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A RAMSEY-TYPE THEOREM AND ITS APPLICATION TO RELATIVES OF HELLY'S THEOREM

by

A. GYÁRFÁS (Budapest)

Introduction

We say that H is a forbidden graph in the set of graphs $\mathfrak{D}$ if no $G \in \mathfrak{D}$ contains a spanned subgraph isomorphic to H.

Forbidden graphs play an important role in many problems of graph theory (interval-graphs, comparability graphs, perfect graphs, etc.). They relate closely to the "covering number" $\alpha(G)$ of a graph G ($\alpha(G)$ denotes the smallest integer k for which G's vertices can be covered with k complete subgraphs of G). Clearly $\alpha(G) \geqq \varphi(G)$ where $\varphi(G)$ denotes the maximal number of G's pairwise independent vertices. Generally $\alpha(G) > \varphi(G)$, moreover $\alpha(G)$ can be arbitrary large if $\varphi(G) \geqq 2$ is fixed ([1]). The graphs for which $\alpha(G) = \varphi(G)$ are investigated in many papers. According to a theorem of Hajnal and Surányi $\alpha(G) = \varphi(G)$ if $G \in \mathcal{G}$, where the n-gons ($n = 4, 5, \ldots$) are forbidden graphs in $\mathcal{G}$ ([2]).

If we take the assumption that for some k the complete k-graph is forbidden in the set of graphs $\mathcal{G}$, then $\alpha(G) \leqq C$ where C depends only on k and $\varphi(G)$ but does not depend on the number of G's vertices ($G \in \mathcal{G}$). This can be derived at once from the following special case of a well-known theorem of Ramsey ([3], $n = 2$, $k_1 = k$, $k_2 = \varphi(G) + 1$):

Theorem (Ramsey). *For every system of natural numbers $k_1, k_2, \ldots, k_n$ there exists a natural number N with the property: if we split the edges of a complete k-tuple G_n into n classes and G_n contains no complete k_i-tuple the edges of which are in the i-th class, then $k \leqq N$.*

Let G_n be a complete graph the edges of which are divided into n classes. Equivalently we can say that the edges of G_n are coloured with n colours. We define $\alpha(G_n)$ as the smallest number t for which $V(G_n)$ can be covered with t complete one-coloured subgraphs of G_n. It is easy to see that if G_n contains no complete k_i-graph, the edges of which are coloured with the i-th colour, then the statement $k \leqq N$ in Ramsey's theorem is equivalent with $\alpha(G_n) \leqq C$, where C depends only on $k_1, k_2, \ldots, k_n$. Theorem 1 provides more general conditions for the boundedness of $\alpha(G_n)$. In Ramsey's theorem the forbidden subgraphs are complete graphs, and we replace them by a more general class $\mathfrak{Q}$ of graphs defined in § 1. It is worth to mention that the class $\mathfrak{Q}$ includes all the forbidden graphs for which we can state that $\alpha(G_n)$ is bounded.

We note that an edge of G_n can be coloured with more than one colour. This is an essential and natural assumption since the forbidden graphs are not the complete graphs only.

In § 2 we formulate a geometrical consequence of Theorem 1 which can be considered as a Helly-type theorem. It arose in connection with a problem of T. GALLAI and this problem was the starting point of the investigations of the present paper.

§ 1

Let $G = (V(G); E(G))$ be a graph. (Here, and in what follows every graph is undirected, loops and multiple edges are not permitted.) The graph $H = (V(H); E(H))$ is said to be a subgraph of G if $V(H) \subset V(G)$ and $(x, y) \in E(H)$ if and only if $(x, y) \in E(G)$ that is H is considered as a subgraph of G if and only if H is "spanned" by some subset of $V(G)$. To indicate that H is a subgraph of G we use the symbol $H \subset G$.

Let $C = \{c_1, c_2, \ldots, c_n\}$ be a set, the elements of C will be called colours. We say that a graph G is coloured with the colours $c_1, c_2, \ldots, c_n$ if we have a function $\mathcal{C}$ on $E(G)$ to the set of non-empty subsets of C. In other words we can say that every edge of G is coloured with at least one colour chosen from the set C. The value of the function $\mathcal{C}$ at the edge (x, y) will be denoted by $\mathcal{C}(x, y)$, that is $\mathcal{C}(x, y)$ denotes the colours with which the edge (x, y) is coloured. If G is a coloured graph and $G' \subset G$ then G' can be considered also as a coloured graph.

