THE SPECTRAL DECOMPOSITION OF SHIFTED CONVOLUTION SUMS

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Abstract

Let π_1, π_2 be cuspidal automorphic representations of $\mathrm{PGL}_2(\mathbb{R})$ of conductor 1 and Hecke eigenvalues $\lambda_{\pi_{1,2}}(n)$, and let $h > 0$ be an integer. For any smooth compactly supported weight functions $W_{1,2} : \mathbb{R}^\times \to \mathbb{C}$ and any $Y > 0$, a spectral decomposition of the shifted convolution sum

$$\sum_{m \pm n = h} \frac{\lambda_{\pi_1}(|m|)\lambda_{\pi_2}(|n|)}{\sqrt{|mn|}} W_1\Big(\frac{m}{Y}\Big) W_2\Big(\frac{n}{Y}\Big)$$

is obtained. As an application, a spectral decomposition of the Dirichlet series

$$\sum_{\substack{m,n \geqslant 1 \\ m-n=h}} \frac{\lambda_{\pi_1}(m)\lambda_{\pi_2}(n)}{(m+n)^s} \Big(\frac{\sqrt{mn}}{m+n}\Big)^{100}$$

is proved for $\Re s > 1/2$ with polynomial growth on vertical lines in the s-aspect and uniformity in the h-aspect.

1. Introduction

Let $G = \mathrm{PGL}_2(\mathbb{R})$, and let $\Gamma = \mathrm{PGL}_2(\mathbb{Z})$. There is a spectral decomposition

$$L^2(\Gamma\backslash G) = \int_\tau V_\tau \, d\tau, \tag{1}$$

where (τ, V_τ) are irreducible automorphic representations of $\Gamma\backslash G$ (including the trivial representation) and $d\tau$ is the spectral measure defined as follows. The trivial representation $(\tau_0, \mathbb{C})$ has spectral measure 1. Each nontrivial representation (τ, V_τ) is generated by a modular form on $\mathscr{H}$ with respect to the full modular group $\mathrm{SL}_2(\mathbb{Z})$;

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hence it corresponds to some Laplacian eigenvalue

$$\lambda_\tau = \frac{1}{4} - \nu_\tau^2 \in \mathbb{R} \quad \text{with } \Re \nu_t \geqslant 0,\ \Im \nu_\tau \geqslant 0.$$

We use the notation

$$\tilde{\lambda}_\tau := 1 + |\lambda_\tau|.$$

A representation (τ, V_τ) generated by a holomorphic or Maass cusp form has spectral measure 1. We can assume that the underlying cusp form is a Hecke eigenform, and we denote by $\lambda_\tau(n)$ its nth Hecke eigenvalue. A representation (τ, V_τ) generated by an Eisenstein series with $\Re\nu_\tau = 0$ and $\Im\nu_\tau > 0$ has spectral measure $\frac{d\nu_\tau}{2\pi i}$, and we denote by $\lambda_\tau(n)$ the divisor sum $\sum_{ab=n}(a/b)^{\nu_\tau}$.

For a function $W \in \mathscr{C}^d(\mathbb{R}^\times)$, we denote

$$\|W\|_{A^d} := \sum_{j=0}^{d} \Big(\int_{\mathbb{R}^\times} (|u| + |u|^{-1})^d \left| \frac{d^j W}{du^j} \right|^2 d^\times u \Big)^{1/2}, \tag{2}$$

provided the integral is finite. Here, $d^\times u := \frac{du}{|u|}$ is the Haar measure on $\mathbb{R}^\times$.

In this article, we obtain a spectral decomposition for shifted convolution sums of Hecke eigenvalues of two arbitrary cusp forms, as well as a spectral decomposition of the corresponding Dirichlet series with polynomial growth estimates on vertical lines and uniform dependence with respect to the shift parameter.

THEOREM 1

Let π_1 and π_2 be arbitrary cuspidal automorphic representations of $\Gamma\backslash G$. Let $a, b, c \geqslant 0$ be arbitrary integers, and let $W_{1,2} : \mathbb{R}^\times \to \mathbb{C}$ be arbitrary functions such that $\|W_{1,2}\|_{A^d}$ exist for $d = 18 + 2a + 2b + 4c$. Then there exist functions $W_\tau : (0, \infty) \to \mathbb{C}$ depending only on $\pi_{1,2}$, $W_{1,2}$, and τ such that the following two properties hold.

- *If $h > 0$ is an arbitrary integer and $Y > 0$ is arbitrary, then one has the decomposition over the full spectrum (excluding the trivial representation)*

$$\sum_{m+n=h} \frac{\lambda_{\pi_1}(|m|)\lambda_{\pi_2}(|n|)}{\sqrt{|mn|}} W_1\Big(\frac{m}{Y}\Big) W_2\Big(\frac{n}{Y}\Big) = \int_{\tau \neq \tau_0} \frac{\lambda_\tau(h)}{\sqrt{h}} W_\tau\Big(\frac{h}{Y}\Big)\, d\tau. \tag{3}$$

- *If $0 < \varepsilon < 1/4$ is arbitrary, then for $y > 0$, one has the uniform bound*

$$\int_{\tau \neq \tau_0} \tilde{\lambda}_\tau^c \left| \Big(y \frac{d}{dy} \Big)^a W_\tau(y) \right| d\tau \ll_{\varepsilon,a,b,c} C_{a,b,c} \min(y^{1/2-\varepsilon}, y^{1/2-b-\varepsilon}), \tag{4}$$

where

$$C_{a,b,c} := (\tilde{\lambda}_{\pi_1} + \tilde{\lambda}_{\pi_2})^{11+a+b+2c} \sum_{d_1+d_2=18+2a+2b+4c} \|W_1\|_{A^{d_1}} \|W_2\|_{A^{d_2}}.$$

Remark 1
The condition on $W_{1,2}$ is satisfied as long as these functions decay rapidly near zero and $\pm\infty$. One can view W as a function on $(0,\infty)$ times (a subset of) the unitary dual of $\Gamma\backslash G$. Implicit in (3) is the fact that this function is integrable in the second variable with respect to the spectral measure. By explicating W, stronger regularity properties follow. Finally, the right-hand side of (3) also captures the summation condition $m-n=h$ if the support of W_2 is on the negative axis.

