# ON THE SUP-NORM OF MAASS CUSP FORMS OF LARGE LEVEL. II

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Abstract. Let $f$ be a Hecke–Maass cuspidal newform of square-free level $N$ and Laplacian eigenvalue $\lambda$. It is shown that $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/12+\epsilon}\|f\|_2$ for any $\epsilon > 0$, with an implied constant depending continuously on $\lambda$. The proof is short and self-contained.

## 1. Introduction

This paper, a sequel to [8], deals with the problem of bounding the $L^\infty$ norm (or sup-norm) in terms of the $L^2$ norm on an arithmetic hyperbolic surface. It is natural to restrict this problem for Hecke–Maass cuspidal newforms, that is, square-integrable joint eigenfunctions of the Laplacian and Hecke operators. In general, establishing bounds between the various $L^p$ norms for automorphic forms is useful in the theory of quantum chaos and subconvexity of $L$-functions, which in turn have deep arithmetic applications. Equally surprising is the fact that one can learn about $L^p$ norms by diophantine principles. Nice examples for this interplay can be found in the works [2–4, 6, 9].

To be specific, we consider the noncompact arithmetic surface $\Gamma_0(N)\backslash\mathfrak{H}$ equipped with its hyperbolic metric and associated measure $dxdy/y^2$; the total volume is then asymptotically equal to $N^{1+o(1)}$. We shall $L^2$-normalize all Hecke–Maass cuspidal newforms with respect to that measure, namely

$$\int_{\Gamma_0(N)\backslash\mathfrak{H}} |f(z)|^2\,\frac{dxdy}{y^2} = 1. \tag{1}$$

We are interested in bounding the sup-norm $\|f\|_\infty$ in terms of the two basic parameters: the Laplacian eigenvalue $\lambda$ and the level $N$. Let us recall briefly the previously known results.

In the $\lambda$-aspect the first nontrivial (and so far unsurpassed) bound is due to Iwaniec and Sarnak [4] who established $\|f\|_\infty \ll_{N,\epsilon} \lambda^{5/24+\epsilon}$ for any $\epsilon > 0$. Their key idea was to make use of the Hecke operators, through the method of amplification, in order to go beyond $\|f\|_\infty \ll \lambda^{1/4}$ which is valid on any Riemannian surface (see Seeger–Sogge [7]).

In this paper we focus mainly on the $N$-aspect and, thanks to the new ideas explained below, we are able to provide a short but self-contained treatment. For several reasons, see [1, 2, 5], the "trivial" bound is $\|f\|_\infty \ll_{\lambda,\epsilon} N^\epsilon$, while the most optimistic bound would be $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/2+\epsilon}$. Here and later, the dependence on $\lambda$ is understood to be continuous. There is no real evidence for the latter bound except that it holds trivially for old forms of level 1. The breakthrough in the $N$-aspect was recently achieved by Blomer–Holowinsky [2, p. 673] who proved $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-25/914+\epsilon}$, at least for square-free $N$. The restriction on $N$ seems difficult to remove: it is needed for a certain application of Atkin–Lehner theory. In [8] the second named author revisited the proof by making a systematic use of geometric arguments, and derived a stronger exponent: $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/22+\epsilon}$. In

2010 *Mathematics Subject Classification.* 11F12,11D45,14G35.

*Key words and phrases.* automorphic forms, trace formula, amplification, diophantine approximation.

The first author was supported by EC grant ERG 239277 and by OTKA grants K 72731, PD 75126. The second author was supported by NSF grant DMS-0635607.

the present paper we exploit Atkin–Lehner theory in a more efficient way, which results in a rather elegant proof and a significantly better estimate for the sup-norm.

**Theorem 1.** *Let $f$ be an $L^2$-normalized Hecke–Maass cuspidal newform of square-free level $N$, trivial nebentypus, and Laplacian eigenvalue $\lambda$. Then for any $\epsilon > 0$ we have a bound*

$$\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/12+\epsilon},$$

*where the implied constant depends continuously on $\lambda$.*

**Remark 1.** By refining the methods of this paper the exponent $-1/12$ can be improved to $-1/6$. The improved result seems to be analogous to the Weyl bound in the subconvexity problem for automorphic $L$-functions, which has been achieved in very few cases. The details will be published in a subsequent paper.

