark that a flow-graph corresponds to a flowchart of a program in a natural way as shown below :

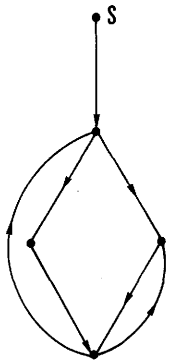



The present paper intends to draw attention to the investigations of properties of two classes of flow-graphs. The first class is where the outdegrees of any vertex is at most 2. We call such a flow-graph *2-bounded.* It is easy to show that every program can be written in such a way that the flow-graph associated with its flowchart is 2-bounded. Section 2 discusses the following problem : what is the necessary and sufficient condition under which an undirected graph can be oriented so that the resulting directed graph is a 2-bounded flow-graph. The problem is generalized to $k$-bounded flow-graphs.

The second class of flow-graphs is the *fully reducible* ones. The notion is due to Cocke and references [1]-[4] are discussing properties and applications of these flow-graphs. The fully reducible flow-graphs can be defined as flow-graphs where every elementary cycle $C$ has a unique entry vertex $V_C$ that is every path from $S$ to a vertex of $C$ has to contain $V_C$. From now on, *FRF* stands instead of fully reducible flow-graphs. The following figure displays two graphs, the first is a *FRF* and the second is not.



Section 3 is devoted to the problem of finding a minimal set of vertices in a *FRF* which, if removed, leaves the graph without cycles. A minimax theorem is proved stating that the cycles of an *FRF* form a normal hypergraph [5]. This problem, in general, is a hard one, the reader can find more material in [6], [7]. A conjecture is also stated for an analogous minimax problem.

2. **2-bounded flow-graphs.** — Here we are concerned with the structure of a 2-bounded flow-graph. Suppose that an undirected graph $G$ given with a special vertex $S$ and we want to orient the edges of $G$ so that it would form a 2-bounded flow-graph with root $S$. Let $V$ be a set of vertices in $G$ and denote by $C_{G-V}$ the number of components in $G - V$ which do not contain $S$. The set of edges in the subgraph spanned by $V$ is denoted by $\varepsilon(V)$.

**Theorem 1.** — *$G$ can be oriented so that it forms a 2-bounded flow-graph if and only if*

$$|\varepsilon(V)| + C_{G-V} \leqq 2\,|V| \quad \textit{for any} \quad V \subset V(G).$$

We can classify further the flowgraphs by introducing the $k$-bounded flowgraphs where the outdegree of any vertex is at most $k$. The case $k = 1$ is not interesting because a 1-bounded flow-graph must be very simple. Theorem 1 can be generalized to $k$-bounded flow-graphs as :

**Theorem 2.** — *$G$ can be oriented so that it forms a $k$-bounded flow-graph if and only if*

$$|\varepsilon(V)| + C_{G-V} \leqq k\,|V| \quad \textit{for any} \quad V \subset V(G).$$

A more general question is when we want an orientation of $G$ where the outdegree of a vertex $x$ is at most $f(x)$. ($f$ is a non negative integer function on the vertices.) In this case the necessary and sufficient condition looks like :

$$|\varepsilon(V)| + C_{G-V} \leqq \sum_{X \in V} f(x) \quad \text{for any} \quad V \subset V(G).$$

The proofs can be found in [8] with several other problems of similar nature. It would be interesting to study some problems in the theory of directed graphs for 2-bounded graphs (or flow-graphs). That class is large enough and in some cases the 2-boundedness does not make the problem any easier.

3. **Minimax-theorem for *FRF*.** — $\nu_V(G)$ is the maximum number of pairwise vertex-disjoint elementary cycles in $G$. $\tau_V(G)$ is the minimal cardinality of a vertex-set of $G$ which, if removed, leaves $G$ without elementary cycles.

**Theorem 3.** — $\nu_V(G) = \tau_V(G)$ *if $G$ is an FRF.*

*Proof.* — Let $G$ be an *FRF* and $T$ be a spanning tree of $G$ rooted at $S$. $T$ defines a partial ordering on the vertices of $G$. Any edge, $XY$ where $X < Y$ defines an elementary cycle with a path of $T$. Let $X_1\,Y_1$ be such an edge with the property that $Y_1$ is minimal. We prove that all cycles which have a common vertex which $C(X_1, Y_1)$ contain $Y_1$. Assume that $C'$ is such a cycle and let $Y$ be the largest vertex of $C'$ on the path $X_1\,Y_1$. Consider the edge $ZY$ of $C'$. If $Z > Y$ or $Z$ and $Y$ are incomparable then $Y$ is an entry vertex of $C(X_1, Y_1)$ which implies $Y_1 = Y$. If $Z < Y$ then the minimality of $Y$ implies $Y_1 = Y$. Therefore $Y_1$ is in $C'$ and the proof is finished. Deleting the edge $X_1\ Y_1$ with all other $XY_1$ edges for which $X < Y_1$ and contracting the path $Y_1\,X_1$ into one point we get $G'$ which is an *FRF* and $\nu_V(G') = \nu_V(G) - 1$. By induction

$$\nu_V(G') = \tau_V(G')$$

and $Y_1$ completes the « blocking-set » of $G'$.

Theorem 3 can be generalized (with essentially the same proof) as :

**Theorem 4.** — *The elementary cycles of an FRF form a normal hypergraph.*

Finally a conjecture is presented. $\nu_e(G)$ and $\tau_e(G)$ is defined on the analogy of $\nu_V(G)$ and $\tau_V(G)$ if the maximal number of *edge-disjoint* cycles are considered and an *edge-set* of minimal cardinality is required to « block » them.

**Conjecture.** — $\nu_e(G) = \tau_e(G)$ *if $G$ is an FRF.*

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