By a result of Kim and Sarnak in [K, Appendix 2], we have

$$|\lambda_\tau(h)| \leqslant d(h)h^\theta$$

for $\theta = 7/64$ and any cuspidal automorphic representation τ of $\Gamma\backslash G$, where $d(n)$ is the number of divisors of n. Thus for two smooth compactly supported functions $W_{1,2}$, the bound (4) immediately gives

$$\sum_{m\pm n=h} \lambda_{\pi_1}(|m|)\lambda_{\pi_2}(|n|)\, W_1\Big(\frac{m}{Y}\Big)W_2\Big(\frac{n}{Y}\Big) \ll_{\varepsilon,\pi_1,\pi_2,W_1,W_2} h^\theta Y^{1/2}(hY)^\varepsilon \tag{5}$$

for any $\varepsilon > 0$, uniformly in $h > 0$ (see, e.g., [B]). The novelty of Theorem 1 is to obtain an exact spectral decomposition of the left-hand side of (3) rather than an upper bound. Thus Theorem 1 develops its full strength when the left-hand side of (3) is averaged over h, as is necessary, for example, to prove subconvex estimates for certain families of L-functions. If π is a cuspidal automorphic representation of G of arbitrary conductor and χ is a primitive Dirichlet character of conductor q, then without much effort, we can deduce the Burgess-type bound

$$L\Big(\frac{1}{2}, \pi\otimes\chi\Big) \ll_{\pi,\varepsilon} q^{(3+2\theta)/8+\varepsilon} \tag{6}$$

by combining a slight generalization of Theorem 1 with amplification and a large sieve inequality (cf. Remark 2). In fact, a similar result can be derived more generally over totally real number fields, where classical methods are much harder to implement. We postpone this discussion to [BH] and give here another application of Theorem 1 which was our initial motivation.

THEOREM 2
Let π_1 and π_2 be arbitrary cuspidal automorphic representations of $\Gamma\backslash G$, and let

$c,k \geqslant 0$ be arbitrary integers satisfying $k > 60 + 12c$. There are holomorphic functions $F_{k,\tau} : \{s : 1/2 < \Re s < 3/2\} \to \mathbb{C}$ depending only on $\pi_{1,2}$, c, k, and τ such that the following two properties hold.

- *If $h > 0$ is an arbitrary integer, then one has the decomposition over the full spectrum (excluding the trivial representation)*

$$\sum_{\substack{m,n\geqslant 1\\ m-n=h}} \frac{\lambda_{\pi_1}(m)\lambda_{\pi_2}(n)\,(mn)^{(k-1)/2}}{(m+n)^{s+k-1}} = h^{1/2-s}\int_{\tau\neq\tau_0} \lambda_\tau(h)\,F_{k,\tau}(s)\,d\tau, \quad \Re s > 1. \tag{7}$$

- *One has the uniform bound*

$$\int_{\tau\neq\tau_0} \tilde{\lambda}_\tau^c\,|F_{k,\tau}(s)|\,d\tau \ll_{\varepsilon,k} (\tilde{\lambda}_{\pi_1}+\tilde{\lambda}_{\pi_2})^{12+4c}\,|s|^{22+4c}, \quad \frac{1}{2}+\varepsilon < \Re s < \frac{3}{2}. \tag{8}$$

Remark 2

Theorems 1 and 2 extend to arbitrary level and to the more general additive constraints $\ell_1 m \pm \ell_2 n = h$ in a straightforward fashion with good control in the $\ell_{1,2}$ -parameters. Here, small technical complications arise from the possible presence of complementary series representations τ and the presence of additional cusps. By [K, Appendix 2], complementary series representations satisfy $0 < \nu_\tau \leqslant \theta$ for $\theta = 7/64$. Accordingly, the right-hand side of (4) becomes (cf. (26))

$$\ll_{\varepsilon,a,b,c} C'_{a,b,c}\,(\ell_1\ell_2)^{1/2+\varepsilon}\,\min(y^{1/2-\theta-\varepsilon},\, y^{1/2-\theta-b-\varepsilon})$$

for any $0 < \varepsilon < 1/4-\theta$, where $C'_{a,b,c}$ is a constant similar to $C_{a,b,c}$ in Theorem 1 but with a larger exponent for $\tilde{\lambda}_{\pi_1}+\tilde{\lambda}_{\pi_2}$ and a sum over larger d_1+d_2. For the Eisenstein spectrum, the exponent of $\ell_1\ell_2$ can be lowered to $1/4+\varepsilon$. Similarly, the right-hand side of (8) becomes, for suitable constants A, B, $C > 0$,

$$\ll_{\varepsilon,k} (\ell_1\ell_2)^A(\tilde{\lambda}_{\pi_1}+\tilde{\lambda}_{\pi_2})^{B+4c}\,|s|^{C+4c}, \quad \frac{1}{2}+\theta+\varepsilon < \Re s < \frac{3}{2}.$$

These extensions enable considerable simplifications in arguments leading to subconvexity results such as (6), which appeared originally in [BHM], or the more difficult case of Rankin-Selberg L-functions treated in [HM]. In this article, we have decided to present the theorems in a special case in order to keep the notational burden minimal and to emphasize the conceptual simplicity of the approach.

Selberg [S] proved in 1965 that for holomorphic cusp forms (i.e., when π_1 and π_2 are in the discrete series), the left-hand side of (7) is meromorphic in s and holomorphic for $\Re s > 1/2$.* (Note that for Γ, poles on the segment $1/2 < s \leqslant 1$ do not occur.) Good

*Selberg adds, regretfully, “We cannot make much use of this function at present” in [S, page 14].

[G1], [G2] was the first to use the spectral decomposition of shifted convolution sums coming from the Fourier coefficients of a holomorphic cusp form for estimating the second moment of the corresponding modular L-function on the critical line. Good's new key ingredient was to show the polynomial growth on vertical lines.

The spectral decomposition in the nonholomorphic case has resisted all attempts so far. Good's method, based on the fact that holomorphic cusp forms are linear combinations of Poincaré series, is not applicable here. Jutila [Ju1] and Sarnak [S1], [S2] independently considered an approximation of the Dirichlet series (7), along with a spectral decomposition, which can be continued to the half-plane $\Re s > 1/2$ with a bound $O(h^{1/2-\sigma+\theta+\varepsilon}|s|^A)$ on vertical lines $\Re s = \sigma$. However, this approximation introduces an error of $O(h^{1-\sigma+\varepsilon}|s|^B)$ which one would like to remove.* Harcos found in his dissertation [H2, Chapter 5] that the error signifies missing harmonics, and he anticipated that analysis on $\Gamma\backslash G$ is key to obtaining a complete system. Independently, Motohashi [M2], [M3] brought the representation theory of $\Gamma\backslash G$ and the Kirillov model into the discussion. In this article, we pursue Motohashi's approach further. One of the new ideas which we employ is the use of Sobolev norms for smooth vectors, inspired by the recent work of Venkatesh [V] which allows for a soft treatment and avoids the difficulties imposed by Poincaré series and the estimation of triple product periods. Sobolev norms also play important roles in related works of Bernstein and Reznikov [BR1]–[BR4] and Cogdell and Piatetski-Shapiro [CP]. Shortly after this article was finished, Motohashi [M4] gave an alternative proof of Theorem 2 which makes regularity properties of the weight functions $F_{k,\tau}(s)$ more transparent.

