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A Helly-type problem in trees

by

A. Gyárfás and J. Lehel
Budapest, Hungary

In our paper we present some results in connection with a problem of T. GALLAI.

$R_1, R_2, \dots, R_c$ will denote c distinct parallel lines in the plane. Let I_k be a closed interval of R_k, then the set $A = \bigcup_{k=1}^{c} I_k$ is said to be a c-interval. I_k is the k-th component of A.

Let $\mathcal{A} = \{A^\nu\}$ be an arbitrary finite family of c-intervals, any two of them having common points. It is well known that in case $c = 1$ the set $\bigcap_{A^\nu \in \mathcal{A}} A^\nu$ is non-empty, or which means the same, there exists a point p so that $p \cap A^\nu \neq \phi$ $(A^\nu \in \mathcal{A})$. (This is Helly's theorem in one-dimension.)

T. GALLAI has posed the problem for c-intervals: to find the least integer $\ell(c)$ for which there is a set $P \subset \bigcup_{k=1}^{c} R_k$ of $\ell(c)$ points that $P \cap A^\nu \neq \phi$ $(A^\nu \in \mathcal{A})$. We may assume that P consists of endpoints of the components of A^ν-s.

In the first part we prove the existence of $\ell(c)$, (Theorem 1.) and we show that $\ell(2) = 2$, $\ell(3) = 4$ (Theorem 2. and Theorem 4.) Theorem 2. was proved by J. Surányi and L. Surányi, independent from us.

Now we formulate the problem which we are dealing with in the second part: instead of distinct lines we suppose that $R_1, R_2, \dots, R_c$ coincide. We define $\ell^*(c)$ in this case on the analogy of $\ell(c)$. It is clear that $\ell(c) \leq \ell^*(c)$. We will prove that $\ell^*(c)$ exists, (Theorem 5.) and $\ell^*(2) = 3$ (Theorem 6.)

Replacing the lines by trees (a tree is a connected graph without circuits) and the intervals by subtrees, all the theorems mentioned above remain true. In particular, if every tree is a path, our problem is equivalent to the original one. This generalization was suggested by L. Lovász, who proved Theorem 2. in this form.

For the sake of simplicity we only sketch the proofs for trees unless they demand different methods as in case of Theorem 6.

I.

Without restriction of generality we may assume that the c-intervals have no common endpoints.

A family of sets is said to be t-independent if the maximal number of its pairwise disjoint sets is t. A family of pairwise intersecting sets is clearly 1-independent. For t-independent c-intervals $\ell_t(c)$ is defined similar to $\ell(c)$. It is obvious that $\ell_1(c) = \ell(c)$. $\ell_t(1) = t$ according to a theorem of Hajnal and Surányi. [1]

The existence of $\ell_t(c)$ follows from

THEOREM 1. $\ell_t(c) \leq \ell_T(c-1) + t$ where $T = [(t+1)^{c-1} - 1] \cdot t$

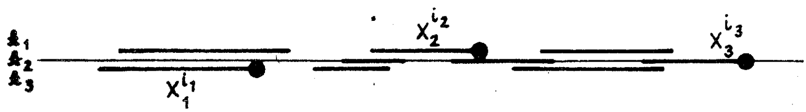
In the proof we use the following

LEMMA: let $\mathfrak{K}_1 = \{X_j^1\}_{j=1}^{t^c}$, $\mathfrak{K}_2 = \{X_j^2\}_{j=1}^{t^c}$, $\dots$, $\mathfrak{K}_t = \{X_j^t\}_{j=1}^{t^c}$

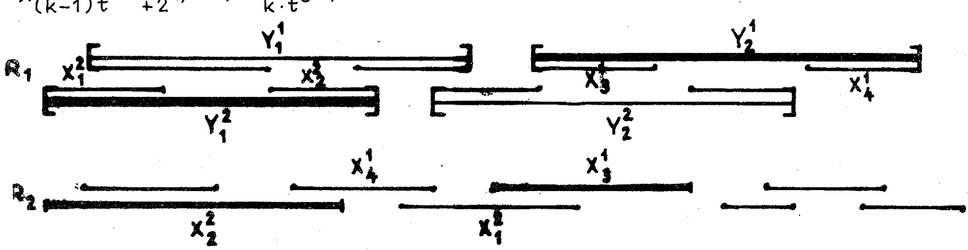
be systems of pairwise disjoint c-intervals. Then we can choose from $\mathfrak{K}_1, \mathfrak{K}_2, \dots, \mathfrak{K}_t$ the c-intervals $X_{j_1}^1, X_{j_2}^2, \dots, X_{j_t}^t$ which are also pairwise disjoint.

