ON RAMSEY-TYPE PROBLEMS

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In the present paper we deal with graphs having a finite number of vertices, single edges and no loops. The number of vertices of the graph G will be denoted by $\pi(G)$, the edge between the vertices A and B by AB. To indicate that a vertex or an edge belongs to the graph, we use the symbol $\in$ A graph is called complete, if any two of its vertices are connected by an edge. Complete graphs with k vertices are called complete k-tuples. The graph having as vertices the points $U_1, U_2, \ldots, U_{k+1}$ and edges $U_1U_2, U_2U_3, \ldots, U_kU_{k+1}$ is a path of length k, U_1, U_{k+1} are its endpoints. Adding the edge $U_{k+1}U_1$ we get a circuit of length $k+1$. $\overline{G}$ will denote the complementary graph or complement of G (i.e. two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G). A graph is connected if for each pair of its vertices there exists a path in G having these vertices as endpoints.

According to the well known theorem of RAMSEY [1] there exists for every system of natural numbers $(k_1, k_2, \ldots, k_r)$ a natural number $N(k_1, k_2, \ldots, k_r)$ with the property that for $n \geqq N(k_1, \ldots, k_r)$ dividing the edges of a complete graph of n vertices into r distinct classes (colouring every edge with one of r different colours) at least for one i $(1 \leqq i \leqq r)$ the i-th class contains a complete k_i – tuple (there exists a one-coloured complete k_i-tuple.) The least value of $N(k_1, \ldots, k_r)$ is unknown for the general case. (For special cases see [2].)

Colouring the edges of complete graphs with r different colours, we may investigate problems about the existence of other special types of one-coloured subgraphs instead of one-coloured k-tuples – as in Ramsey-theorem. In this paper we shall consider two different types of graphs, namely:

a) paths of given length and
b) connected graphs.

Let $g(k, l)$ denote the least integer for which in case $\pi(G) \geqq g(k, l)$ either G contains a path of length k, or $\overline{G}$ one of length l.

Our main purpose is to prove the following

THEOREM 1. *For $k \geqq l$ we have*

$$g(k, l) = k + \left[\frac{l+1}{2}\right]. \tag{1}$$

Considering the other special case of this type of problems, let $f_r(n)$ denote the greatest integer with the property, that colouring the edges of a complete n-tuple g with r colours arbitrarily, there exists always a one-coloured connected subgraph with at least $f_r(n)$ vertices.

It is easy to see the following remark of P. ERDŐS: if a graph is not connected then its complement is connected, i.e. $f_2(n) = n$. We shall prove

THEOREM 2.

$$f_3(n) = \left[\frac{n+1}{2}\right]. \tag{2}$$

Now we turn to the proof of Theorem 1. First we prove $g(k, l) \leqq k + \left[\frac{l+1}{2}\right]$ by induction on k. For $k=1$ the Theorem evidently holds and let us suppose that for all k-s less than this the statement is true. Let us consider a graph G with $k + \left[\frac{l+1}{2}\right]$ vertices. If $l<k$, then for any subgraph of G with $k-1+\left[\frac{l+1}{2}\right]$ points holds that either itself contains a path of length $k-1$, or its complement a path of length l. For $l=k$ we consider a subgraph with $k-1+\left[\frac{l}{2}\right]$ points. This or its complement contains a path of length $k-1$. Thus in every case can be supposed, that the length of the longest path of G is $k-1$. Let $U_1, U_2, \ldots, U_k$ be the consecutive vertices of such a path and $U \mathrel{\overline{=}} \{U_1, \ldots, U_k\}$. We denote the remaining vertices by $V_1, \ldots, V_{\left[\frac{l+1}{2}\right]}$ and the set of them by $V = \left\{V_1, \ldots, V_{\left[\frac{l+1}{2}\right]}\right\}$.

It clearly holds that

(i) for all $V_i \in V$ either $V_iU_j \in \bar{G}$ or $V_iU_{j+1} \in \bar{G}$

(ii) for all $V_i \in V$ $V_iU_1 \in \bar{G}$ and $V_iU_k \in \bar{G}$

(iii) for $V_{i1}, V_{i2}, V_{i3} \in V$ and $U_j, U_{j+1} \in U$

at least one of the latest points is connected in $\bar{G}$ with at least two of V_{i1}, V_{i2}, V_{i3}.