If the graph G is coloured with $c_1, c_2, \ldots, c_n$ then $G(c_i)$ denotes the graph the vertex set of which is $V(G)$ and $(x, y) \in E(G(c_i))$ if and only if $c_i \in \mathcal{C}(x, y)$. The set of complete graphs coloured with n colours is denoted by $\mathcal{K}_n$

Let G be a graph coloured with $c_1, c_2, \ldots, c_n$. We say that G is covered with the complete graphs $G_1, G_2, \ldots, G_p$ if $V(G) = \bigcup_{j=1}^{p} V(G_j)$ and for all j there exists an i so that $G_j(c_i)$ is a complete graph. $\alpha(G)$ will denote the smallest integer k for which G can be covered with k complete graphs.

Now we define a family $\mathfrak{Q}$ of graphs as follows: let k, l be non-negative integers, at least one of them differs from 0. The graph Q_k^l is defined in the following way: $|V(Q_k^l)| = 2k + l$ and the complement of Q_k^l contains k edges no two of them have a common endpoint. The graphs Q_0^l (the complete l-tuple) and Q_k^0 will be denoted shortly by Q^l and Q_k. The set of graphs in the form Q_l^k will be denoted by $\mathfrak{Q}$. Some members of $\mathfrak{Q}$ can be seen in the following figure.

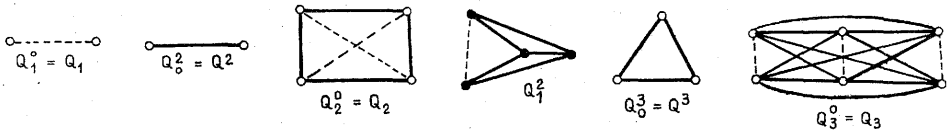

Fig. 1

An important property of $\mathfrak{Q}$ is that if $A \in \mathfrak{Q}$ and $B \subset A$ then $B \in \mathfrak{Q}$ $(B \neq \emptyset)$. Now we can formulate Theorem 1 and Theorem 2.

THEOREM 1. *For every system of graphs $G_1, G_2, \ldots, G_n \in \mathfrak{Q}$ there exists an $\alpha_0 = \alpha_0(G_1, G_2, \ldots, G_n)$ with the property: if the complete graph G is coloured with $c_1, c_2, \ldots, c_n$ and $\alpha(G) > \alpha_0$ then for some i $(1 \leqq i \leqq n)$ $G(c_i)$ has a subgraph isomorphic to G_i.*

REMARK 1. It is worth to mention that Theorem 1 is the best possible in the following sense: if for some family $\mathfrak{R}$ of graphs Theorem 1 holds then $\mathfrak{R} \subset \mathfrak{Q}$. We prove that at the end of the paper.

REMARK 2. If we want to formulate a theorem on the analogy of Theorem 1 for k-graphs then the family $\mathfrak{Q}$ will be smaller: in case of k-graphs it contains only the complete graphs and the empty graph of k points. (This shows that Ramsey's theorem can not be stated in case of k-graphs.)

COROLLARY (Ramsey's theorem [3]). *For every system of natural numbers $k_1, k_2, \ldots, k_n$ there exists an $n_0 = n_0(k_1, k_2, \ldots, k_n)$ with the property: if the complete graph G is coloured with $c_1, c_2, \ldots, c_n$ and $|V(G)| > n_0$ then for some i $G(c_i)$ contains a complete k_i-graph.*

PROOF. If $G_i = Q^{k_i}$ in Theorem 1 then the number $n_0 = \alpha_0(Q^{k_1}, \ldots, Q^{k_n}) \times \times \max\limits_{1 \leqq i \leqq n} k_i$ has the required property.

The set of vertices $H \subseteq V(G)$ is said to be independent if for all $x, y \in H$ $(x, y) \notin E(G)$. Theorem 1 can be stated in the following form:

THEOREM 2. *For every system of graphs $G_1, G_2, \ldots, G_n \in \mathfrak{Q}$ and for every natural number t there exists a natural number $\alpha_0 = \alpha_0(G_1, G_2, \ldots, G_n, t)$ with the property: if G is a graph coloured with $c_1, c_2, \ldots, c_n$ and no $t+1$ vertices of G are independent, moreover $\alpha(G) > \alpha_0$ then for some i $(1 \leqq i \leqq n)$ $G(c_i)$ contains the graph G_i.*

PROOF. If we colour every edge of G's complement with c_{n+1} we can apply Theorem 1 with $G_{n+1} = Q^{t+1}$ and it is obvious that the number $\alpha_0(G_1, \ldots, G_n, t) = t . \alpha_0(G_1, G_2, \ldots, G_n, Q^{t+1})$ has the required property.