Finally, we note that there are other important techniques for understanding shifted convolution sums. The archetypes of shifted convolution sums are the additive divisor sums, which have been studied extensively.† These special sums arise from Eisenstein series rather than cusp forms, and their spectral decomposition is very explicitly known by the work of Motohashi [M1] (cf. [JM, Lemma 4]). In the general case, variants of the circle method with the Kloosterman refinement have been particularly successful (see [DFI1], [DFI2], [Ju2], [H1], [HM], [B], [BHM]). Recently, Venkatesh [V] developed a geometric method, based on equidistribution and mixing, which can be applied for shifted convolution sums of higher rank. In our context, the left-hand side of (3) is an automorphic period over a closed horocycle, so that [V, Theorem 3.2] implies a weaker but nontrivial version of (5).

*For a striking recent application of the approximate spectral decomposition, see [LLY]. For a careful analysis of the error term, see [Ju3].

†In fact, one can trace the history back to various elegant identities published by Jacobi [J] in 1829.

2. Bounds for the discrete spectrum

2.1. Kirillov model and Sobolev norms

We use the notation

$$n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad a(u) := \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \qquad k(\theta) := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and we think of these matrices as elements of $G = \mathrm{PGL}_2(\mathbb{R})$. In addition, we write

$$e(x) := e^{2\pi i x}, \quad x \in \mathbb{R}.$$

Let (π, V_π) be a representation generated by a cusp form on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}$. Then (π, V_π) is contained in $L^2(\Gamma\backslash G)$; hence it is equipped with a canonical inner product given by

$$\langle \phi_1, \phi_2 \rangle := \int_{\Gamma\backslash G} \phi_1(g)\overline{\phi_2(g)}\, dg, \tag{9}$$

where $dg := dx\left(\frac{du}{u^2}\right)\left(\frac{d\theta}{\pi}\right)$ for $g = n(x)a(u)k(\theta)$ is the Haar measure on G. The Kirillov model $\mathscr{K}(\pi)$ realizes π in $V_{\mathscr{K}(\pi)} := L^2(\mathbb{R}^\times, d^\times u)$, which is equipped with its own canonical inner product, given by

$$\langle W_1, W_2 \rangle := \int_{\mathbb{R}^\times} W_1(u)\overline{W_2(u)}\, d^\times u. \tag{10}$$

Let $\|\cdot\|$ denote the norms determined by these inner products.

For a smooth vector $\phi \in V_\pi^\infty$, we define the corresponding smooth vector $W_\phi \in V_{\mathscr{K}(\pi)}^\infty$ as

$$W_\phi(u) := \int_0^1 \phi\big(n(x)a(u)\big)\, e(-x)\, dx, \quad u \in \mathbb{R}^\times. \tag{11}$$

It gives rise to the Fourier decomposition

$$\phi\big(n(x)a(u)\big) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\lambda_\pi(|n|)}{\sqrt{|n|}} W_\phi(nu)\, e(nx), \quad x \in \mathbb{R},\ u \in \mathbb{R}^\times, \tag{12}$$

where $\lambda_\pi(n)$ denotes the nth Hecke eigenvalue of the cusp form on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}$ which generates (π, V_π). This follows from Shalika's multiplicity-one theorem combined with standard facts about Hecke operators (see [CP, Sections 4.1, 6.2] and [DFI3, (6.14), (6.15)]). We have the uniform bound (see [DFI3, Proposition 19.6])

$$\sum_{1\leqslant n\leqslant x} |\lambda_\pi(n)|^2 \ll_\varepsilon x(x|\nu_\pi|)^\varepsilon. \tag{13}$$

By Kirillov's theorem (see [CP, Sections 4.2 – 4.4]), the canonical inner products of V_π and $V_{\mathscr{K}(\pi)}$ are related by a proportionality constant depending only on π,

$$\langle \phi_1, \phi_2 \rangle = C_\pi \langle W_{\phi_1}, W_{\phi_2} \rangle, \quad \phi_1, \phi_2 \in V_\pi^\infty. \tag{14}$$

The relations (12) and (14) can also be verified by classical means (see [DFI3, Section 4] and [BM, Sections 2, 4]).

We can evaluate the proportionality constant C_π as follows. Let

$$E(g, s) := \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} U_s(\gamma g)$$

with $U_s(n(x)a(u)k(\theta)) := |u|^s$ and $\Gamma_\infty := \{n(x) : x \in \mathbb{Z}\}$ denote the standard weight-zero Eisenstein series on G, and let $\phi \in V_\pi^\infty$ be any vector of pure weight (cf. Section 2.2). Then for $\Re s > 1$, we have, by the Rankin-Selberg unfolding technique, (12), and Parseval,

$$\begin{aligned}
\big\langle |\phi(g)|^2, E(g, s) \big\rangle &= \frac{1}{2} \int_{\mathbb{R}^\times} \int_0^1 \left| \phi\big(n(x)a(u)\big) \right|^2 |u|^{s-2} \, dx \, du \\
&= \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{|\lambda_\pi(|n|)|^2}{|n|} \int_{\mathbb{R}^\times} |W_\phi(nu)|^2 \, |u|^{s-1} \, d^\times u \\
&= \sum_{n=1}^\infty \frac{|\lambda_\pi(n)|^2}{n^s} \int_{\mathbb{R}^\times} |W_\phi(u)|^2 \, |u|^{s-1} \, d^\times u.
\end{aligned}$$

Taking residues of both sides at $s = 1$ yields

$$\|\phi\|^2 \operatorname*{res}_{s=1} E(g, s) = \|W_\phi\|^2 \operatorname*{res}_{s=1} \frac{L(s, \pi \otimes \tilde{\pi})}{\zeta(2s)}.$$

Using (14), we conclude that*

$$C_\pi = \frac{\operatorname{vol}(\Gamma \backslash G)}{\zeta(2)} L(1, \operatorname{Ad}^2 \pi).$$

In particular, by (13) and [HL, Theorem 0.2], we know that

$$\tilde{\lambda}_\pi^{-\varepsilon} \ll_\varepsilon C_\pi \ll_\varepsilon \tilde{\lambda}_\pi^{\varepsilon} \tag{15}$$

for any $\varepsilon > 0$.

