ON RAMSEY-TYPE PROBLEMS

By

L. GERENCSÉR and A. GYÁRFÁS
Eötvös Loránd University, Budapest
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In the present paper we deal with graphs having a finite number of vertices, single edges and no loops. The number of vertices of the graph $G$ will be denoted by $\pi(G)$, the edge between the vertices $A$ and $B$ by $AB$. To indicate that a vertex or an edge belongs to the graph, we use the symbol $\in$. A graph is called complete, if any two of its vertices are connected by an edge. Complete graphs with $k$ vertices are called complete $k$-tuples. The graph having as vertices the points $U_1, U_2, \ldots, U_{k+1}$ and edges $U_1U_2, U_2U_3, \ldots, U_kU_{k+1}$ is a path of length $k$, $U_1, U_{k+1}$ are its endpoints. Adding the edge $U_{k+1}U_1$ we get a circuit of length $k+1$. $\overline{G}$ will denote the complementary graph or complement of $G$ (i.e. two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in $G$). A graph is connected if for each pair of its vertices there exists a path in $G$ having these vertices as endpoints.

According to the well known theorem of RAMSEY [1] there exists for every system of natural numbers $(k_1, k_2, \ldots, k_r)$ a natural number $N(k_1, k_2, \ldots, k_r)$ with the property that for $n \geqslant N(k_1, \ldots, k_r)$ dividing the edges of a complete graph of $n$ vertices into $r$ distinct classes (colouring every edge with one of $r$ different colours) at least for one $i (1 \leqslant i \leqslant r)$ the $i$-th class contains a complete $k_i$-tuple (there exists a one-coloured complete $k_i$-tuple.) The least value of $N(k_1, \ldots, k_r)$ is unknown for the general case. (For special cases see [2].)

Colouring the edges of complete graphs with $r$ different colours, we may investigate problems about the existence of other special types of one-coloured subgraphs instead of one-coloured $k$-tuples — as in Ramsey-theorem. In this paper we shall consider two different types of graphs, namely:

a) paths of given length and
b) connected graphs.

Let $g(k, l)$ denote the least integer for which in case $\pi(G) \geqslant g(k, l)$ either $G$ contains a path of length $k$, or $\overline{G}$ one of length $l$.

Our main purpose is to prove the following
Theorem 1. For \( k \geq l \) we have

\[
g(k, l) = k + \left[ \frac{l+1}{2} \right].
\]

Considering the other special case of this type of problems, let \( f_r(n) \) denote the greatest integer with the property, that colouring the edges of a complete \( n \)-tuple \( g \) with \( r \) colours arbitrarily, there exists always a one-coloured connected subgraph with at least \( f_r(n) \) vertices.

It is easy to see the following remark of P. Erdős: if a graph is not connected then its complement is connected, i.e. \( f_2(n) = n \). We shall prove

Theorem 2.

\[
f_3(n) = \left\lceil \frac{n+1}{2} \right\rceil.
\]

Now we turn to the proof of Theorem 1. First we prove \( g(k, l) = k + \left[ \frac{l+1}{2} \right] \) by induction on \( k \). For \( k = 1 \) the Theorem evidently holds and let us suppose that for all \( k \)-s less than this the statement is true. Let us consider a graph \( G \) with \( k + \left[ \frac{l+1}{2} \right] \) vertices. If \( l < k \), then for any subgraph of \( G \) with \( k - 1 + \left[ \frac{l+1}{2} \right] \) points holds that either it contains a path of length \( k - 1 \), or its complement a path of length \( l \). For \( l = k \) we consider a subgraph with \( k - 1 + \left[ \frac{l}{2} \right] \) points. This or its complement contains a path of length \( k - 1 \). Thus in every case can be supposed, that the length of the longest path of \( G \) is \( k - 1 \). Let \( U_1, U_2, \ldots, U_k \) be the consecutive vertices of such a path and \( U = \{U_1, \ldots, U_k\} \). We denote the remaining vertices by \( V_1, \ldots, V_{\left\lceil \frac{l+1}{2} \right\rceil} \) and the set of them by \( V = \{V_1, \ldots, V_{\left\lceil \frac{l+1}{2} \right\rceil}\} \).

It clearly holds that

(i) for all \( V_i \in V \) either \( V_iU_j \in \overline{G} \) or \( V_iU_{j+1} \in \overline{G} \)
(ii) for all \( V_i \in V \) \( V_iU_1 \in \overline{G} \) and \( V_iU_k \in \overline{G} \)
(iii) for \( V_{i1}, V_{i2}, V_{i3} \in V \) and \( U_j, U_{j+1} \in U \)

at least one of the latest points is connected in \( \overline{G} \) with at least two of \( V_{i1}, V_{i2}, V_{i3} \).