It is worth to mention the special case, when $n = 1$.

COROLLARY 1. *Let $H \in \mathfrak{Q}$ and t be a natural number. There exists a natural number $n_0 = n_0(H, t)$ with the property: if G is a graph which does not contain H and no $t+1$ vertices of G are independent, then $\alpha(G) \leqq n_0$.*

Finally we formulate Corollary 1 in the following form:

COROLLARY 2. *Let r, s be natural numbers, $s \geqq 2$. There exists a natural number $n_0 = n_0(r, s)$ so that if a graph G contains no complete r-tuple and the complement of G contains no Q_s then $\chi(G) \leqq n_0$, where $\chi(G)$ denotes the chromatic number of G.*

§ 2

Let $\mathcal{A}$ be a family of subsets of a set E. We will denote by $\gamma(\mathcal{A})$ the smallest integer k for which the following assertion holds: if we have an arbitrary finite system $\mathcal{F} \subset \mathcal{A}$ any two members of it having non-empty intersection then $\mathcal{F}$ can be split into k (or fewer) subfamilies each having non-empty intersection. Let $\gamma(\mathcal{A}) = \infty$ if no such a k exists. We can say that $\gamma(\mathcal{A})$ is the smallest number of needles required to pierce all members of any finite family of pairwise intersecting sets in $\mathcal{A}$ (cf. [4], p.128).

The following problem was posed by T. GALLAI: Let $E_1, E_2, \ldots, E_n$ denote n pairwise disjoint copies of the real line R^1. An n-interval is a set which is expressible as the union of n closed intervals of $E_1, E_2, \ldots, E_n$ respectively. The set of n-intervals is denoted by $\mathfrak{I}_n$. The problem is: to find the numbers $\gamma(\mathfrak{I}_n)$ $(n = 1, 2, \ldots)$.

GALLAI's problem can be formulated in an other form which is a new variant of the problems concerning the existence of common transversals (cf. [4], p.129—132).

We call an $n-1$ dimensional hyperplane A of R^n a p-transversal if A is perpendicular to a coordinate axis. $\mathcal{S}^n$ will denote the family of parallelotopes in R^n with edges parallel to the coordinate axes. It is easy to see that $\gamma(\mathfrak{I}_n)$ is the smallest integer k for which the following assertion holds: if $\mathcal{F} \subset \mathcal{S}^n$ is an arbitrary finite family any two members of which admit a common p-transversal then $\mathcal{F}$ can be split into k (or fewer) subfamilies each of them admits a common p-transversal.

Clearly $\gamma(\mathfrak{I}_1) = 1$ (Helly's theorem) and the results $\gamma(\mathfrak{I}_2) = 2$, $\gamma(\mathfrak{I}_3) = 4$ are proved in [5]. It seems to be difficult to find $\gamma(\mathfrak{I}_n)$ for $n \geqq 4$.

Now we introduce the sum of general families. Let $E_1, E_2, \ldots, E_n$ be pairwise disjoint sets and $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be families of subsets of $E_1, E_2, \ldots, E_n$ respectively. Let $E = \bigcup_{i=1}^{n} E_i$ and $\mathcal{A} = \{A \subset E : A_i = \bigcup_{i=1}^{n} A_i, A_i \in \mathcal{A}_i\}$. The family $\mathcal{A}$ is said to be the sum of the families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and it is denoted by $\sum_{i=1}^{n} \mathcal{A}_i$. Because of the disjointness of the sets $E_1, E_2, \ldots, E_n$, two members of $\sum_{i=1}^{n} \mathcal{A}_i$ have common points if and only if for at least one i $(1 \leqq i \leqq n)$ the i-th "components" have common points.

A graph G is called forbidden in the family $\mathcal{A}$ if G's vertices can not be represented by $\mathcal{A}$'s members, that is $V(G)$ can not be injected to $\mathcal{A}$ so that the images of $x, y \in V(G)$ are intersecting if and only if $(x, y) \in E(G)$. Using the well-known notion of the intersection graphs, we can say that G is forbidden in $\mathcal{A}$ if G is not an intersection graph of $\mathcal{A}$.