*Here, the adjoint square is the same as the symmetric square.

We introduce Sobolev norms for vectors in V_π^∞ and $V_{\mathscr{K}(\pi)}^\infty$ in terms of the derived action of the Lie algebra $\mathfrak{g}$ of G. We consider the usual basis of $\mathfrak{g}$ consisting of

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad L := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and note the commutation relations $[H, R] = 2R$, $[H, L] = -2L$, $[R, L] = H$. Then for smooth vectors $\phi \in V_\pi^\infty$ and $W \in V_{\mathscr{K}(\pi)}^\infty$, we define

$$\|\phi\|_{S^d} := \sum_{\mathrm{ord}(\mathscr{D}) \leqslant d} \|\mathscr{D}\phi\| \qquad \text{and} \qquad \|W\|_{S^d} := \sum_{\mathrm{ord}(\mathscr{D}) \leqslant d} \|\mathscr{D}W\|, \tag{16}$$

where $\mathscr{D}$ ranges over all monomials in H, R, L of order at most d in the universal enveloping algebra $U(\mathfrak{g})$. We can see how $U(\mathfrak{g})$ acts on $\mathscr{K}(\pi)$: H acts by $2u\frac{d}{du}$, and R acts by $2\pi i u$ (cf. [Bu, page 155]). The Casimir element $H^2 + 2RL + 2LR = H^2 - 2H + 4RL$ acts by $-4\lambda_\pi$; hence RL acts by $-\lambda_\pi + u^2\frac{d^2}{du^2}$, and therefore L acts by $(2\pi i)^{-1}\big(-\lambda_\pi u^{-1} + u\frac{d^2}{du^2}\big)$. This way, we obtain the estimate (see [V, Lemma 8.4])

$$\|W\|_{S^d} \ll_d \tilde{\lambda}_\pi^d \, \|W\|_{A^{2d}}. \tag{17}$$

Here, the norm $\|\cdot\|_{A^d}$ was defined in (2).

2.2. Normalized Whittaker functions

In order to understand the behavior of $W_\phi(u)$ near zero, we decompose $\phi \in V_\pi$ into pure weight pieces

$$\phi = \sum_{p \in \mathbb{Z}} \phi_p, \tag{18}$$

where $\phi_p \in V_\pi^\infty$ satisfies

$$\phi_p\big(gk(\theta)\big) = e^{2ip\theta}\phi_p(g), \quad g \in G,\ \theta \in \mathbb{R}. \tag{19}$$

Convergence of (18) is understood in the L^2-norm. Correspondingly, W_ϕ decomposes in $V_{\mathscr{K}(\pi)}$ as

$$W_\phi = \sum_{p \in \mathbb{Z}} W_{\phi_p}. \tag{20}$$

Note that Parseval gives

$$\|\phi\|^2 = \sum_{p \in \mathbb{Z}} \|\phi_p\|^2. \tag{21}$$

It is known (cf. [BM, (2.14)–(2.27)]) that each $W_{\phi_p}(u)$ is a constant multiple of some normalized Whittaker function

$$\tilde{W}_{p,\pi}(u) := \frac{\varepsilon_{p,\pi}(\operatorname{sgn}(u))\, W_{\operatorname{sgn}(u)p,\nu_\pi}(|u|)}{|\Gamma(1/2-\nu_\pi+\operatorname{sgn}(u)p)\Gamma(1/2+\nu_\pi+\operatorname{sgn}(u)p)|^{1/2}}, \quad u \in \mathbb{R}^\times, \tag{22}$$

where $\varepsilon_{p,\pi} : \{\pm 1\} \to \{z : |z| = 1\}$ is a suitable phase factor, $W_{\alpha,\beta}$ is the standard Whittaker function (see [WW, Chapter XVI]), and the right-hand side is understood to be zero if one of $1/2 \pm \nu_\pi + \operatorname{sgn}(u)p$ is a nonpositive integer. By [BM, Section 4], the functions $\tilde{W}_{p,\pi} : \mathbb{R}^\times \to \mathbb{C}$ form an orthonormal basis of $L^2(\mathbb{R}^\times, d^\times u)$; therefore

$$|W_{\phi_p}(u)| = \|W_{\phi_p}\|\, |\tilde{W}_{p,\pi}(u)|, \quad u \in \mathbb{R}^\times. \tag{23}$$

We can choose the parameter ν_π so that $\Re\nu_\pi \geqslant 0$. Then for $\tilde{W}_{p,\pi}(u) \neq 0$, we have

$$\frac{\Gamma(1/2+\nu_\pi+\operatorname{sgn}(u)p)}{\Gamma(1/2-\nu_\pi+\operatorname{sgn}(u)p)} \ll (|p|+|\nu_\pi|+1)^{2\Re\nu_\pi},$$

so that the uniform bound [BM, (4.5)] (whose proof applies in all cases) yields

$$\tilde{W}_{p,\pi}(u) \ll |u|^{1/2}\left(\frac{|u|}{|p|+|\nu_\pi|+1}\right)^{-1-\Re\nu_\pi} \exp\left(-\frac{|u|}{|p|+|\nu_\pi|+1}\right), \quad u \in \mathbb{R}^\times. \tag{24}$$

If (π, V_π) belongs to the principal series (i.e., $\Re\nu_\pi = 0$), then for $0 < \varepsilon < 1$, we also have the uniform bound (cf. [BM, (4.3)])

$$\tilde{W}_{p,\pi}(u) \ll_\varepsilon (|p|+|\nu_\pi|+1)\, |u|^{1/2-\varepsilon}, \quad u \in \mathbb{R}^\times. \tag{25}$$

Indeed, for $|u| \geqslant 1$, the bound (24) is stronger, while for $|u| < 1$, it is an immediate consequence of [BM, (4.2)] and [HM, Appendix]. We show below that for $0 < \varepsilon < 1/4$, this bound holds true even when (π, V_π) belongs to the discrete series. We note that representations belonging to the complementary series (i.e., $0 < \nu_\pi < 1/2$) do not occur in (1), but for completeness, we record an analogue of (25) for this case, valid for $0 < \varepsilon < 1$ (cf. (24), [BM, (4.2)], [HM, Appendix]):

$$\tilde{W}_{p,\pi}(u) \ll_\varepsilon (|p|+|\nu_\pi|+1)^{1+\nu_\pi}\, |u|^{1/2-\nu_\pi-\varepsilon}, \quad u \in \mathbb{R}^\times. \tag{26}$$

