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A RAMSEY-TYPE PROBLEM IN DIRECTED AND BIPARTITE GRAPHS

by

A. GYÁRFÁS and J. LEHEL (Budapest)

Let k and l be natural numbers $k \geqq l$. It was proved in [1] that if the edges of a complete graph G of $k + \left[\frac{l+1}{2}\right]$ vertices are coloured with two colours (for example with red and blue) then G contains a red path of length k or a blue path of length l, and the number $k + \left[\frac{l+1}{2}\right]$ cannot be replaced by a smaller one. The complete graph in this theorem is undirected. In the present paper we investigate the cases when G is a tournament (directed asymmetric complete graph) or a directed symmetric complete graph, or a complete (undirected) bipartite graph.

With respect to tournaments our result is a generalization of a well-known theorem of L. RÉDEI [2].

THEOREM 1. *Let T be a tournament of $\prod_{i=1}^{n} k_i + 1$ vertices the edges of which are coloured with n different colours. Then for some i $(1 \leqq i \leqq n)$ T contains a path of length k_i every edges of which are coloured with the i-th colour.*

REMARK. The following example shows that Theorem 1 is the best possible. Let the vertices of the tournament T^* be the n-tuples $(t_1, t_2, \ldots, t_n)$ where the t_i-s are integers satisfying the inequalities $1 \leqq t_i \leqq k_i$. There is an edge from $(t_1, \ldots, t_n)$ to $(t_1', \ldots, t_n')$ coloured with the i-th colour if and only if $t_i < t_i'$ and $t_j = t_j'$ for $j < i$. T^* has $\prod_{i=1}^{n} k_i$ vertices and contains no path of length k_i coloured with the i-th colour.

PROOF of Theorem 1. We prove by induction on n.

(i) If $n = 1$ then Theorem 1 reduces to the theorem of RÉDEI.

(ii) Assuming that Theorem 1 holds for some n, let T be a tournament of $\prod_{i=1}^{n+1} k_i + 1$ vertices the edges of which is coloured with $(n + 1)$ colours. Consider the subgraph $\bar{T}$ of T spanned by the edges of the $(n + 1)$-th colour. If the graph $\bar{T}$ contains no path of length k_{n+1}, then according to a theorem of T.

[1] Now and henceforward the term path means elementary path.

GALLAI [3] $\overline{T}$ is k_{n+1} colourable that is the vertices of $\overline{T}$ can be divided into k_{n+1} classes so that the edges connect only the vertices from different classes. This means that T's vertices can be split into k_{n+1} subtournaments each of them coloured with the colours 1, 2, . . ., n. There exists a class of at least $\prod_{i=1}^{n} k_i + 1$ elements, so by the inductive hypothesis it contains for some i $(1 \leqq i \leqq n)$ a path of length k_i coloured with the i-th colour, so the theorem follows.

THEOREM 2. *Let G be a directed symmetric complete graph of $k+l-1$ $(k, l \geqq 2)$ vertices the edges of which are coloured with two colours. Then G contains a path of length k or a path of length l coloured with the first or the second colour, respectively.*

REMARK. If G^* is a directed symmetric complete graph of $k+l-2$ vertices, let $V(G^*) = A \cup B$ where $A \cap B = \emptyset$, $|A| = k-1$. The edge $(u, v) \in E(G^*)$ is coloured with the first (second) colour if and only if $u \in A$ $(u \in B)$.

This example shows that Theorem 2 is the best possible.

Theorem 2 follows at once from the following theorem of H. RAYNAUD [4].

THEOREM R. *If G is a directed symmetric complete graph the edges of which are coloured with two colours, then G contains a Hamiltonian circuit which is the union of two one-coloured paths.*

$G(m, n)$ will denote a bipartite (undirected) graph the vertices of which are divided in two classes of m and n elements respectively. (Edges connect only vertices from different classes.) The complement of $G(m, n)$ is the bipartite graph $\overline{G}(m, n)$ defined by the vertices of $G(m, n)$ and by the edges which connect vertices from different classes and are not contained in $G(m, n)$. $K(m, n) = G(m, n) \cup \overline{G}(m, n)$ is a complete bipartite graph. In the formulation of Theorem 3 we use the graph and its complement instead of the two colours.

THEOREM 3. *If k and l are odd natural numbers then $G\left(\frac{k+l}{2}, \frac{k+l}{2}\right)$ contains a path of length k or its complement contains a path of length l.*

REMARK 1. Similar theorems can be stated if at least one of k and l is even:

$$\text{if}\left\{\begin{array}{lll} & k \text{ is even},\ l = k+1 & G(k, k+1) \\ k < l-1; & k \text{ is even},\ l \text{ is odd} & G\left(\frac{k+l-1}{2}, \frac{k+l-1}{2}\right) \\ k < l; & k \text{ is odd},\ l \text{ is even} \quad \text{then} & G\left(\frac{k+l-1}{2}, \frac{k+l+1}{2}\right) \\ & k = l \quad k, l \text{ are even} & G(k-1, k+1) \\ & k \neq l \quad k, l \text{ are even} & G\left(\frac{k+l}{2}-1, \frac{k+l}{2}\right) \end{array}\right\} \text{contains}$$

a path of length k or its complement contains a path of length l. Their proofs can be given on the analogy of Theorem 3 so they are omitted.