Now we can obtain a sufficient condition for $\gamma(\sum_{i=1}^{n} \mathcal{A}_i) < \infty$ from Theorem 1 (apart from some trivial cases it is easy to see that $\gamma(\mathcal{A}_i) < \infty$ $(i = 1, 2, \ldots, n)$ is a necessary condition for the finiteness of $\gamma(\sum_{i=1}^{n} \mathcal{A}_i)$).

THEOREM 3. *Let* $Q_{k_i}^{l_i} \in \mathfrak{Q}$ *be a forbidden graph in* $\mathcal{A}_i$ *and* $\gamma(\mathcal{A}_i) < \infty$ $(1 \leqq i \leqq n)$ *then* $\gamma(\sum_{i=1}^{n} \mathcal{A}_i) < \infty$.

PROOF. Let $\mathcal{F} \subset \sum_{i=1}^{n} \mathcal{A}_i$ be an arbitrary finite system any two members of it having non-empty intersection. The elements of $\mathcal{F}$ can be set into an one-to-one correspondence to the points of a graph G. Two points $a, b \in V(G)$ are connected with an edge coloured with i if and only if the corresponding elements A, B of $\sum_{i=1}^{n} \mathcal{A}_i$ have common points in their i-th component. So $G \in \mathcal{K}_n$ and Theorem 1 guarantees that G can be covered with α_0 complete graphs and since $\gamma(\mathcal{A}_i) < \infty$ the proof is complete.

A family of sets is said to be t-independent if the maximal number of its pairwise disjoint sets is t. Clearly the 1-independence means that any two sets have non-empty intersection. We denote by $\gamma_t(\mathcal{A})$ the smallest integer k for which any finite t-independent family $\mathcal{F} \subset \mathcal{A}$ can be split into k (or fewer) subfamilies each of them having non-empty intersection. It is obvious that $\gamma_1(\mathcal{A}) = \gamma(\mathcal{A})$. Now Theorem 2 leads to a generalization of Theorem 3:

THEOREM 4. *If* $Q_{k_i}^{l_i} \in \mathfrak{Q}$ *is a forbidden graph in* $\mathcal{A}_i$ *and* $\gamma(\mathcal{A}_i) < \infty$ $(i = 1, 2, \ldots, n)$ *then* $\gamma_t(\sum_{i=1}^{n} \mathcal{A}_i) < \infty$ *for every natural number* t.

Now we describe families for which Theorem 3 and Theorem 4 can be applied.

(i) Let $\mathfrak{J}$ be the family of all closed intervals of the real line R^1. $\gamma(\mathfrak{J}) = 1$ and it is easy to see that the graph Q_2 is forbidden in $\mathfrak{J}$ (cf. [6]). (If $\mathcal{A}_1 = \mathcal{A}_2 = \ldots = \mathcal{A}_n = \mathfrak{J}$ then $\gamma_t(\Sigma \mathcal{A}_i) = \gamma_t(\mathfrak{J}_n) < \infty$ from Theorem 4.)

(ii) We denote by $\mathcal{P}^m$ (as above) the family of parallelotopes in R^m with edges parallel to the coordinate axes. It can be seen that the graph Q_{m+1} is forbidden in $\mathcal{P}^m$ but Q_m is not (cf. [7]). $\gamma(\mathcal{P}^m) = 1$ is obvious.

(iii) Let T be a tree (connected graph without circuit) and the family of the subtrees of T will be denoted by $\mathcal{F}$. It is easy to prove that $\gamma(\mathcal{F}) = 1$ and the graph Q_2 is forbidden in $\mathcal{F}$.

(iv) Let K be a convex polygon in R^2. The family of all polygons which can be obtained as the image of K under a positive homothety will be denoted by $\mathcal{H}(K)$. It is well-known that $\gamma(\mathcal{H}(K)) < \infty$ ([4]). We prove that for some $k = k(K)$ the graph Q_k is forbidden in $\mathcal{H}(K)$. In the proof we use the following simple

PROPOSITION. *If K_1 and K_2 are two convex polygons in R^2 and $K_1 \cap K_2 = \emptyset$ then K_1 and K_2 can be separated by a line L which is parallel to an edge of K_1 or to an edge of K_2* ([8]).