If (π, V_π) belongs to the discrete series, then $\nu_\pi = \ell - 1/2$, where $\ell \geqslant 1$ is an integer. For $|p| < \ell$, we have $\tilde{W}_{p,\pi} = 0$. For $|p| \geqslant \ell$, it follows from (24) via $\exp(-t) \ll t^{-1}$ (or from [BM, (2.16)] by two integrations by parts) that

$$\tilde{W}_{p,\pi}(u) \ll |3p|^{\ell+3/2}\, |u|^{-\ell-1}, \quad u \in \mathbb{R}^\times. \tag{27}$$

The Mellin transform of $W_{p,\pi}(u)$ satisfies the Jacquet-Langlands local functional equation (see [BM, (4.11)]), which is reflected in the convolution identity (see [BM, (4.9)])

$$\tilde{W}_{p,\pi}(u) = (-1)^p \int_0^\infty j_{\ell-1/2}(y)\, \tilde{W}_{p,\pi}\Big(\frac{y}{u}\Big)\, d^\times y, \quad u \in \mathbb{R}^\times,$$

where

$$j_{\ell-1/2}(y) := (-1)^\ell\, 2\pi \sqrt{y}\, J_{2\ell-1}(4\pi\sqrt{y}), \quad y > 0.$$

Using the bound [BM, (2.46)] for the kernel $j_{\ell-1/2}(y)$, we can conclude that

$$\tilde{W}_{p,\pi}(u) \ll \int_0^\infty \min(y^{1/4}, y^\ell)\, \Big|\tilde{W}_{p,\pi}\Big(\frac{y}{u}\Big)\Big|\, d^\times y, \quad u \in \mathbb{R}^\times.$$

We split the integral at $y = |3pu|$ and estimate the two pieces separately. On the one hand, by the Cauchy-Schwarz inequality and $\|\tilde{W}_{p,\pi}\| = 1$,

$$\begin{aligned}\int_0^{|3pu|} \cdots &\leqslant \Big\{ \int_0^{|3pu|} \min(y^{1/2}, y^{2\ell})\, d^\times y \Big\}^{1/2} \\ &\ll \min(|3pu|^{1/4}, |3pu|^\ell).\end{aligned}$$

On the other hand, by (27),

$$\begin{aligned}\int_{|3pu|}^\infty \cdots &\ll |3p|^{\ell+3/2} \int_{|3pu|}^\infty \min(y^{1/4}, y^\ell) \Big(\frac{y}{|u|}\Big)^{-\ell-1} d^\times y \\ &\ll |p|^{1/2} \min(|3pu|^{1/4}, |3pu|^\ell).\end{aligned}$$

All in all, we see that for any $0 < \varepsilon < 1/4$, we have

$$\tilde{W}_{p,\pi}(u) \ll |p|^{1/2} \min(|3pu|^{1/4}, |3pu|^\ell) \leqslant |p|^{1/2} |3pu|^{1/2-\varepsilon}, \quad u \in \mathbb{R}^\times.$$

This implies (25), as claimed.

2.3. Bounds for smooth vectors

We can derive a bound for $\|\phi\|_\infty$ in terms of a suitable Sobolev norm of ϕ (cf. (16)). Let $y_0 := |p| + |\nu_\pi| + 1$. By (12), (19), and (23)–(25), we have

$$\begin{aligned}\|\phi_p\|_\infty &\leqslant \sup_{y>1/2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{|\lambda_\pi(|n|)|}{\sqrt{|n|}} |W_{\phi_p}(ny)| \\ &\ll_\varepsilon \|W_{\phi_p}\| \sup_{y>1/2} \Big\{ \sum_{1\leqslant n\leqslant y_0/y} \frac{|\lambda_\pi(n)|}{\sqrt{n}} y_0 (ny)^{1/2-\varepsilon} \\ &\qquad + \sum_{n>y_0/y} \frac{|\lambda_\pi(n)|}{\sqrt{n}} \frac{y_0}{\sqrt{ny}} e^{-ny/y_0} \Big\}.\end{aligned}$$

Together with (13)–(15), we find that

$$\|\phi_p\|_\infty \ll_\varepsilon \|W_{\phi_p}\| \sup_{y>1/2} \frac{y_0^2}{\sqrt{y}}\big(yy_0(1+|\nu_\pi|)\big)^\varepsilon \ll (|p|+|\nu_\pi|+1)^{2+\varepsilon}\,\|\phi_p\|.$$

Since $\phi \in V_\pi^\infty$, the norm $\|\phi_p\|$ decays fast in p, which enables us to derive an analogue of this inequality for ϕ. Indeed, $R-L \in \mathfrak{g}$ acts by the differential operator $\frac{d}{d\theta}$ (cf. [Bu, page 155]); hence it follows via (19) that

$$\|\phi_p\|_\infty \ll_\varepsilon \tilde{\lambda}_\pi^{1+\varepsilon}(1+|p|)^{-1+\varepsilon}\,\|(1+R-L)^3\phi_p\|.$$

We apply $(1+R-L)^3 \in U(\mathfrak{g})$ on the weight decomposition (18); then by Parseval (cf. (21)), the Cauchy-Schwarz inequality, and (18), we obtain the uniform bound*

$$\|\phi\|_\infty \ll_\varepsilon \tilde{\lambda}_\pi^{1+\varepsilon}\,\|\phi\|_{S^3}. \tag{28}$$

In a similar fashion, we can derive pointwise bounds for W_ϕ. First, we combine (23), (25), (14), and (15) to see that for any $0<\varepsilon<1/4$,

$$W_{\phi_p}(u) \ll_\varepsilon (|p|+|\nu_\pi|+1)^{1+\varepsilon}\,\|\phi_p\|\,|u|^{1/2-\varepsilon}, \quad u\in\mathbb{R}^\times,$$

and then we use the action of $R-L\in\mathfrak{g}$ to conclude that

$$W_{\phi_p}(u) \ll_\varepsilon \tilde{\lambda}_\pi^{1/2+\varepsilon}(1+|p|)^{-1+\varepsilon}\,\|(1+R-L)^2\phi_p\|\,|u|^{1/2-\varepsilon}, \quad u\in\mathbb{R}^\times.$$

We apply $(1+R-L)^2 \in U(\mathfrak{g})$ on the weight decomposition (18); then by Parseval (cf. (21)), the Cauchy-Schwarz inequality, and (20), we obtain the uniform bound

$$W_\phi(u) \ll_\varepsilon \tilde{\lambda}_\pi^{1/2+\varepsilon}\,\|\phi\|_{S^2}\,|u|^{1/2-\varepsilon}, \quad u\in\mathbb{R}^\times.$$

Finally, by replacing ϕ by $(\pm1+H^2+2RL+2LR)^c R^b H^a\phi$, we obtain the more general inequality (cf. comments after (16))

$$\left(u\frac{d}{du}\right)^a W_\phi(u) \ll_{\varepsilon,a,b,c} \tilde{\lambda}_\pi^{1/2-c+\varepsilon}\,\|\phi\|_{S^{2+a+b+2c}}\,\min(|u|^{1/2-\varepsilon},|u|^{1/2-b-\varepsilon}), \quad u\in\mathbb{R}^\times, \tag{29}$$

for any $0<\varepsilon<1/4$ and any integers $a,b,c\geqslant 0$.