REMARK 2. Theorem 3 is the best possible (as well as the modifications described in the previous remark). This is shown by an example. If $m < \frac{k+l}{2}$, let $A \cup B$ and C be the "upper" and "lower" classes of $G(m, n)$ respectively, where $|A| \leqq \frac{k-1}{2}$, $|B| \leqq \frac{l-1}{2}$, $|A \cup B| = m$, $|C| = n$. G contains the edges between A and C (Fig. 1). Since in $K(x, n)$ the length of the longest

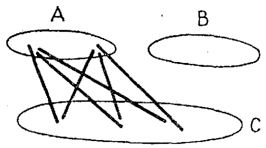


Fig. 1

path is less than or equal to $2x$, $G(m, n)$ (and $\bar{G}(m, n)$) contains no path of length k (of length l).

PROOF of Theorem 3. Let $G = G\left(\frac{k+l}{2}, \frac{k+l}{2}\right)$. We say that a sequence of distinct vertices $S = (A_1, \ldots, A_r, X, B_1, \ldots, B_s)$ of G is a bipath[2] of length $r+s$ if $(A_i, A_{i+1}) \in E(G)$ for $i = 1, \ldots, r-1$ $(A_r, X) \in E(G)$; $(X, B_1) \notin E(G)$ and $(B_i, B_{i+1}) \in E(G)$ for $i = 1, \ldots, s-1$ ($r = 0$ or $s = 0$ is permitted). A_1 and B_s are the endpoints of the bipath, X is the midpoint of it. The sequences $S_1 = (A_1, \ldots, A_r, X)$ and $S_2 = (X, B_1, \ldots, B_s)$ are called the branches of the bipath S. S is called Hamiltonian bipath if it contains every vertex of G.

Let S be a bipath of G the length of which is maximal. If it is Hamiltonian then the length of the branch S_1 is at least k or the length of the branch S_2 is at least l and the theorem follows. So we can assume that S is not a Hamiltonian bipath. Logically it can belong to one of the following three types:

(i) A_1 and B_s belong to different classes of $V(G)$ (Fig. 2).[3] Then we can

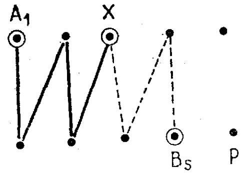


Fig. 2

[2] Sometimes S is also used to denote the set of vertices of the bipath.
[3] On our figures A_1 is always in the "upper" class.

select a vertex P from $V(G) - S$ in the class which does not contain X.

$$\text{If} \quad \begin{array}{l} (X, P) \in E(G) \\ (X, P) \notin E(G) \end{array} \quad \text{then} \quad \begin{array}{l} S' = (B_1, B_2, \ldots, B_s, A_1, A_2, \ldots, X, P) \\ S'' = (A_r, A_{r-1}, \ldots, A_1, B_s, \ldots, B_2, B_1, X, P) \end{array}$$

is a bipath the length of which is greater then $r + s$ and this contradicts to the maximality of S.

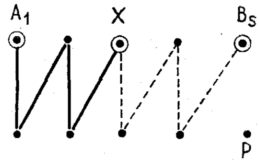


Fig. 3

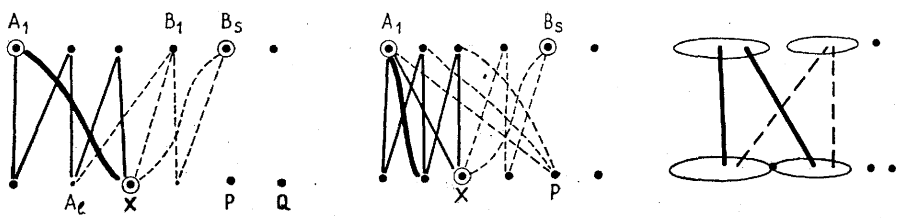
(ii) A_1, B_s and X belong to the same class of $V(G)$ (Fig. 3). Then we can select a vertex P from $V(G) - S$ in the "lower" class. $(P, B_s) \in E(G)$ $(P, A_1) \notin E(G)$ by the maximality of S.

$$\text{If} \quad \begin{array}{l} (X, P) \in E(G) \\ (X, P) \notin E(G) \end{array} \quad \text{then} \quad \begin{array}{l} S' = (A_1, A_2, \ldots, A_r, X, P, B_s, \ldots, B_1) \\ S'' = (B_s, B_{s-1}, \ldots, B_1, X, P, A_1, \ldots, A_r) \end{array}$$

is a bipath the length of which is greater then $r + s$ and this also contradicts to the maximality of S.