**Remark 2.** The dependence on $\lambda$ can be made polynomial and, combining the ideas of the present paper with those of [4], a hybrid version can be established, improving on [2, Theorem 2].

Let us explain briefly the new ingredients compared to previous approaches. The earlier works [2,4,8] made great use, when estimating $|f(z)|$, of a fine analysis of the diophantine properties of the point $z \in \mathfrak{H}$. The present paper bypasses that analysis by focusing on a point $z \in \mathfrak{H}$ with highest imaginary part such that $|f(z)| = \|f\|_\infty$. The key observation is that such a point $z \in \mathfrak{H}$ always has good diophantine properties (Lemma 1) which allows a more efficient treatment of the counting problem that lies at the heart of the argument (Lemma 3). Another feature is that we exploit better the structure of the amplifier.

**Acknowledgements.** We thank the referees for their careful reading and valuable suggestions on exposition. We also thank Harald Helfgott and Guillaume Ricotta for helpful discussions and comments. The results in this paper were found during the "Workshop on Analytic Number Theory" held at the Institute for Advanced Study in March 2010.

## 2. Atkin–Lehner operators and a gap principle

We let $\mathrm{GL}_2(\mathbb{R})^+$ act on the upper-half plane $\mathfrak{H} = \{x + iy \in \mathbb{C} : y > 0\}$ by the usual fractional linear transformations. Throughout this paper we shall assume that the level $N$ is square-free. For each divisor $M \mid N$ we consider the matrices $W_M \in M_2(\mathbb{Z})$ of determinant $M$ such that

$$W_M \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \qquad \text{and} \qquad W_M \equiv \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \pmod M.$$

Scaling the $W_M$'s by $1/\sqrt{M}$ we obtain matrices in $\mathrm{SL}_2(\mathbb{R})$ which we call the *Atkin–Lehner operators*. The Atkin–Lehner operators together for all $M \mid N$ form a subgroup $A_0(N)$ of $\mathrm{SL}_2(\mathbb{R})$ containing $\Gamma_0(N)$ as a normal subgroup. The quotient group $A_0(N)/\Gamma_0(N)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\omega(N)}$, where $\omega(N)$ is the number of distinct prime factors of $N$. Our Hecke–Maass newform $f$ is an eigenvector for the Atkin–Lehner operators with eigenvalues $\pm 1$, therefore in examining its sup-norm we can restrict to the following fundamental domain for $A_0(N)$.

**Definition 1.** Let $\mathcal{F}(N)$ be the set of $z \in \mathfrak{H}$ such that $\operatorname{Im} z \geqslant \operatorname{Im} \delta z$ for all $\delta \in A_0(N)$.

The main new idea in this paper is the observation that the elements of $\mathcal{F}(N)$ have good diophantine properties. The following lemma takes the role of Dirichlet approximation in [2,8].

**Lemma 1.** *Let $z \in \mathcal{F}(N)$. Then we have*

$$\operatorname{Im} z \geqslant \frac{\sqrt{3}}{2N}, \tag{2}$$

*and for any* $(c,d)\in\mathbb{Z}^2$ *distinct from* $(0,0)$ *we have*

$$|cz+d|^2 \geqslant \frac{1}{N}. \tag{3}$$

*Proof.* We prove (2) first. By the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ we can represent $z$ as

$$z=\begin{pmatrix} a & b\\ c & d\end{pmatrix} w,\qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}),\qquad \operatorname{Im} w\geqslant \frac{\sqrt{3}}{2}.$$

By a standard result (see [1, Lemma 2.1], or [8, Lemma 2.1] for a more direct proof) we can find $\begin{pmatrix} a & b'\\ c & d'\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$ and some divisor $M\mid N$ such that

$$\begin{pmatrix} a & b'\\ c & d'\end{pmatrix}\begin{pmatrix} M^{1/2} & 0\\ 0 & M^{-1/2}\end{pmatrix}\in A_0(N).$$

Clearly, there is an $n\in\mathbb{Z}$ such that

$$\begin{pmatrix} a & b'\\ c & d'\end{pmatrix}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1 & n\\ 0 & 1\end{pmatrix},$$

so that $w':=(w-n)/M$ satisfies

$$z=\begin{pmatrix} a & b'\\ c & d'\end{pmatrix}\begin{pmatrix} M^{1/2} & 0\\ 0 & M^{-1/2}\end{pmatrix} w',\qquad \operatorname{Im} w'\geqslant \frac{\sqrt{3}}{2M}.$$

By Definition 1 we have $\operatorname{Im} z\geqslant \operatorname{Im} w'$, hence (2) follows.