Let us denote by $d_1, d_2, \ldots, d_r$ the directions determined by the edges of K. We assert that the graph Q_{r+1} is forbidden in $\mathcal{H}(K)$. Assume, on the contrary that there exist elements $A_1, A_2, \ldots, A_{r+1}, B_1, B_2, \ldots, B_{r+1}$ in $\mathcal{H}(K)$ for which $A_i \cap B_i = \emptyset$ $(i = 1, 2, \ldots, r+1)$ and $A_i \cap B_j \neq \emptyset$ $(i \neq j)$. Let A_i^m and B_i^m denote the projections of A_i and B_i to the line e_m, where e_m is perpendicular to the direction d_m. Since $A_i \cap B_i = \emptyset$ therefore the proposition guarantees that for all i there exists an $m(i)$ for which $A_i^{m(i)} \cap B_i^{m(i)} = \emptyset$. This implies the existence of an i and of a j $(i \neq j)$ for which $m(i) = m(j) = m_0$, that is $A_i^{m_0} \cap B_i^{m_0} = \emptyset$, $A_j^{m_0} \cap B_j^{m_0} = \emptyset$ but $A_i^{m_0} \cap A_j^{m_0}$, $A_i^{m_0} \cap B_j^{m_0}$, $B_i^{m_0} \cap A_j^{m_0}$, $B_i^{m_0} \cap B_j^{m_0}$ are obviously nonempty sets, and this is a contradiction since the graph Q_2 is forbidden in $\mathcal{I}$ (cf. (i)). We note that if $\mathcal{C}$ denotes the set of circles of radius 1 in the plane then Q_k is not forbidden in $\mathcal{C}$. This can be seen from the following example: let the centres of the $2k$ circles be the vertices of a regular $2k$-gon inscribed to a circle of radius $1-\varepsilon$ (ε is a sufficiently small positive number).

(v) Let $\mathcal{L}$ be a family of lines in R^3, no three of them lying in a plane. It is well-known that $\gamma(\mathcal{L}) = 1$ and the graph Q_1^2 is obviously forbidden in $\mathcal{L}$.

(vi) Let $\mathcal{A}_i$ be a family of subsets of the set E_i $(i = 1, \ldots, s)$. The family $\mathcal{A} = \{A \subset \mathop{\times}\limits_{i=1}^{s} E_i : A = \mathop{\times}\limits_{i=1}^{s} A_i, A_i \in \mathcal{A}_i\}$ is said to be the Cartesian product of the families $\mathcal{A}_i$ and it is denoted by $\mathop{\times}\limits_{i=1}^{s} \mathcal{A}_i$. It is easy to prove that if $\gamma(\mathcal{A}_i) < \infty$ and some graph $G \in \mathcal{Q}$ is forbidden in $\mathcal{A}_i$ $(i = 1, 2, \ldots, s)$ then the family $\mathop{\times}\limits_{i=1}^{s} \mathcal{A}_i$ inherits these properties. So starting from the examples previously given, we can obtain further examples of families which satisfy the assumptions of Theorem 3 and Theorem 4.

Theorem 4 (and so Theorem 3) does not provide necessary condition for $\gamma_i (\sum_{i=1}^{n} \mathcal{A}_i) < \infty$. This is shown by the following example. Let $\mathcal{I}(s)$ denote the family of all non-empty subsets of R^1 which are the union of s or fewer closed intervals. J. LEHEL and I proved that $\gamma(\mathfrak{I}(s)) < \infty$ ([5]). On the other hand it is easy to see that no member of $\mathfrak{Q}$ is forbidden in $\mathfrak{I}(s)$ if $s \geq 2$, but $\gamma_t(\sum_{i=1}^{n} \mathfrak{I}(s_i)) < \infty$ for all $s_1, s_2, \ldots, s_n, t$.

Now we turn to the proof of Theorem 1. It is sufficient to prove Theorem 1 in case of $G_1, G_2, \ldots, G_n \in \{Q_k\}_{k=1}^{\infty}$ since for every graph $G \in \mathfrak{Q}$ there exists an integer k so that $G \subset Q_k$ (actually $Q_k^l \subset Q_{k+l}$). Therefore we may assume that $G_1 = Q_{k_1}, \ldots, G_n = Q_{k_n}$.