Consider a maximal path of $\bar{G}$ not containing U_1, U_k with the property that any edge of it connects a point of U with a point of V, and its endpoints are in V; let us denote the endpoints by A and B, and the path by S. If S contains all points of V, then by adding the edges U_1A, BU_k we have a path of length $2\left[\frac{l+1}{2}\right] \geqq l$ in $\bar{G}$. So we may suppose that the set of points V not contained by S is not empty. Let this set be called W. Consider a maximal path q of $\bar{G}$ not containing U_1, U_k and having no common points with S, such that any edge of it connects a point of U with a point of W and the endpoints of it, called by C and D, are in W. We show that all points of V are contained either in S or in q. Suppose that $X \in V$ but $X \notin S$, $X \notin q$. It is clear, that the number of vertices of S and q in U is at most $\left[\frac{l+1}{2}\right]-3 < \left[\frac{k-3}{2}\right] = \left[\frac{k-2-1}{2}\right]$ since $l \leqq k$. So there exist two points $U_i, U_{i+1} \in \{U_2, \ldots, U_{k-1}\}$ which do not belong either to S or to q. Applying (iii) for $A, C, X \in V$ and $U_i, U_{i+1} \in U$ we have a contradiction to the maximal properties of S and q.

So the sum of the length of S and q is $2\left[\frac{l+1}{2}\right]-4$. We add them the edges U_1A, BU_k, U_kC, DU_1 and so we have a circuit of length $2\left[\frac{l+1}{2}\right]$ in $\bar{G}$. For odd l this contains a desired path with length l. For even l an easy reasoning shows that there are $U_i, U_{i+1} \in U$ which do not belong to this circuit. Hence one of them is connected with a vertex of the circuit (see (i)) and so we have again a path with length l in $\bar{G}$. That completes the proof.

Now we give examples for graphs G with $k+\left[\frac{l+1}{2}\right]-1$ points that have no path of length k, and for them at the same time $\bar{G}$ have no path of length l.

a) Let G consist of the disjoint graphs H_1, H_2 with k and $\left[\frac{l+1}{2}\right]-1$ points respectively, where the graph H_1 is complete.

b) For even l we can leave one of the edges of H_1. These graphs possess obviously the desired property.[1]

Now we turn to the proof of Theorem 2. We consider a classification of the edges of a complete graph G into three classes, i.e. let the edges of G be coloured with red, yellow and blue colours. So we get the graphs G_r, G_y and G_b formed by the red. yellow and blue edges respectively. We say that a subgraph is for example red-connected if it is a connected subgraph of G_r. Let us take a maximal red-connected subgraph R. It may be supposed that R is not empty and $\pi(R) < \pi(G) = n$. Let B be a point of G such that $B \notin R$. Since R is a maximal connected subgraph of G_r, BR_i is not red for $R_i \in R$. So one may suppose that there are at least $\frac{1}{2}\pi(R)$ points of R which are connected with B by blue edges. Let V denote the set of these points of R and W be the maximal blue-connected subgraph that contains B. If Y is a point such that $Y \notin R$ and $Y \notin W$ then YV_i is yellow for $V_i \in V$. Let Q denote the maximal yellow-connected subgraph that contains Y. If there is no such Y, Q denotes the empty set. R, W, Q contain together all points of G. Namely any points $S \notin R$ is connected with a

[1] The weaker result $g(k, l) \leqq k+l$ can be easily proved. Let us consider any vertex P and a pair of paths of G and $\bar{G}$ without common vertices except P. It can be proved that a pair of paths with maximal sum of lengths contains all points. (Maximality with respect to all P and all pairs.) From that the statement follows.

$V_i \in V$ either by a blue or a yellow edge, i.e. either $S \in W$ or $S \in Q$. In both cases

$$\pi(W)+\pi(Q) \geqq (\pi(G)-\pi(R))+2\pi(V) \geqq n-\pi(R)+2\frac{\pi(R)}{2} = n,$$

and then

$$\max(\pi(W), \pi(Q)) \geqq \frac{n}{2}$$

which completes the proof of $f_3(n) \geqq \left[\frac{n+1}{2}\right]$.

For the proof of $f_3(n) \leqq \left[\frac{n+1}{2}\right]$ we prove the more general

LEMMA. For odd r, $n = (r+1)\nu$ $(\nu = 1, 2, \ldots)$

$$f_r(n) \leqq \frac{2}{r+1} n. \tag{3}$$

Here we use the following theorem:

The edges of a complete graph G_0 with $2k$ vertices can be coloured with $2k-1$ colours so that the edges having common vertices have different colours ([3]). Let H be a graph with $r+1$ vertices, and consider a colouring mentioned above for $2k = r+1$. Let us replace any vertex of H by an arbitrarily coloured complete ν-tuple. Let the edges which connect vertices from two different ν-tuples have the same colour as the edge connecting the corresponding vertices in H. This graph clearly satisfies the requirements and this proves (3).

References

[1] RAMSEY, F. P., *Proc. London Math. Soc.*, 2, **30** (1930), 264–286.
[2] GREENWOOD-GLEASON, *Canadian Journal of Math.*, **7** (1955), 1–7.
KÉRY GERZSON, *Matematikai Lapok*, **15** (1964), 204–223.
[3] RINGEL, Färbungsprobleme, 32.