Now we prove (3). We can assume $(c,d)=1$, then as above we can find $\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$ and some divisor $M\mid N$ such that

$$\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}\begin{pmatrix} M^{1/2} & 0\\ 0 & M^{-1/2}\end{pmatrix}\in A_0(N).$$

Defining

$$w:=\begin{pmatrix} M^{-1/2} & 0\\ 0 & M^{1/2}\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix} z,$$

we see that

$$\operatorname{Im} w=\frac{\operatorname{Im} z}{M\,|cz+d|^2}.$$

By Definition 1 we have $\operatorname{Im} z\geqslant \operatorname{Im} w$, hence (3) follows. □

## 3. Amplification and the pretrace formula

We shall combine the amplification method of Duke–Friedlander–Iwaniec with the pretrace formula of Selberg. This reduces Theorem 1 to a counting problem about integral matrices, which is solved in the next section using the gap principle discussed in the previous section.

We embed our cuspidal newform $f$ into an *orthonormal basis* $(f_j)_{j\geqslant 0}$ of Hecke–Maass eigenforms of level $N$ and trivial nebentypus, with $f_0$ a constant function and $f_j$ cuspidal otherwise. The Laplacian eigenvalue of $f_j$ is denoted by $\frac14+r_j^2$ with $r_j\in\mathbb{R}\cup[-\frac{i}{2},\frac{i}{2}]$, and the Hecke eigenvalues of $f_j$ are denoted by $\lambda_j(n)$. Correspondingly, the Laplacian eigenvalue and Hecke eigenvalues of $f$ are denoted by $\frac14+r_f^2$ and $\lambda_f(n)$.

The basic idea of amplification in the current setting is the inequality

$$h(r_f)\left|\sum_{l\geqslant 1} x_l\lambda_f(l)\right|^2 |f(z)|^2\leqslant \sum_{j\geqslant 0} h(r_j)\left|\sum_{l\geqslant 1} x_l\lambda_j(l)\right|^2 |f_j(z)|^2+\text{ cont. },$$

where $h:\mathbb{R}\cup[-\frac{i}{2},\frac{i}{2}]\to\mathbb{R}_+$ is a positive even smooth function of rapid decay, $(x_l)$ is a sequence of complex numbers supported on finitely many $l$'s coprime with $N$, and "+ cont." stands for an analogous positive contribution of the continuous spectrum. After squaring out the $l$-sum on the right hand side and applying the Hecke relations we arrive at

$$h(r_f)\left|\sum_{l\geqslant 1}x_l\lambda_f(l)\right|^2|f(z)|^2\leqslant\sum_{l\geqslant 1}|y_l|\left|\sum_{j\geqslant 0}\lambda_j(l)h(r_j)\,|f_j(z)|^2+\text{ cont. }\right|,$$

where

$$y_l:=\sum_{\substack{d\mid(l_1,l_2)\\ l=l_1l_2/d^2}}x_{l_1}\overline{x_{l_2}}=\sum_{\substack{d\geqslant 1\\ l=l_1l_2}}x_{dl_1}\overline{x_{dl_2}}. \tag{4}$$

Assume now that $h$ is the Selberg transform of a smooth and rapidly decaying point pair invariant $k(z,z')=k(u(z,z'))$, as in [4, (1.1)–(1.2)], with

$$u(z,z'):=\frac{|z-z'|^2}{\operatorname{Im}(z)\operatorname{Im}(z')},\qquad z,z'\in\mathfrak{H}. \tag{5}$$

Then the spectral $j$-sum can be rewritten as a geometric sum

$$\sum_{j\geqslant 0}\lambda_j(l)h(r_j)\,|f_j(z)|^2+\text{ cont. }=\frac{1}{\sqrt{l}}\sum_{\gamma\in M(l,N)}k(\gamma z,z),$$

where $M(l,N)$ is the set of matrices $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_2(\mathbb{Z})$ with $\det(\gamma)=l$ and $c\equiv 0\,(N)$. The identity follows by applying the $l$-th Hecke operator on the spectral expansion (cf. [4, (1.3)–(1.4)])

$$\sum_{j\geqslant 0}h(r_j)f_j(z)\overline{f_j(w)}+\text{ cont. }=\sum_{\gamma\in\Gamma_0(N)}k(\gamma z,w). \tag{6}$$