Consider a maximal path of \( \overline{G} \) not containing \( U_1, U_k \) with the property that any edge of it connects a point of \( U \) with a point of \( V \), and its endpoints are in \( V \); let us denote the endpoints by \( A \) and \( B \), and the path by \( S \). If \( S \) contains all points of \( V \), then by adding the edges \( U_1A, BU_k \) we have a path of length \( 2 \left[ \frac{l+1}{2} \right] \geq l \) in \( \overline{G} \). So we may suppose that the set of points \( V \) not contained by \( S \) is not empty. Let this set be called \( W \). Consider a maximal path \( q \) of \( G \) not containing \( U_1, U_k \) and having no common points with \( S \), such that any edge of it connects a point of \( U \) with a point of \( W \) and the endpoints of it,
called by $C$ and $D$, are in $W$. We show that all points of $V$ are contained either in $S$ or in $q$. Suppose that $X \in V$ but $X \notin S$, $X \notin q$. It is clear, that the number of vertices of $S$ and $q$ in $U$ is at most $\left\lfloor \frac{l+1}{2} \right\rfloor - 3 < \left\lfloor \frac{k-3}{2} \right\rfloor = \left\lfloor \frac{k-2-1}{2} \right\rfloor$ since $l \leq k$. So there exist two points $U_i, U_{i+1} \in \{U_2, \ldots, U_{k-1}\}$ which do not belong either to $S$ or to $q$. Applying (iii) for $A, C, X \in V$ and $U_i, U_{i+1} \in U$ we have a contradiction to the maximal properties of $S$ and $q$.

So the sum of the length of $S$ and $q$ is $2 \left\lfloor \frac{l+1}{2} \right\rfloor - 4$. We add them the edges $U_1 A, BU_k, U_k C, DU_1$ and so we have a circuit of length $2 \left\lfloor \frac{l+1}{2} \right\rfloor$ in $\overline{G}$. For odd $l$ this contains a desired path with length $l$. For even $l$ an easy reasoning shows that there are $U_i, U_{i+1} \in U$ which do not belong to this circuit. Hence one of them is connected with a vertex of the circuit (see (i)) and so we have again a path with length $l$ in $\overline{G}$. That completes the proof.

Now we give examples for graphs $G$ with $k + \left\lfloor \frac{l+1}{2} \right\rfloor - 1$ points that have no path of length $k$, and for them at the same time $\overline{G}$ have no path of length $l$.

a) Let $G$ consist of the disjoint graphs $H_1, H_2$ with $k$ and $\left\lfloor \frac{l+1}{2} \right\rfloor - 1$ points respectively, where the graph $H_1$ is complete.

b) For even $l$ we can leave one of the edges of $H_1$. These graphs possess obviously the desired property.$^1$

Now we turn to the proof of Theorem 2. We consider a classification of the edges of a complete graph $G$ into three classes, i.e. let the edges of $G$ be coloured with red, yellow and blue colours. So we get the graphs $G_r, G_y$ and $G_b$ formed by the red, yellow and blue edges respectively. We say that a subgraph is for example red-connected if it is a connected subgraph of $G_r$. Let us take a maximal red-connected subgraph $R$. It may be supposed that $R$ is not empty and $\pi(R) < \pi(G) = n$. Let $B$ be a point of $G$ such that $B \notin R$. Since $R$ is a maximal connected subgraph of $G$, $BR_i$ is not red for $R_i \in R$. So one may suppose that there are at least $\frac{1}{2} \pi(R)$ points of $R$ which are connected with $B$ by blue edges.

Let $V$ denote the set of these points of $R$ and $W$ be the maximal blue-connected subgraph that contains $B$. If $Y$ is a point such that $Y \notin R$ and $Y \notin W$ then $YV_i$ is yellow for $V_i \in V$. Let $Q$ denote the maximal yellow-connected subgraph that contains $Y$. If there is no such $Y$, $Q$ denotes the empty set. $R, W, Q$ contain together all points of $G$. Namely any points $S \notin R$ is connected with a

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$^1$ The weaker result $g(k, l) \leq k+l$ can be easily proved. Let us consider any vertex $P$ and a pair of paths of $G$ and $\overline{G}$ without common vertices except $P$. It can be proved that a pair of paths with maximal sum of lengths contains all points. (Maximality with respect to all $P$ and all pairs.) From that the statement follows.
V_i \in V$ either by a blue or a yellow edge, i.e. either $S \in W$ or $S \in Q$. In both cases
\[
\pi(W) + \pi(Q) \geq (\pi(G) - \pi(R)) + 2\pi(V) \geq n - \pi(R) + 2 \frac{\pi(R)}{2} = n,
\]
and then
\[
\max(\pi(W), \pi(Q)) \geq \frac{n}{2}
\]
which completes the proof of $f_3(n) \equiv \left\lfloor \frac{n+1}{2} \right\rfloor$.

For the proof of $f_3(n) \equiv \left\lfloor \frac{n+1}{2} \right\rfloor$ we prove the more general

**Lemma.** For odd $r$, $n = (r+1)v$ ($v = 1, 2, \ldots$)

(3)

\[
f_r(n) \equiv \frac{2}{r+1} n.
\]

Here we use the following theorem:

The edges of a complete graph $G_0$ with $2k$ vertices can be coloured with $2k - 1$ colours so that the edges having common vertices have different colours ([3]). Let $H$ be a graph with $r+1$ vertices, and consider a colouring mentioned above for $2k = r+1$. Let us replace any vertex of $H$ by an arbitrarily coloured complete $\nu$-tuple. Let the edges which connect vertices from two different $\nu$-tuples have the same colour as the edge connecting the corresponding vertices in $H$. This graph clearly satisfies the requirements and this proves (3).

**References**