We prove by induction on $k_1, k_2, \ldots, k_n$. If $k_i = 1$ for some i then Theorem 1 holds for $\alpha_0(k_1, k_2, \ldots, k_n) = 1$.

Let us suppose that Theorem 1 holds for the systems $Q_{k_1}, Q_{k_2}, \ldots, Q_{k_{i-1}}$ $Q_{k_i-1}, Q_{k_{i+1}}, \ldots, Q_{k_n}$ $(k_i \geq 2,\ 1 \leq i \leq n)$ that is there exists the number $\alpha_0^i = \alpha_0^i(Q_{k_1}, \ldots, Q_{k_{i-1}}, Q_{k_i-1}, Q_{k_{i+1}}, \ldots, Q_{k_n})$ for all i $(1 \leq i \leq n)$. The proof is based on the existence of certain types of subgraphs in G.

Let C be a set of colours. We define the C-property for coloured graphs in the following way:

(i) If $C = \{c\}$ then the graph H enjoys the C-property if $V(H) = \{p, q\}$ and $c \notin \mathfrak{C}(p, q)$.

(ii) Let $C = \{c_1, c_2, \ldots, c_m, c_{m+1}\}$. We say that the coloured graph H enjoys the C-property if there exists an $x_H \in V(H)$ so that the graph $H - x_H$ can be written in the form $\bigcup_{i=1}^{m+1} T_i$, where the graph T_i enjoys the $C - \{c_i\}$-property and $c_i \notin \mathfrak{C}(x_H, y)$ if $y \in V(T_i)$.

It is clear from the definition that if H enjoys the C-property then $|V(H)| \leq K$, where K is a constant depending only on $|C|$. Now we formulate a simple lemma concerning the C-property.

LEMMA. *Let $G = H \cup \{y\}$ be a coloured graph, the edges between H and $\{y\}$ are coloured with $c_1, \ldots, c_n$ and assume that H enjoys the C-property $(C = \{c_1, \ldots, c_n\})$. Then for some c_j and $p, q \in V(H)$ $c_j \in \mathfrak{C}(y, p) \cap \mathfrak{C}(y, q)$ and $c_j \notin \mathfrak{C}(p, q)$.*

PROOF. We prove the lemma by induction on $|C|$. If $|C| = 1$ the lemma obviously true. Assume the lemma to be true for $|C| = n - 1$ and let $H = x_H \cup \bigcup_{i=1}^{n} T_i$ and $c_{i_0} \in \mathfrak{C}(y, x_H)$. If $c_{i_0} \in \mathfrak{C}(y, q)$ for some $q \in V(T_{c_{i_0}})$ then $c_{i_0}, p = x_H, q$ have the desired property. So we can assume that $c_{i_0} \notin \mathfrak{C}(y, q)$ for all $q \in V(T_{c_{i_0}})$. Now we can apply the inductive hypothesis for the graph $G' = T_{c_{i_0}} \cup \{y\}$ and the lemma follows.

We continue the proof of Theorem 1. Let G be a graph coloured with $c_1, c_2, \ldots, c_n$ and suppose that $Q_{k_i} \not\subseteq G(c_i)$. We investigate two cases.

1. G contains a subgraph H enjoying the C-property, where $C = \{c_1, c_2, \ldots, c_n\}$. Let $p, q \in V(H)$. We define the graph $A^i_{p,q}$ as the subgraph of G spanned by the set

$$\{y \in V(G) - V(H) \colon c_i \in \mathcal{C}(y, p) \cap \mathcal{C}(y, q),\ c_i \notin \mathcal{C}(p, q)\}\,.$$

The graph $A^i_{p,q}$ can be covered with α^i_0 (or fewer) complete graphs by the inductive hypothesis since $A^i_{p,q}\,(c_j)$ does not contain Q_{k_j} for $j \neq i$ and $A^i_{p,q}(c_i)$ does not contain Q_{k_i-1}. (If $Q_{k_i-1} \subset A^i_{p,q}(c_i)$ then the subgraph of G spanned by the set $V(Q_{k_i-1}) \cup \{p \cup q\}$ would contain Q_{k_i}.) Moreover, the lemma induces that $\bigcup_{\substack{1\le i\le n\\ p,q\in V(H)}} V(A^i_{p,q}) = V(G) - V(H)$ and as it has been already mentioned $|V(H)| \le K(n)$, so the number of the different $A^i_{p,q}$-s is at most $n\binom{K(n)}{2}$. Therefore the covering of G $G = \bigcup_{p_j\in V(H)} \{p_j\} \cup \bigcup_{\substack{1\le i\le n\\ p,q\in V(H)}} A^i_{p,q}$ consists of at most $K(n) + n\binom{K(n)}{2} \max_i \alpha^i_0 = N_1$ complete graphs.