It follows that

$$h(r_f)\left|\sum_{l\geqslant 1}x_l\lambda_f(l)\right|^2|f(z)|^2\leqslant\sum_{l\geqslant 1}\frac{|y_l|}{\sqrt{l}}\sum_{\gamma\in M(l,N)}|k(\gamma z,z)|\,.$$

The choice of $h$ is irrelevant as we are only interested in the dependence of $\|f\|_\infty$ on $N$. For instance, we can work with [4, (1.5)] or [8, (2.9)]. As for the amplifier, we shall use the one in [9, § 4.1] which is a modification of the classical construction of Iwaniec. The modification enables to bypass the use of bounds towards the Ramanujan conjecture. Let

$$\mathcal{P}:=\{p\text{ prime}:\,p\nmid N\text{ and }\Lambda\leqslant p\leqslant 2\Lambda\}\quad\text{and}\quad\mathcal{P}^2:=\{p^2:\,p\in\mathcal{P}\}$$

be a large set of unramified primes and the set of their squares, and define

$$x_l:=\begin{cases}\operatorname{sgn}(\lambda_f(l)), & \text{if } l\in\mathcal{P}\cup\mathcal{P}^2\\ 0, & \text{otherwise.}\end{cases}$$

The main property of this sequence (amplifier) is that

$$\left|\sum_{l\geqslant 1}x_l\lambda_f(l)\right|\gg\Lambda^{1-\epsilon},$$

which follows at once from the relation $\lambda_f(l)^2-\lambda_f(l^2)=1$ forcing $\max(|\lambda_f(l)|,|\lambda_f(l^2)|)\geqslant 1/2$. Here and later $\epsilon$ is any positive quantity which may vary from line to line, and all implied constants may depend on $\epsilon$. In addition, $y_l$ defined in (4) satisfies:

- $y_1\ll\Lambda$ and $|y_l|\leqslant 4$ for $l>1$;

- $y_l = 0$ for $1 < l < \Lambda$ and for $l > 16\Lambda^4$;
- $y_l = 0$ for $l > 8\Lambda^3$ unless $l$ is a square in $[\Lambda^4, 16\Lambda^4]$.

We conclude, for any $\Lambda \geqslant 1$,

$$\begin{aligned}\Lambda^{2-\epsilon}\,|f(z)|^2 \ll_{r_f} \Lambda &\sum_{\gamma\in M(1,N)} |k(\gamma z,z)| \\ &+ \sum_{\Lambda\leqslant l\leqslant 8\Lambda^3} \frac{1}{\sqrt{l}} \sum_{\gamma\in M(l,N)} |k(\gamma z,z)| + \sum_{\substack{\Lambda^4\leqslant l\leqslant 16\Lambda^4\\ l\text{ square}}} \frac{1}{\sqrt{l}} \sum_{\gamma\in M(l,N)} |k(\gamma z,z)|\,,\end{aligned} \tag{7}$$

where the implied constant depends continuously on the spectral parameter $r_f$.

As discussed in the previous section, the supremum of $f$ is attained on the set $\mathcal{F}(N)$. By the rapid decay of $k$, the contribution of $\gamma$'s with $u(\gamma z,z) > N^\epsilon$ can be bounded by $\ll_A N^{-A}$ for any $A > 0$, which is negligible. For the remaining $\gamma$'s with $u(\gamma z,z) \leqslant N^\epsilon$, we shall simply use $k(\gamma z,z) \ll 1$. Therefore it remains to estimate, uniformly on $z \in \mathcal{F}(N)$, the number of integral matrices $\gamma$ satisfying certain constraints. This is achieved in Lemma 3 below.

## 4. A counting problem about integral matrices

We begin with the counting of parabolic matrices for which the argument is easier. We call a $2\times 2$ matrix *parabolic* if its characteristic polynomial is a square.