We conclude that if S is a maximal bipath then it belongs to the type:

(iii) A_1, B_s belong to the same class. X belongs to the "lower" one. In this case the lengths of the branches of S are odd numbers. If the length of S_1 is not less then k or the length of S_2 is not less than l, our theorem follows. Otherwise the length of S_1 (S_2) is not greater than $k - 2$ ($l - 2$) so the length of the bipath S is at most $k + l - 4$ that is S contains at most $k + l - 3$ vertices. So we can select two different vertices P, Q from the "lower" class of $V(G)$ so that $P, Q \notin S$ (Fig. 4). Now we assert that the graph spanned by S_1 is a complete bipartite graph, and the graph spanned by S_2 contains no edges.



(Case A and B) (Case C and D) (Case E and F)

Fig. 4

A. $(A_1, X) \in E(G)$ since in case of $(A_1, X) \notin E(G)$ the bipath $(A_r, A_{r-1}, \dots, A_1, X, B_1, \dots, B_s)$ would be one of maximal length and its midpoint A_1 is in the same class than its endpoints (cf. (ii)), this is a contradiction. $(X, B_s) \notin E(G)$ follows in the same way.

B. $(B_1, A_l) \notin E(G)$ (l is even). If we assume that $(B_1, A_l) \in E(G)$ for some l then the maximal bipath $(A_{l-1}, A_{l-2}, \dots, A_1, X, A_r, A_{r-1}, \dots, A_l, B_1, \dots, B_s)$ has its midpoint B_1 in the same class than its endpoints (cf. (ii)) and that leads to contradiction.

C. $(P, A_l) \notin E(G)$ for odd l. If $(P, A_l) \in E(G)$ then the length of the bipath $(P, A_l, A_{l-1}, \dots, A_1, X, A_r, \dots, A_{l+1}, B_1, \dots, B_s)$ would be greater than $r + s$.

D. The circuit $A_1 A_2 \dots A_r X A_1$ contains all edges of the form (A_j, A_{j+3}).[4] If for example $(A_j, A_{j+3}) \notin E(G)$ then the length of the bipath $(A_{j-1}, A_{j-2}, \dots, A_1, X, A_r, A_{r-1}, \dots, A_{j+4}, Q, A_{j+2}, P, A_j, A_{j+3}, B_1, B_2, \dots, B_s)$ would be greater than the length of S.

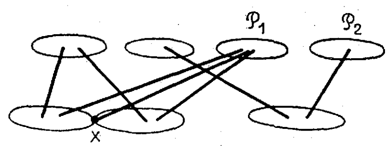
E. Let A_i and A_j be the vertices of S_1 belonging to different classes of $V(G)$. We prove that $(A_i, A_j) \in E(G)$. It is enough to consider the case when $i + 3 \leqq j$ and A_j is in the "upper" class. If $(A_i, A_j) \notin E(G)$ then the maximal bipath $(A_{i+1}, \dots, A_{j-2}, A_{j+1}, \dots, A_{r-1}, A_r, X, A_1, \dots, A_{i-2}, A_{i-1}, P, A_j, A_i, B_1, B_2, \dots, B_s)$ has its midpoint A_{i-1} in the same class of $V(G)$ than its endpoints (cf. (ii)) and this is a contradiction. $(A_{j+1}, A_{j-2}) \in E(G)$ by D and $(P, A_{i-1}), (P, A_j) \notin E(G)$ by C. So we conclude that the graph spanned by S_1 is a complete bipartite graph. The graph spanned by S_2 contains no edges, this can be proved by the same reasoning (applying propositions analogous to A, B, C, D, E).

The following assertions (included in F, G, H) can be easily checked.

F. $(A_i, B_j) \in E(G)$ if i is odd and j is even, $(A_i, B_j) \notin E(G)$ if i is even and j is odd. We split the vertices of $V(G) - S$ belonging to the same class of $V(G)$ as A_1 into two parts as follows:

$$P \in \mathcal{P}_1 \quad \text{if} \quad (X, P) \in E(G),$$

$$P \in \mathcal{P}_2 \quad \text{if} \quad (X, P) \notin E(G).$$



(Case G and H)

Fig. 5

G. (P, A_i) and $(P, B_i) \in E(G)$ for $P \in \mathcal{P}_1$ (i is even),

(P, A_i) and $(P, B_i) \notin E(G)$ for $P \in \mathcal{P}_2$ (i is even).

[4] j is considered to be equal to j' if $j \equiv j' \pmod{r+1}$, $X = A_{r+1}$.

H. $(P, R) \notin E(G)$ if $P \in \mathcal{S}_1$ and $R \in V(G) - (S \cup \mathcal{S}_1 \cup \mathcal{S}_2)$,

$(P, R) \in E(G)$ if $P \in \mathcal{S}_2$ and $R \in V(G) - (S \cup \mathcal{S}_1 \cup \mathcal{S}_2)$.

Considering the structure of the graphs spanned by S_1 and S_2 and applying the propositions F, G, H, C and the proposition analogous to C ($(P, B_l) \in E(G)$ if l is odd and $P \in V(G) - S$) we conclude that G can be written as the union of two complete bipartite graphs G_1 and G_2. It is easy to see that G_1 or G_2 has a path of length k, or the complement of G contains a path of length l. So Theorem 3 follows.

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MTA SZÁMÍTÁSTECHNIKAI KÖZPONTJA
H–1250 BUDAPEST
ÚRI U. 49.
HUNGARY