2. We assume that G contains no subgraph with the C-property. In this case the existence of $\alpha_0(k_1, k_2, \ldots, k_n)$ can be proved by induction on n. The case $n = 1$ is trivial. Suppose that the assertion is true for $n-1$, we define L_{c_i} as follows:

$L_{c_i} = \{x \in V(G) \colon \{y \in V(G) \colon c_i \notin \mathcal{C}(x, y)\}$ contains no subgraph with the $C - \{c_i\}$-property$\}$.

Clearly $\bigcup_{i=1}^{n} L_{c_i} = V(G)$ since G contains no subgraph with the C-property. We may assume that there exist $p_1, p_2 \in L_{c_i}$ for which $C_i \notin \mathcal{C}(p_1, p_2)$. We divide L_{c_i} into three parts:

$$X^1_{c_i} = \{x \in L_{c_i} : c_i \notin \mathcal{C}(p_1, x)\},\ X^2_{c_i} = \{x \in L_{c_i} : c_i \notin \mathcal{C}(p_2, x)\},\ Y_{c_i} = L_{c_i} - (X^1_{c_i} \cup X^2_{c_i}).$$

The graph spanned by Y_{c_i} can be covered with α^i_0 complete graphs by the inductive hypothesis concerning $k_1, k_2, \ldots, k_n$. Now we consider the graphs spanned by the sets $X^1_{c_i}$ and $X^2_{c_i}$. In consequence of the definition of L_{c_i} neither of them contains subgraphs with the $C - \{c_i\}$ property. So we can apply the inductive hypothesis concerning n since $|C - \{c_i\}| = n - 1$. We conclude that $\alpha(G) \le N_2$ where N_2 depends only on the α^i_0-s and n.

So we can define $\alpha_0 = \alpha_0(Q_{k_1}, Q_{k_2}, \ldots, Q_{k_n}) = \max(N_1, N_2)$ and the proof of Theorem 1 is complete.

Finally we prove the assertion stated in the first remark after Theorem 1. Let $\mathfrak{R}$ be a set for which Theorem 1 holds, and assume that $\mathfrak{R} - \mathfrak{Q} \neq \emptyset$.

Let $H \in \mathfrak{R} - \mathfrak{Q}$. In the complement of H there exists two edges with common endpoints. Thus H contains at least one of the graphs shown in the following figure:

Fig. 2

We prove that Theorem 1 does not hold if $n = 2$ and $G_1 = G_2 = H_1$. Let us define the graph $G'_k \in \mathfrak{K}_2$ in the following way: $V(G'_k) = \{a_{i,j}\}_{i,j=1}^{k}$ The edge connecting the points a_{ij} and a_{mn} is coloured with c_1 if $i \neq m$ and coloured with c_2 if $j \neq n$. It is easy to see that $G'_k\ (c_i)$ $(i = 1,2)$ does not contain H_1 and $\alpha(G'_k) = k$.

We prove that Theorem does not hold if $n = 2$ and $G_1 = G_2 = H_2$. We define the graph $G''_k \in \mathfrak{K}_2$ in the following way: let S_k be a graph containing no triangles and the chromatic number of which is k. (Such graph exists, cf., for example, [1].) Let $|S_k| = n_k$, and A be a copy of S_k. Let us replace the vertices of A by $B_1, B_2, \ldots, B_{n_k}$ where B_i is a copy of S_k. All edges between B_i and B_j are coloured with c_1 if the corresponding vertices of A are connected by an edge. The edges of B_i are coloured with c_2 and the remaining pairs of points are connected with edges coloured with both c_1 and c_2. The graph G''_k obtained in this way enjoys the property: $G''_k\ (c_i)$ does not contain H_2 $(i = 1, 2)$, $\alpha(G''_k) = k$.

So we got a contradiction since H contains at least one of the graphs H_1, H_2.

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MTA SZÁMÍTÁSTECHNIKAI KÖZPONTJA
H–1250 BUDAPEST
ÚRI U. 49.
HUNGARY