**Lemma 2.** *Assume that $z = x+iy \in \mathcal{F}(N)$ and $1 \leqslant l < y^{-2}N^{-\epsilon}$. The only parabolic matrices $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in M_2(\mathbb{Z})$ of determinant $l$ such that*

$$c \equiv 0\,(N) \quad \textit{and} \quad u(\gamma z,z) \leqslant N^\epsilon,$$

*are the multiples of the identity, i.e. $l$ is a square and $\gamma = \pm\begin{pmatrix}\sqrt{l} & 0\\ 0 & \sqrt{l}\end{pmatrix}$.*

*Proof.* The matrix $\gamma$ stabilizes a certain cusp $\mathfrak{a} \in \mathbb{P}^1(\mathbb{Q})$. Since $N$ is square-free, there exists an element $\sigma_{\mathfrak{a}} \in A_0(N)$ that sends $i\infty$ to $\mathfrak{a}$. The matrix $\gamma' := \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}$ stabilizes the cusp $i\infty$ and has determinant $l$, hence it is of the form

$$\gamma' = \pm\begin{pmatrix}\sqrt{l} & b\\ 0 & \sqrt{l}\end{pmatrix}$$

for some $b \in \mathbb{Z}$. In particular, $l$ is a perfect square. Furthermore, for $z' := \sigma_{\mathfrak{a}}^{-1}z$,

$$u(\gamma'z',z') = u(\gamma z,z) \leqslant N^\epsilon. \tag{8}$$

By assumption, $z' = x' + iy'$ satisfies $y' \leqslant y$. The distance between $\gamma'z' = x' + \frac{b}{\sqrt{l}} + iy'$ and $z'$ is

$$u(\gamma'z',z') = \frac{b^2}{ly'^2} \geqslant \frac{b^2}{ly^2}.$$

By (8) and the bound on $l$, this implies $b = 0$. ∎

We shall now count general matrices, with special attention to square determinants.

**Lemma 3.** *Let $z = x+iy \in \mathcal{F}(N)$ and set $K := 1 + LNy^2$.*

*(i) The number of matrices $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in M_2(\mathbb{Z})$ such that*

$$1 \leqslant \det(\gamma) \leqslant L, \quad c \equiv 0\,(N), \quad u(\gamma z,z) \leqslant N^\epsilon,$$

*is uniformly bounded by*

$$\ll KL^{1/2}N^\epsilon + KLN^\epsilon/(Ny).$$

*(ii) Assume that $K \leqslant N^{1-\epsilon}$. The number of matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ such that*

$$1 \leqslant \det(\gamma) \leqslant L \text{ is a square}, \quad c \equiv 0\,(N), \quad u(\gamma z, z) \leqslant N^\epsilon,$$

*is uniformly bounded by*

$$\ll K L^{1/2} N^\epsilon.$$

*Proof.* We write $l := \det(\gamma)$, and we observe that (5) gives

$$u(\gamma z, z) = \frac{\left|-cz^2 + (a-d)z + b\right|^2}{l y^2}.$$

Let us fix $c$ and examine the points $-cz^2 + (a-d)z + b$ for all the different pairs $(a-d, b)$ that occur. By $1 \leqslant l \leqslant L$ and $u(\gamma z, z) \leqslant N^\epsilon$ these points lie in a (Euclidean) disc of side length $< N^\epsilon \sqrt{K/N}$, while by Lemma 1 the minimal distance between them is at least $1/\sqrt{N}$. Therefore we can see, by decomposing the disc into equal squares of diameter slightly less than $1/\sqrt{N}$, that for any fixed $c$ the number of different pairs $(a-d, b)$ is $\ll KN^\epsilon$. In particular, for $c = 0$ we have $\ll KN^\epsilon$ pairs $(a-d, b)$ satisfying $1 \leqslant ad \leqslant L$, hence the number of $\gamma$'s with $c = 0$ is $\ll KL^{1/2}N^\epsilon$.