*Less-explicit versions of this bound were derived by Bernstein and Reznikov [BR2, Proposition 4.1] and Venkatesh [V, Lemma 9.3] in more general contexts, without recourse to Whittaker functions.

3. Bounds for the continuous spectrum

Let (π, V_π) be a representation generated by an Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}$. As (π, V_π) is not contained in $L^2(\Gamma\backslash G)$, we cannot use the definition (9) as in the cuspidal case. Nevertheless, we can still define the Kirillov model $\mathscr{K}(\pi)$ and use the definitions (10) and (11). It turns out that for the purpose of spectral decomposition, the right analogue of (9) reads*

$$\langle \phi_1, \phi_2 \rangle := \pi^{-1} |\zeta(1+2\nu_\pi)|^2 \, \langle W_{\phi_1}, W_{\phi_2} \rangle, \tag{30}$$

and then we have (14) and the *lower bound part of* (15) with

$$C_\pi := \pi^{-1} |\zeta(1+2\nu_\pi)|^2.$$

Note that $\nu_\pi = 0$ does not occur for a nonzero Eisenstein series. Let $\|\cdot\|$ denote the norms on V_π and $V_{\mathscr{K}(\pi)}$ determined by these inner products; then (16) defines the corresponding Sobolev norms $\|\cdot\|_{S^d}$ on V_π^∞ and $V_{\mathscr{K}(\pi)}^\infty$.

For a smooth vector $\phi \in V_\pi^\infty$, we have the following analogue of (12):

$$\phi\big(n(x)a(u)\big) = W_{\phi,0}(y) + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\lambda_\pi(|n|)}{\sqrt{|n|}} W_\phi(nu)\, e(nx), \quad x \in \mathbb{R},\ u \in \mathbb{R}^\times, \tag{31}$$

where

$$W_{\phi,0}(u) := \int_0^1 \phi\big(n(x)a(u)\big)\, dx, \quad u \in \mathbb{R}^\times,$$

and

$$\lambda_\pi(n) := \sum_{ab=n} \Big(\frac{a}{b}\Big)^{\nu_\pi}.$$

Equations (18)–(21) hold true as in the cuspidal case.

It is known (cf. [BM, (3.31), (2.16)]) that each $W_{\phi_p}(u)$ is a constant multiple of some normalized Whittaker function of the form (22); therefore we can conclude, exactly as in the cuspidal case, that

$$\Big(u\,\frac{d}{du}\Big)^a W_\phi(u) \ll_{\varepsilon,a,b,c} \tilde{\lambda}_\pi^{1/2-c+\varepsilon}\, \|\phi\|_{S^{2+a+b+2c}} \min(|u|^{1/2-\varepsilon}, |u|^{1/2-b-\varepsilon}), \quad u \in \mathbb{R}^\times, \tag{32}$$

for any $0 < \varepsilon < 1/4$ and any integers $a, b, c \geqslant 0$.

*We apologize to the reader that π denotes a constant and a representation at the same time.

4. Proof of Theorem 1

Let (π_1, V_{π_1}) and (π_2, V_{π_2}) be arbitrary cuspidal automorphic representations of $\Gamma\backslash G$, and let $W_{1,2} : \mathbb{R}^\times \to \mathbb{C}$ be arbitrary smooth functions of compact support. There are unique smooth vectors $\phi_i \in V_{\pi_i}^\infty$ such that $W_{\phi_i} = W_i$. If $h > 0$ is an arbitrary integer and $Y > 0$ is arbitrary, then we have, by (12),

$$\sum_{m+n=h} \frac{\lambda_{\pi_1}(|m|)\lambda_{\pi_2}(|n|)}{\sqrt{|mn|}} W_1\Big(\frac{m}{Y}\Big) W_2\Big(\frac{n}{Y}\Big) = \int_0^1 (\phi_1\phi_2)\big(n(x)a(Y^{-1})\big)\, e(-hx)\, dx.$$

Let us decompose spectrally the smooth vector $\phi_1\phi_2 \in L^2(\Gamma\backslash G)$ according to (1):

$$\phi_1\phi_2 = \int_\tau \psi_\tau\, d\tau.$$

This decomposition is unique and converges in the topology defined by the Sobolev norms $\|\cdot\|_{S^d}$ (see [CP, Propositions 1.3, 1.4]). In particular, $\psi_\tau \in V_\tau^\infty$, and the decomposition is compatible with the actions of G and $\mathfrak{g}$. The explicit knowledge of the projections $L^2(\Gamma\backslash G) \to V_\tau$ yields (cf. [BM, (2.14), (3.31), Lemma 2]), in combination with Plancherel, (9), and (30),

$$\|\mathscr{D}(\phi_1\phi_2)\|^2 = \int_\tau \|\mathscr{D}\psi_\tau\|^2\, d\tau, \quad \mathscr{D} \in U(\mathfrak{g}). \tag{33}$$

By [CPS, Corollary to Lemma 1.1], the functional $\phi \mapsto W_\phi(h)$ is continuous in the topology defined by the Sobolev norms $\|\cdot\|_{S^d}$ (this also follows from (29), (32), and Plancherel); hence the above imply, in combination with (12) and (31), that

$$\sum_{m+n=h} \frac{\lambda_{\pi_1}(|m|)\lambda_{\pi_2}(|n|)}{\sqrt{|mn|}} W_1\Big(\frac{m}{Y}\Big) W_2\Big(\frac{n}{Y}\Big) = \int_{\tau\neq\tau_0} \frac{\lambda_\tau(h)}{\sqrt{h}} W_\tau\Big(\frac{h}{Y}\Big)\, d\tau,$$

where $W_\tau := W_{\psi_\tau}$. This is just (3). On the right-hand side, we have, by (29) and (32),

$$\tilde{\lambda}_\tau^c \Big(y\frac{d}{dy}\Big)^a W_\tau(y) \ll_{\varepsilon,a,b,c} \tilde{\lambda}_\tau^{-3/2+\varepsilon} \|\psi_\tau\|_{S^{6+a+b+2c}} \min(y^{1/2-\varepsilon}, y^{1/2-b-\varepsilon}), \quad y > 0.$$