PROOF: we prove by induction on c.

(i) In case of $c=1$ we may assume the disjoint (1-)intervals to follow one another from left to the right. Let us choose the index $i_1 \in \{1,2,\dots,t\}$ for which the right endpoint of $X_1^{i_1}$ is to the extreme left. We continue the process by choosing an $i_2 \in \{1,2,\dots,t\} - \{i_1\}$ for which the right endpoint of $X_2^{i_2}$ is to the extreme left, and so on. This procedure obviously leads to an interval-system of the desired property.



(ii) Supposing the Lemma to be true for $c-1$, let $\{X_j^i\}_{j=1}^{t^c}$ be a system of c-intervals satisfying the assumptions of the Lemma. $(i=1,2,\dots,t)$. We may suppose that for every i the first components of $X_1^i, X_2^i, \dots, X_{t^c}^i$ follow one another from left to the right on R_1. Let Y_k^i be the convex hull of the union of the first components of $X^i_{(k-1)t^{c-1}+1}$, $X^i_{(k-1)t^{c-1}+2}, \dots, X^i_{k\cdot t^{c-1}}$.



The intervals Y_k^i are pairwise disjoint for all i, so because of (i) we can choose the pairwise disjoint intervals $Y_{j_1}^1, \dots, Y_{j_t}^t$. We apply the inductive hypothesis for the last components of the system $\{X_\nu^i\}_{\nu=(j_i-1)t^{c-1}+1}^{j_i t^{c-1}}$.

The t pairwise disjoint c-intervals with their first components are t pairwise disjoint c-intervals, which completes the proof.

Proof of Theorem 1.: let $\{A^\nu\}$ be a t-independent system of c-intervals, $A^\nu = I^\nu \cup B^\nu$ where I^ν is the first component of A^ν (B^ν is a $(c-1)$-

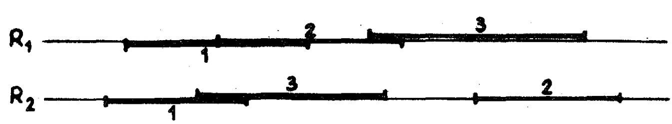
interval). We define a sequence of points of R_1 as follows: $P_0 = -\infty$ and P_r is to the extreme right with the property: the system $\{B^\nu : I^\nu \subset (P_{r-1}, P_r)\}$ is $[(t+1)^{c-1} - 1]$ -independent. ((a,b) and $(a,b]$ mean open and half-closed intervals). As a consequence of the definition of P_r, there exists an $I^{\nu_r} \subset (P_{r-1}, P_r)$ so that the right endpoint of I^{ν_r} is P_r, moreover the system $\{B^\nu : I^\nu \subset (P_{r-1}, P_r]\}$ is $(t+1)^{c-1}$ -independent, so it contains a pairwise disjoint subsystem $X_1^r, \dots, X^r_{(t+1)^{c-1}}$. It follows from the Lemma (applying to $t+1$ and $c-1$) that $P_{t+1} = +\infty$. We decompose the system $\{A^\nu\}$ into two parts:

$$\{A^\nu\} = \bigcup_{r=1}^{t+1} \{A^\nu : I^\nu \subset (P_{r-1}, P_r)\} \cup \{A^\nu : P_s \in I^\nu \text{ for some } s \quad (1 \leqq s \leqq t)\}$$

It is clear that the system of (c-1)-intervals B^ν belonging to the A^ν-s of the first part of the decomposition is $t[(t+1)^{c-1} - 1]$ -independent and the theorem follows. The simple consequence of this theorem is

THEOREM 2. $\ell(2) = 2$

PROOF: because of $\ell(2) = \ell_1(2) \leqq \ell_1(1) + 1 = 2$ we only have to prove $\ell(2) \geqq 2$ which is obvious from the following simple example:

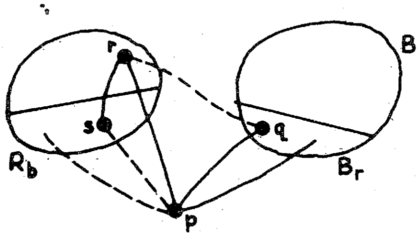


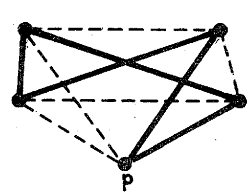
We will prove that Theorem 2. also holds under more general assumptions. For this purpose we need the notion of the interval-graph. A finite graph G is called an interval-graph if its vertices are in an one-to-one correspondence to an interval-system of the real line, and two vertices are connected if and only if the corresponding intervals have common points. A well-known property of interval-graphs is that they do not contain circuits without chords. Let's suppose that we have an one-to-one correspondence between the vertices of a complete graph G and the members of a system of pairwise intersecting 2-intervals. The edge between two vertices of G is coloured with red (with blue) if the corresponding 2-intervals meet each other in their first (second) components. Clearly G is an interval-graph if we consider only its edges of one colour. Now it is easy to see that Theorem 2. follows from the following

THEOREM 3. Let G be a complete graph, and its edges $E(G)$ coloured with red and blue. (An edge may be coloured with both colours.) Let us suppose that G contains neither blue nor red circuits of length 4 and 5 without chords. Then $V(G) = V(R) \cup V(B)$ where R and B complete red and blue subgraphs. (We denote the set of H's vertices by $V(H)$).

PROOF: by induction on the number of G's vertices. For $|V(G)| = 2$ the theorem holds. Supposing that it is true for $|V(G)| = n$, let G be a graph of the desired property with $|V(G)| = n+1$. If $p \in V(G)$ then $V(G-\{p\}) = V(R) \cup V(B)$. Let R_b and B_r be the subgraphs of R and B with which p is connected only with blue and red edges respectively. It is clear that $|V(R_b)|$ and $|V(B_r)| \neq 0$, otherwise we have nothing to prove. Let us choose the decomposition of $V(G-\{p\})$ so that $|V(B_r)| + |V(R_b)|$ should be minimal. Let us consider a $q \in V(B_r)$. Because of the minimality there exists a blue edge (q,r) between q and R. Let s be an element of $V(R_b)$. If $r \notin V(R_b)$ then (q,s) is not red, otherwise p,q,s,r,p would be

not red

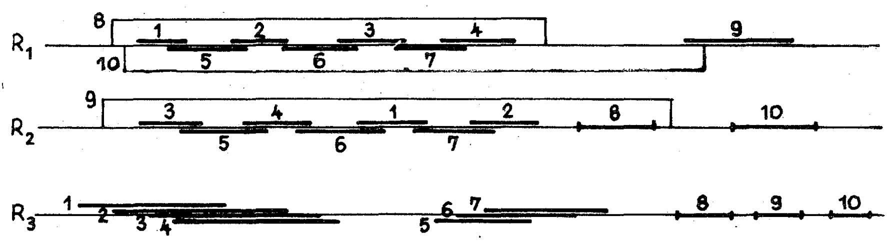




a red circuit without chords. Analogously we may suppose that in case of $u \in V(R_b)$ there exists a $v \in V(B_r)$ such that (u,v) is not blue. Now it follows that there is a circuit in G with points alternately to $V(R_b)$ and $V(B_r)$ and its edges are alternately blue and red. Let us consider the circuit C of minimal lenght of such type. It is easy to see that C is of length 4, so p and the points of C determine a circuit of length 5 without chords. This contradiction proves the theorem.

THEOREM 4. $\ell(3) = 4$

PROOF: the following example shows that $\ell(3) \geqq 4$:



Let $\mathfrak{A} = \{A^\nu\}$ be a system of 3-intervals, any two of them having common points. $A^\nu = I_1^\nu \cup I_2^\nu \cup I_3^\nu$, $P \in R_1$ and $Q \in R_2$ are two points given arbitrarily.

Let us decompose $\mathfrak{A}$ as follows: $\mathfrak{A} = \mathfrak{A}_{11}(P,Q) \cup \mathfrak{A}_{12}(P,Q) \cup$
$\cup \mathfrak{A}_{21}(P,Q) \cup \mathfrak{A}_{22}(P,Q) \cup B(P,Q)$ where

$$\mathfrak{A}_{11}(P,Q) = \{ A^\nu : I_1^\nu \subset (-\infty, P) \;\; I_2^\nu \subset (-\infty, Q) \}$$

$$\mathfrak{A}_{12}(P,Q) = \{ A^\nu : I_1^\nu \subset (-\infty, P) \;\; I_2^\nu \subset (Q, +\infty) \}$$

$$\mathfrak{A}_{21}(P,Q) = \{ A^\nu : I_1^\nu \subset (P, +\infty) \;\; I_2^\nu \subset (-\infty, Q) \}$$