For $c \neq 0$ we also use that the imaginary part of $-cz^2 + (a-d)z + b$ equals $(a-d-2cx)y$, whence $u(\gamma z, z) \leqslant N^\epsilon$ forces

$$|a - d - 2cx| \leqslant L^{1/2} N^\epsilon. \tag{9}$$

This shows, in combination with the alternate form

$$u(\gamma z, z) = \frac{\left| l - |cz+d|^2 - (cz+d)(a-d-2cx)\right|^2}{l c^2 y^2},$$

that

$$\left| l - |cz+d|^2 \right| \leqslant L^{1/2} N^\epsilon \bigl(|cy| + |cz+d|\bigr) \leqslant 2L^{1/2}N^\epsilon\, |cz+d|\,.$$

This is only possible when $|cz + d| \leqslant 3L^{1/2}N^\epsilon$. Taking imaginary parts yields $c \ll L^{1/2}N^\epsilon/y$, while taking real parts and using also (9) gives $a + d \ll L^{1/2}N^\epsilon$. By $c \equiv 0\,(N)$ there are $\ll L^{1/2}N^\epsilon/(Ny)$ choices for $c \neq 0$ and each choice gives $\ll KN^\epsilon$ possibilities for the pair $(a-d, b)$. As $a + d$ can only take $\ll L^{1/2}N^\epsilon$ values, we conclude that the number of $\gamma$'s with $c \neq 0$ is $\ll KLN^\epsilon/(Ny)$.

If we further assume that $K \leqslant N^{1-\epsilon}$ and $l$ is square (still under $c \neq 0$), then we use that the triple $(c, a-d, b)$ determines

$$(a-d)^2 + 4bc = (a+d)^2 - 4l.$$

This value is nonzero by Lemma 2 and bounded above by $LN^\epsilon$ by the earlier bound $a + d \ll L^{1/2}N^\epsilon$. Furthermore, by Lemma 1, we have that $LN^\epsilon \ll y^{-2} \ll N^2$ and since $a + d$ is the mean of the divisor pair $a + d \pm 2\sqrt{l}$, we have that $a + d$ can only take $\ll N^\epsilon$ values for any triple $(c, a-d, b)$. Therefore the number of $\gamma$'s with $c \neq 0$ and square determinant is $\ll KL^{1/2}N^\epsilon/(Ny) \ll KL^{1/2}N^\epsilon$. Here we used Lemma 1 again.

□

## 5. Proof of Theorem 1

Since $f$ is a cusp form, it decays rapidly at each cusp. As a first step we shall need a quantitative bound at the cusp $\infty$. We recall the following standard result (see [8, § 3.2] or [2, (92) & (27)]).

**Lemma 4.** *Uniformly in $x + iy \in \mathfrak{H}$ and $N$,*

$$f(x+iy) \ll_{\lambda,\epsilon} N^{-1/2+\epsilon} y^{-1/2},$$

*where the implied constant depends continuously on the Laplacian eigenvalue $\lambda$ of $f$.*

In particular, in proving Theorem 1 we can restrict to $x + iy \in \mathcal{F}(N)$ with

$$N^{-1} \ll y \ll N^{-5/6}. \tag{10}$$

Note that the lower bound is automatic by Lemma 1.

Now we apply Lemma 3 to (7). Using $k(\gamma z, z) \ll 1$ and a dyadic decomposition in $l$, we infer

$$f(x+iy) \ll_{\lambda,\epsilon} \Lambda^{-1+\epsilon} N^{\epsilon} \left(\Lambda(1 + Ny^2) + (1 + L_1 Ny^2)(1 + L_1^{1/2}/(Ny)) + (1 + L_2 Ny^2)\right)^{1/2}$$

for some

$$\Lambda \ll L_1 \ll \Lambda^3 \qquad \text{and} \qquad \Lambda^4 \ll L_2 \ll \Lambda^4.$$

If we specify the amplifier length as

$$\Lambda := (Ny^2)^{-1/3},$$

then by (10) we have $N^{2/9} \ll \Lambda \ll N^{1/3}$ and also

$$\begin{aligned} f(x+iy) &\ll_{\lambda,\epsilon} N^{\epsilon} \Lambda^{-1} \left(\Lambda + \Lambda^{3/2}/(Ny) + \Lambda^4 Ny^2\right)^{1/2} \\ &\ll_{\lambda,\epsilon} N^{\epsilon} \left(N^{1/6} y^{1/3} + N^{-5/12} y^{-1/3}\right) \\ &\ll_{\lambda,\epsilon} N^{-1/12+\epsilon}. \end{aligned}$$

As $|f(x+iy)|$ attains the value $\|f\|_\infty$ at some $x + iy \in \mathcal{F}(N)$, the proof of Theorem 1 is complete.

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