A combination of the Cauchy-Schwarz inequality, Weyl's law, and (33) then shows that

$$\int_{\tau\neq\tau_0} \tilde{\lambda}_\tau^c \Big|\Big(y\frac{d}{dy}\Big)^a W_\tau(y)\Big|\, d\tau \ll_{\varepsilon,a,b,c} \|\phi_1\phi_2\|_{S^{6+a+b+2c}} \min(y^{1/2-\varepsilon}, y^{1/2-b-\varepsilon}), \quad y > 0. \tag{34}$$

Using (28) and the Leibniz rule for derivations, we see that

$$\|\phi_1\phi_2\|_{S^d} \ll_{\varepsilon,d} (\tilde{\lambda}_{\pi_1} + \tilde{\lambda}_{\pi_2})^{1+\varepsilon} \sum_{d_1+d_2=d+3} \|\phi_1\|_{S^{d_1}} \|\phi_2\|_{S^{d_2}}$$

for any integer $d \geqslant 0$, which implies, by (14), (15), and (17), that

$$\|\phi_1\phi_2\|_{S^d} \ll_{\varepsilon,d} (\tilde{\lambda}_{\pi_1} + \tilde{\lambda}_{\pi_2})^{d+4+\varepsilon} \sum_{d_1+d_2=2d+6} \|W_1\|_{A^{d_1}} \|W_2\|_{A^{d_2}}.$$

We combine this inequality with (34) to arrive at (4). □

In retrospect, we can see that this proof works for all test functions $W_{1,2} : \mathbb{R}^\times \to \mathbb{C}$ whose norms $\|W_{1,2}\|_{A^d}$ exist for $d = 18 + 2a + 2b + 4c$.

5. Proof of Theorem 2

Let (π_1, V_{π_1}) and (π_2, V_{π_2}) be arbitrary cuspidal automorphic representations of $\Gamma\backslash G$, and let $c, k \geqslant 0$ be arbitrary integers. Let $q > 0$ be an integer to be determined later in terms of c and k. Consider the function $G : [0, \infty) \to \mathbb{C}$, defined by

$$G(t) := \begin{cases} \{t(1-t)\}^q, & 0 \leqslant t \leqslant 1, \\ 0, & t > 1, \end{cases}$$

and consider its Laplace transform (composed with $z \mapsto -z$), defined by

$$\tilde{G}(z) := \int_0^\infty G(t)\, e^{zt}\, dt, \quad z \in \mathbb{C}.$$

Note that $\tilde{G}(z)$ is entire, and by successive integration by parts, it satisfies the uniform bound

$$\tilde{G}(z) \ll_q |z|^{-q-1}, \quad \Re z = 1. \tag{35}$$

If $m, n \geqslant 1$ are arbitrary integers and $Y > 0$ is arbitrary, then by (35) and the theory of the Laplace transform, we have the identity

$$\left(\frac{mn}{Y^2}\right)^{k/2} G\left(\frac{m+n}{Y}\right) = \frac{1}{2\pi i} \int_{(1)} \tilde{G}(z)\, W_k\left(\frac{m}{Y}, z\right) W_k\left(\frac{n}{Y}, z\right) dz, \tag{36}$$

where $W_k : [0, \infty) \times \mathbb{C} \to \mathbb{C}$ is the function defined by

$$W_k(t, z) := t^{k/2} e^{-zt}, \quad t \geqslant 0,\ z \in \mathbb{C}. \tag{37}$$

By Theorem 1, we can see that if k is sufficiently large in terms of c, then there exist functions $W_{k,\tau} : (0, \infty) \times \{z : \Re z > 0\} \to \mathbb{C}$ depending only on $\pi_{1,2}$, k, and τ such that the following two properties hold for all z with $\Re z > 0$. If $h > 0$ is an arbitrary integer and $Y > 0$ is arbitrary, then we have the decomposition over the full spectrum (excluding the trivial representation)

$$\sum_{\substack{m,n\geqslant 1\\ m-n=h}} \frac{\lambda_{\pi_1}(m)\lambda_{\pi_2}(n)}{\sqrt{mn}} W_k\Big(\frac{m}{Y}, z\Big) W_k\Big(\frac{n}{Y}, z\Big) = \int_{\tau\neq\tau_0} \frac{\lambda_\tau(h)}{\sqrt{h}} W_{k,\tau}\Big(\frac{h}{Y}, z\Big)\, d\tau, \qquad (38)$$

and we have the uniform bounds

$$\int_{\tau\neq\tau_0} \tilde{\lambda}_\tau^c |W_{k,\tau}(y,z)|\, d\tau \ll_{\varepsilon,c} C_{0,1,c}(z) \min(y^{1/2-\varepsilon}, y^{-1/2-\varepsilon}), \quad y > 0,\ \Re z > 0,$$

where $0 < \varepsilon < 1/4$ is arbitrary and

$$C_{0,1,c}(z) := (\tilde{\lambda}_{\pi_1} + \tilde{\lambda}_{\pi_2})^{12+2c} \sum_{d_1+d_2=20+4c} \|W_k(\cdot, z)\|_{A^{d_1}} \|W_k(\cdot, z)\|_{A^{d_2}}.$$

Here, we use the convention that $W_k(u, z) = 0$ for $u < 0$. By (2) and (37), the right-hand side exists as long as $k > 60 + 12c$. Under this condition, (38) is justified, and we conclude that

$$\int_{\tau\neq\tau_0} \tilde{\lambda}_\tau^c |W_{k,\tau}(y,z)|\, d\tau$$
$$\ll_{\varepsilon,k} (\tilde{\lambda}_{\pi_1} + \tilde{\lambda}_{\pi_2})^{12+4c} |z|^{20+4c} \min(y^{1/2-\varepsilon}, y^{-1/2-\varepsilon}), \quad y > 0,\ \Re z = 1. \qquad (39)$$