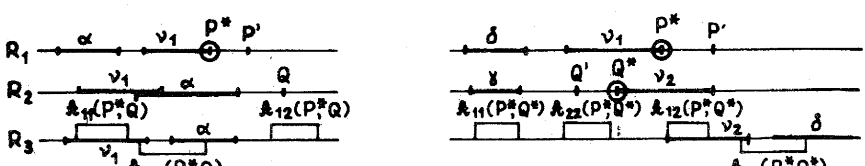
$$\mathfrak{A}_{22}(P,Q) = \{ A^\nu : I_1^\nu \subset (P, +\infty) \;\; I_2^\nu \subset (Q, +\infty) \}$$

$$B(P,Q) = \{ A^\nu : (I_1^\nu \cup I_2^\nu) \cap \{P,Q\} \neq \emptyset \}$$

In the proof we often use the following simple proposition: let $\{X_i\}$ and $\{Y_i\}$ be two systems of intervals, $X_i \cap Y_j \neq \phi$ for all i and j, then $\cap X_i$ or $\cap Y_j$ is non-empty.

The third components of the 3-intervals from $\mathfrak{A}_{11}(P,Q)$ and $\mathfrak{A}_{22}(P,Q)$ satisfy the assumptions of this proposition, so we conclude that one of them has common points. The same reasoning can be applied to $\mathfrak{A}_{12}(P,Q)$ and $\mathfrak{A}_{21}(P,Q)$. Because of the symmetry we may suppose that the two systems are $\mathfrak{A}_{11}(P,Q)$ and $\mathfrak{A}_{12}(P,Q)$ that is $\bigcap_{A^\nu \in \mathfrak{A}_{11}(P,Q)} I_3^\nu \neq \phi$ and $\bigcap_{A^\nu \in \mathfrak{A}_{12}(P,Q)} I_3^\nu \neq \phi$.

We wish to emphasize that the two systems depend on P and Q which is denoted shortly by $\{P,Q\} \rightarrow \langle 11,12 \rangle$. Now we want to define the points P^* and Q^* so that $\{P^*, Q^*\} \rightarrow \langle 11,12,21,22 \rangle$ holds. Let $P^* = \max_{\{P,Q\}\rightarrow\langle 11,12\rangle} P$ (The lines are considered as sets ordered in the usual way.) P^* is the right endpoint of one and only one interval. Let Q be an arbitrary point of R_2 for which $\{P^*,Q\} \rightarrow \langle 11,12 \rangle$. It is clear from the definition of P^* that if $(P^*, P']$ contains no endpoints of I_1^ν-s, then $\{P',Q\} \rightarrow \langle 11,12 \rangle$ is false, so for example $\bigcap_{A^\nu \in \mathfrak{A}_{11}(P',Q)} I_3^\nu$ is empty. This means that if P^* is the right endpoint of the component $I_1^{\nu_1}$ then $A^{\nu_1} \in \mathfrak{A}_{11}(P',Q)$ and there exists an I_1^α where $A^\alpha \in \mathfrak{A}_{11}(P^*,Q)$ so that $I_3^{\nu_1} \cap I_3^\alpha = \phi$. Thus $\bigcap_{A^\nu \in \mathfrak{A}_{22}(P^*,Q)} I_3^\nu \neq \phi$ that is $\{P^*,Q\} \rightarrow \langle 11,12,22 \rangle$.



Q^* is defined similarly: $Q^* = \min_{\{P^*,Q\}\rightarrow\langle 11,12,22\rangle} Q$ (Q^* is the left endpoint of one and only one interval.)

If $A^{\nu_1} \in \mathfrak{A}_{12}(P', Q^*)$ then $\{P', Q^*\} \to \langle 11,12,22 \rangle$ is false, so there exists an $A^{\beta} \in \mathfrak{A}_{12}(P^*, Q^*)$ for which $I_3^{\nu_1} \cap I_3^{\beta} = \phi$, that is $\{P^*, Q^*\} \to \langle 11,12,22,21 \rangle$.

It follows that $A^{\nu_1} \in \mathfrak{A}_{11}(P', Q^*)$ since $Q^* \notin I_2^{\nu_1}$, so there is an $A^{\gamma} \in \mathfrak{A}_{11}(P', Q^*)$ so that $I_3^{\nu_1} \cap I_3^{\gamma} = \phi$. If $[Q', Q^*)$ does not contain endpoints, then $\{P^*, Q'\} \to \langle 11,12,22 \rangle$ is false because of the minimality of Q^*. Let Q^* be the left endpoint of the interval $I_2^{\nu_2}$. In case of $A^{\nu_2} \in \mathfrak{A}_{22}(P^*, Q')$ every element of the set $\{I_3^{\nu}: A^{\nu} \in \mathfrak{A}_{22}(P^*, Q')\}$ meets I_3^{γ} and $I_3^{\nu_1}$, this also holds for $I_3^{\nu_2}$ which contradicts to the minimality of Q^*.

We conclude that $A^{\nu_2} \in \mathfrak{A}_{12}(P^*, Q')$ so we can find an $A^{\delta} \in \mathfrak{A}_{12}(P^*, Q^*)$ so that $I_3^{\delta} \cap I_3^{\nu_2} = \phi$. Thus $\{P^*, Q^*\} \to \langle 11,12,22,21 \rangle$. $\{I_3^{\nu}: A^{\nu} \in \mathfrak{A}_{11}(P^*, Q^*) \cup \mathfrak{A}_{22}(P^*, Q^*)\}$ and $\{I_3^{\nu}: A^{\nu} \in \mathfrak{A}_{12}(P^*, Q^*) \cup \mathfrak{A}_{21}(P^*, Q^*)\}$ are interval-systems of non-empty intersection, so we can choose the points R and S from these intersections. It is clear that for all A^{ν}, $\{P^*, Q^*, R, S\} \cap A^{\nu} \neq \phi$ which proves the theorem.

We close the first part with a note.

Let us suppose that $\mathfrak{A}$ is a system of c-intervals, any k of them having common points. $\ell^k(c)$ is defined in a similar manner as $\ell(c)$ above. It is easy to see that $\ell^k(c) = 1$ if $k \geq 2c$. The following scheme shows the values of $\ell^k(c)$ known to us.

k \ c	2	3	4	5	
2	2	4			
3	2	3			
4	1	2	2		
5	1	2	2		
6	1	1	2	2	

II.

Let A^{ν} be a subset of the real line R which is expressible as the union of c disjoint closed intervals. A^{ν} is also called shortly c-interval. Let

$\ell_t^*(c)$ be the least integer so that for every t-independent finite family $\mathcal{A} = \{A^\nu\}$ there exists a set $\{P_i\} \subset R$ with the property: $\{P_i\} \cap A^\nu \neq \phi$ if $A^\nu \in \mathcal{A}$ and $|\{P_i\}| = \ell_t^*(c)$. Obviously $\ell_t^*(1) = \ell_t(1) = t$.

The existence of $\ell_t^*(c)$ follows from

THEOREM 5. $\ell_t^*(c) \leqq [\ell_t^*(c-1)]^{c-1} \cdot \ell_t(c) + \sum_{i=1}^{c-1} [\ell_t^*(c-1)]^i$

This estimate is generally very far from being the best. We only sketch the proof:

If $A^\nu = I_1^\nu \cup I_2^\nu \cup \ldots \cup I_c^\nu$ (the components are indexed from left to the right) and $\mathcal{B}_1 = \{B_1^\nu\}$ is the system of (c-1)-intervals derived from A^ν by replacing I_1^ν and I_2^ν with their convex hull. There exists a set $\mathcal{P}_1 \subset R$ of $\ell_t^*(c-1)$ elements having the property:

$$P \cap B_1^\nu \neq \phi \quad \text{if} \quad B_1^\nu \in \mathcal{B}_1 \text{ and } P \in \mathcal{P}_1$$

We delete the sets A^ν for which $P \cap A^\nu \neq \phi$ and repeat the same reasoning by taking the system $\mathcal{B}_2 = \{B_2^\nu\}$ where B_2^ν is the $c-1$-interval of components I_1^ν, $\mathrm{conv}\{I_2^\nu \cup I_3^\nu\}\, I_4^\nu, \ldots, I_c^\nu$, which yields the set $\mathcal{P}_2$. We choose a point Q of $\mathcal{P}_2$, and delete from $\mathcal{B}_2$ the sets B_2^ν for which $Q \cap B_2^\nu \neq \phi$. Applying this method $c-1$ times, we see the remaining system $\mathcal{A}'$ "separated" by the points $P, Q, \ldots$ and as a consequence of Theorem 1. there is a set $\mathcal{P}$ (depending on the choice of $P, Q, \ldots$) of $\ell_t(c)$ elements enjoying the property: $\mathcal{P} \cap A^\nu \neq \phi$ if $A^\nu \in \mathcal{A}'$. This is repeated for every choice of $P \in \mathcal{P}_1$, $Q \in \mathcal{P}_2, \ldots$ and the theorem follows.