The functions $W_{k,\tau}(y, z)$ furnished by the proof of Theorem 1 also have good behavior for individual τ. To see this, denote by $\phi_{i,z} \in V_{\pi_i}$ the two vectors corresponding to the left-hand side of (38) in the proof of Theorem 1. By (29) and (32), we have

$$W_{k,\tau}(y,z) \ll_{\varepsilon,\tau} \|\psi_{\tau,z}\|_{S^3} \min(y^{1/2-\varepsilon}, y^{-1/2-\varepsilon}), \quad y > 0,\ \Re z > 0,$$

where $\psi_{\tau,z} \in V_\tau$ is the projection of $\phi_{1,z}\phi_{2,z}$ on V_τ. Applying (14) and (30) and then applying (17),

$$\|\psi_{\tau,z}\|_{S^3} \ll_\tau \|W_{k,\tau}(\cdot, z)\|_{S^3} \ll_\tau \|W_{k,\tau}(\cdot, z)\|_{A^6}, \quad \Re z > 0,$$

whence by (37), we can conclude that

$$W_{k,\tau}(y,z) \ll_{\varepsilon,k,\tau} |z|^6 \min(y^{1/2-\varepsilon}, y^{-1/2-\varepsilon}), \quad y > 0,\ \Re z = 1. \qquad (40)$$

In an almost identical fashion,

$$y\frac{d}{dy} W_{k,\tau}(y,z) \ll_{\varepsilon,k,\tau} |z|^6 y^{1/2-\varepsilon}, \quad y > 0,\ \Re z = 1. \qquad (41)$$

Finally, by (29), (32), (14), (30), and (17),

$$\begin{aligned} W_{k,\tau}(y,z) - W_{k,\tau}(y,z') &\ll_{\tau,y} \|\psi_{\tau,z} - \psi_{\tau,z'}\|_{S^2} \\ &\ll_{\tau,y} \|W_{k,\tau}(\cdot,z) - W_{k,\tau}(\cdot,z')\|_{S^2} \\ &\ll_{\tau,y} \|W_{k,\tau}(\cdot,z) - W_{k,\tau}(\cdot,z')\|_{A^4}, \quad \Re z,\ \Re z' > 0, \end{aligned}$$

whence by (37), we can conclude that

$$\lim_{z'\to z} W_{k,\tau}(y,z') = W_{k,\tau}(y,z), \quad y > 0,\ \Re z > 0. \tag{42}$$

By (35) and (40)–(42), the integral

$$H_{k,\tau}(y) := \frac{1}{2\pi i}\int_{(1)} \tilde{G}(z)\, W_{k,\tau}(y,z)\, dz, \quad y > 0, \tag{43}$$

defines a differentiable function $H_{k,\tau} : (0,\infty) \to \mathbb{C}$ for all τ and satisfies the uniform bound

$$H_{k,\tau}(y) \ll_{\varepsilon,k,\tau} \min(y^{1/2-\varepsilon}, y^{-1/2-\varepsilon}), \quad y > 0, \tag{44}$$

as long as $q > 6$. Then (35)–(39) and two applications of Fubini's theorem show that

$$\sum_{\substack{m,n\geqslant 1\\ m-n=h}} \frac{\lambda_{\pi_1}(m)\lambda_{\pi_2}(n)}{\sqrt{mn}}\Big(\frac{mn}{Y^2}\Big)^{k/2} G\Big(\frac{m+n}{Y}\Big) = \int_{\tau\neq\tau_0} \frac{\lambda_\tau(h)}{\sqrt{h}} H_{k,\tau}\Big(\frac{h}{Y}\Big)\, d\tau,$$

as long as $q > 20 + 4c$. We specify $q := 21 + 4c$ and evaluate the Mellin transform in Y at $1-s$ of both sides:

$$\begin{aligned} &\int_0^\infty Y^{1-s-k} \sum_{\substack{m,n\geqslant 1\\ m-n=h}} \lambda_{\pi_1}(m)\lambda_{\pi_2}(n)\,(mn)^{(k-1)/2}\, G\Big(\frac{m+n}{Y}\Big)\frac{dY}{Y} \\ &\quad = \int_0^\infty Y^{1-s} \int_{\tau\neq\tau_0} \frac{\lambda_\tau(h)}{\sqrt{h}} H_{k,\tau}\Big(\frac{h}{Y}\Big)\, d\tau\, \frac{dY}{Y}. \end{aligned} \tag{45}$$

The left-hand side is absolutely convergent for $\Re s > 1$, and by Fubini's theorem, it equals

$$\hat{G}(s+k-1) \sum_{\substack{m,n\geqslant 1\\ m-n=h}} \frac{\lambda_{\pi_1}(m)\lambda_{\pi_2}(n)\,(mn)^{(k-1)/2}}{(m+n)^{s+k-1}}, \quad \Re s > 1.$$

The right-hand side is absolutely convergent for $1/2 < \Re s < 3/2$ by (43), (35), and (39), and by Fubini's theorem, it equals

$$h^{1/2-s}\int_{\tau\neq\tau_0}\lambda_\tau(h)\,\hat{H}_{k,\tau}(s-1)\,d\tau,\qquad \frac{1}{2}<\Re s<\frac{3}{2}.$$

The Mellin transforms $\hat{H}_{k,\tau}(s-1)$ are holomorphic functions in the strip $1/2 < \Re s < 3/2$ by (44), and by (43), (35), and (39), they satisfy the uniform bound

$$\int_{\tau\neq\tau_0}\tilde{\lambda}_\tau^c\,|\hat{H}_{k,\tau}(s-1)|\,d\tau \ll_{\varepsilon,k} (\lambda_{\pi_1}+\lambda_{\pi_2})^{12+4c},\qquad \frac{1}{2}+\varepsilon<\Re s<\frac{3}{2}. \tag{46}$$

Finally, we observe that

$$\begin{aligned}\hat{G}(s+k-1) &= \int_0^1 \left\{t(1-t)\right\}^q t^{s+k-2}\,dt\\ &= \frac{\Gamma(s+k+q-1)\,\Gamma(q+1)}{\Gamma(s+k+2q)},\qquad \Re s > 1-k-q,\end{aligned}$$

and therefore

$$\hat{G}(s+k-1) \gg_k |s|^{-q-1},\qquad \frac{1}{2}<\Re s<\frac{3}{2}.$$

We put

$$F_{k,\tau}(s) := \frac{\hat{H}_{k,\tau}(s-1)}{\hat{G}(s+k-1)},\qquad \frac{1}{2}<\Re s<\frac{3}{2},$$

and then the statements of Theorem 2 are immediate. In particular, (7) follows from the comparison of the two sides in (45), while (8) is a consequence of (46). □

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