From this theorem $\ell^*(2) \leqq \ell^*(1) \cdot 2 + 1 = 3$. This is the best estimate of $\ell^*(2)$ as will be shown in Theorem 6.

Now we generalize the problems considered above in the following manner. We replace the real lines, intervals and c-intervals by trees, subtrees and forests of c components. (c-forests for the sake of shortness.)

All the theorems stated in part I. remain true in this generalized setting. We define some notions on the analogy of the ones having played an important role in the previous proofs.

Let G be a tree and F a subtree of G. The complement of F with respect to G being a forest, we denote by $F'(x)$ the component containing $x \in V(F)$. $x \in V(F)$ is said to be an extreme point of F if and only if $F'(x) \neq \{x\}$.

The assumption that two intervals have no common endpoints is replaced by the next one: if $F_1, F_2 \subset G$ then F_1 and F_2 do not contain common extreme points.

In the first part, intervals often appeared in extremal position. This is replaced by such an $F \subset G$ that $F'(x)$ is maximal.

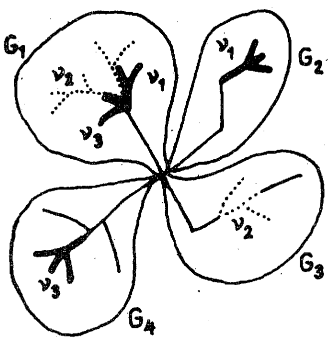
At last we may clearly assume the $\ell_t(c)$ ($\ell_t^*(c)$) vertices to be extreme.

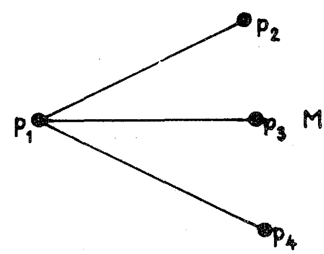
The existence of $\ell_t^*(c)$ in this general case also holds as a consequence of Theorem 1. The proof of the next theorem is much more difficult than in case of intervals.

THEOREM 6. $\ell^*(2) = 3$

PROOF: we assert that $\ell^*(2) \leqq 3$. Let $\mathfrak{F} = \{F^\nu\}$ be a finite family of 2-forests, any two of them having non-empty intersection. The components of F^ν are denoted by H_1^ν and H_2^ν. It is clear that there is an (unique) S^ν path of G connecting an $x \in V(H_1^\nu)$ and a $y \in V(H_2^\nu)$, so that $V(S^\nu) \cap V(H_1^\nu) = \{x\}$ and $V(S^\nu) \cap V(H_2^\nu) = \{y\}$. The graphs $T^\nu = F^\nu \cup S^\nu$ are substrees of G, any two of them having common points, so we can choose a point S from $\bigcap_\nu T^\nu$.

Now we split $\mathfrak{F}$ into two parts: $\mathfrak{F}' = \{F^\nu : S \in F^\nu\}$ $\mathfrak{F}'' = \mathfrak{F} - \mathfrak{F}'$. We may assume (by changing the indices if nessecery) that if $F^\nu \in \mathfrak{F}'$ then $S \in H_1^\nu$. The edges of G incident to S divide G into subtrees $G_1, G_2, \dots, G_k$. ($G_i \cap G_j = S$) It is obvious that in case of $F^\nu \in \mathfrak{F}''$ H_1^ν and H_2^ν are subgraphs of different G_i-s. It may be supposed that every G_i contains some component of F^ν. ($F^\nu \in \mathfrak{F}''$). We define a graph M in order to show the distribution of F^ν '-s components in the G_i-s. $V(M) = \{p_1, p_2, \ldots, p_k\}$ and $(p_i, p_j) \in E(M)$ if and only if there exists an F^ν, the components of which are in G_i and G_j.





It is clear that any two edges of M have a common vertex and the degree of every point is at least one. Consequently M is a triangle or a star. If M is a star as shown on the figure, we define G' deleting from G_1 the point S with the edge incident to S and $G'' = \bigcup_{i=2}^{k} G_i$. Now for every $F^\nu \in \mathfrak{F}''$ the two components of F^ν are in G' and G'' respectively so the existence of an $X \in V(G')$ and of an $Y \in V(G'')$ such that $\{X, Y\} \cap V(F^\nu) \neq \emptyset$ follows from Theorem 2. Thereby $\{X, Y, S\} \cap V(F^\nu) \neq \emptyset$ if $F^\nu \in \mathfrak{F}$ which proves our assertion.

In the sequel we assume M to be a triangle, that is we have $G_1 \cup G_2 \cup G_3 = G$. We refine the splitting $\mathfrak{F} = \mathfrak{F}' \cup \mathfrak{F}''$ in the following manner: $\mathfrak{F}' = \bigcup_{i=1}^{3} \mathfrak{F}_i$ and $\mathfrak{F}'' = \mathfrak{F}_{12} \cup \mathfrak{F}_{23} \cup \mathfrak{F}_{31}$ where

$\mathfrak{F}_i = \{ F^\nu : F^\nu \in \mathfrak{F}', H_2^\nu \subset G_i \}$ $(i = 1,2,3)$

$\mathfrak{F}_{12} = \{ F^\nu : F^\nu \in \mathfrak{F}''$, the components of F^ν are in G_1 and $G_2 \}$

$\mathfrak{F}_{23} = \{ F^\nu : F^\nu \in \mathfrak{F}''$, the components of F^ν are in G_2 and $G_3 \}$

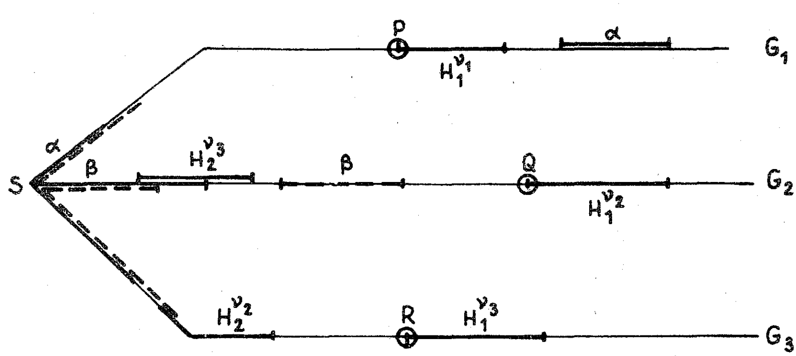
$\mathfrak{F}_{31} = \{ F^\nu : F^\nu \in \mathfrak{F}''$, the components of F^ν are in G_3 and $G_1 \}$

It is easy to prove that we can chose P, Q, R from $V(G_1), V(G_2)$ and $V(G_3)$ resp. with the property: $\{P,Q,R\} \cap V(F^\nu) \neq \phi$ if $F^\nu \in \mathfrak{F}''$. The details are left to the reader.

If P, Q, R are vertices of G_1, G_2 and G_3 respectively, then $G(S,P,Q,R)$ will denote the union of the (unique) paths $\overline{SP}$, $\overline{SQ}$, $\overline{SR}$. (This is the smallest subtree of G containing the vertices S, P, Q, R).

Let $G(S,P,Q,R)$ be minimal for set-theoretic inclusion with the property: $\{P,Q,R\} \cap V(F^\nu) \neq \phi$ if $F^\nu \in \mathfrak{F}''$.

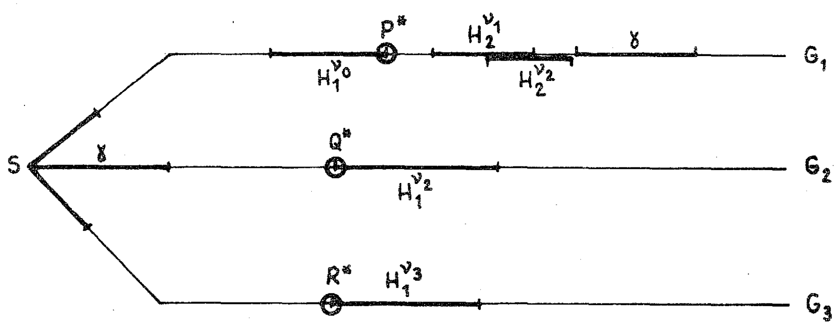
We prove that $\{P,Q,R\} \cap V(F^\nu) \neq \phi$ if $F^\nu \in \mathfrak{F}''$ and $F^\nu \in \mathfrak{F}_i \cup \mathfrak{F}_j$ for suitable i and j $(i \neq j)$. If this does not hold, then there is, for example, an $F^\alpha \in \mathfrak{F}_1$ and an $F^\beta \in \mathfrak{F}_2$ so that $\{P,Q,R\} \cap V(F^\alpha \cup F^\beta) = \phi$. Let $H_1^{\nu_1}$, $H_1^{\nu_2}$ and $H_1^{\nu_3}$ be the (unique) subtrees with extreme points P, Q, R.



If $F^{\nu_3} \in \mathfrak{F}_{23}$ then $H_2^{\nu_3}$ meets H_1^α (in the interior of path $\overline{SQ}$ by minimality) and so $H_2^{\nu_3} \cap H_1^{\nu_2} = \phi$. It follows that $F^{\nu_2} \in \mathfrak{F}_{23}$, and the same argument shows $H_2^{\nu_2}$ to be disjoint from $H_1^{\nu_3}$; this leads to contradiction. Replacing F^{α} by F^{β}, the same argument can be repeated and the assertion follows.

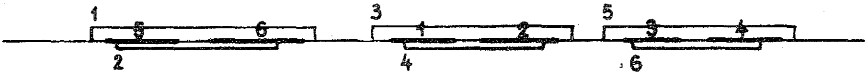
We may assume (by symmetry): $\{P,Q,R\} \cap V(F^{\nu}) \neq \phi$ when $F^{\nu} \in \mathfrak{F}'' \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$. This is the property for which we choose a minimal tree $G(S,Q^*,R^*)$.

If $\{Q^*,R^*\} \cap V(F^{\nu}) \neq \phi$ for all $F^{\nu} \in \mathfrak{F}_1$ we have nothing to prove. Let F^{γ} be such that $H_1^{\gamma} \cap \{Q^*,R^*\} = \phi$ $F^{\gamma} \in \mathfrak{F}_1$, $H_1^{\nu_2}$ and $H_1^{\nu_3}$ will denote the two components belonging to the extreme points Q^* and R^*. Obviously $F^{\nu_2} \in \mathfrak{F}_{12}$ and $F^{\nu_3} \in \mathfrak{F}_{31}$ otherwise we have a contradiction. (Using F^{γ} as above.) Clearly $H_2^{\nu_2} \cap H_2^{\nu_3} \neq \phi$. We choose P^* as close as possible to the tree $H_2^{\nu_2} \cup H_2^{\nu_3}$ with the propety: $\{P^*,Q^*,R^*\} \cap F^{\nu} \neq \phi$ for all $F^{\nu} \in \mathfrak{F}'' \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$.



By the minimum-property of $G(S,Q^*,R^*)$ $P^* \notin H_2^{\nu_2} \cup H_2^{\nu_3}$. P^* is the extreme point of a tree $H_1^{\nu_0}$, but the assumptions $H_1^{\nu_0} \in \mathfrak{F}_2$, $H_1^{\nu_0} \in \mathfrak{F}_3$, $H_1^{\nu_0} \in \mathfrak{F}_{12}$ lead to contradiction.

So $\{P^*,Q^*,R^*\} \cap F^{\nu} \neq \phi$ for all $F^{\nu} \in \mathfrak{F}$. We conclude that $\ell^*(2) \leqq 3$. The following example proves that $\ell^*(2) \geqq 3$:



Finally we present a conjecture: if G is a tree and $\mathfrak{F} = \{F^{\nu}\}$ is a finite family of its c-forests, any $c+1$ of them having non-empty intersection, then there exists a set $P \subset V(G)$ so that $|P| = c$ and $P \cap F^{\nu} \neq \phi$ for all $F^{\nu} \in \mathcal{F}$. We can prove this only in case of $c \leqq 3$, and if G is a path. (c is arbitrarily given)

L. Surányi's result is worth mentioning (oral communication); it gives a necessary and sufficient condition for a graph G to be representable as a system of subtrees of a suitable tree (between two vertices of G, there is an edge iff the corresponding subtrees have common vertices): A graph G is representable if and only if it does not contain any circuit without chords.

Using this theorem we can easily obtain the following two corollaries of our first and fourth theorem:

COROLLARY 1. Let G be a complete graph, and its edges coloured with c different colours. (An edge may be coloured with more than one colour.) Let us suppose that G does not contain any circuit of one colour without chords. Then there exists an integer $f(c)$ (which does not depend on $|V(G)|$) so that $V(G) = \bigcup_{i=1}^{f(c)} V(G_i)$ where G_i is a complete graph of one colour.

COROLLARY 2. If $c=3$ then the least value of $f(3)$ is 4.

REFERENCE

[1] A. HAJNAL-J.SURÁNYI: Über die Auflösung von Graphen in vollständige Teilgraphen Ann. Univ. Sci. Budapest, 1. 958, p. 113.