A supplement to Chebotarev's density theorem

(based on joint work with K. Soundararajan)

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The density of split primes (1 of 2)

Notation

$$\mathcal{H}_\sigma := \{s \in \mathbb{C} : \Re(s) > \sigma\}$$

Dedekind (1894) associated a zeta function to any number field L:

$$\zeta_L(s) := \sum_{\mathfrak{I}} \frac{1}{N(\mathfrak{I})^s} = \prod_{\mathfrak{P}} \left(1 - \frac{1}{N(\mathfrak{P})^s}\right)^{-1}, \qquad s \in \mathcal{H}_1.$$

As proved by Hecke (1918), this function is meromorphic on $\mathbb{C}$ with a simple pole at $s = 1$ and no other pole. Moreover, it satisfies a functional equation, generalizing the work of Riemann (1859).

By taking the logarithmic derivative of both sides, we obtain

$$-\frac{\zeta_L'}{\zeta_L}(s) = \sum_{\mathfrak{P}} \sum_{r=1}^{\infty} \frac{\log N(\mathfrak{P})}{N(\mathfrak{P})^{rs}} \approx \sum_{\mathfrak{P}} \frac{\log N(\mathfrak{P})}{N(\mathfrak{P})^s},$$

where $f(s) \approx g(s)$ means that $f(s) - g(s)$ is a Dirichlet series converging absolutely in $\mathcal{H}_{1/2}$.

The density of split primes (2 of 2)

Let $L/\mathbb{Q}$ be a Galois extension. Then with a bit of algebraic number theory we see that

$$-\frac{\zeta_L'}{\zeta_L}(s) \approx \sum_{\mathfrak{P}} \frac{\log N(\mathfrak{P})}{N(\mathfrak{P})^s} \approx \sum_{p \text{ splits completely in } L} [L:\mathbb{Q}] \frac{\log p}{p^s}.$$

In particular, the right-hand side is meromorphic on $\mathcal{H}_{1/2}$ with simple poles, and $s = 1$ is a pole:

$$\sum_{p \text{ splits completely in } L} \frac{\log p}{p^s} \sim \frac{1}{[L:\mathbb{Q}]} \cdot \frac{1}{s-1}, \qquad s \to 1.$$

Compare with the special case $L = \mathbb{Q}$. The other poles are the zeros of $\zeta_L(s)$ in $\mathcal{H}_{1/2}$. According to the generalized Riemann hypothesis, there is no such zero. This is equivalent to:

$$\sum_{\substack{p \leqslant x \\ p \text{ splits completely in } L}} \log p = \frac{x}{[L:\mathbb{Q}]} + O_{L,\varepsilon}(x^{1/2+\varepsilon}), \qquad \varepsilon > 0.$$

Dirichlet's theorem on primes (1 of 2)

Now let $L = \mathbb{Q}(e^{2\pi i/q})$. Then the previous findings become a special case of Dirichlet's theorem on primes:

$$\sum_{p \equiv 1 \pmod{q}} \frac{\log p}{p^s} \approx \frac{1}{\varphi(q)} \cdot -\frac{\zeta_L'}{\zeta_L}(s) \sim \frac{1}{\varphi(q)} \cdot \frac{1}{s-1}, \qquad s \to 1.$$

Question

How about the density of $p \equiv a \pmod{q}$ for $(a, q) = 1$?

Hint

$$\frac{\zeta_L(s)}{\zeta_{\mathbb{Q}}(s)} = \prod_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} L(s, \chi_{\mathrm{prim}}).$$

The factors on the right-hand side are entire functions, hence so is the left-hand side. They do not vanish at the point $s = 1$.

Dirichlet's theorem on primes (2 of 2)

Dirichlet (1837) realized that the non-vanishing at $s = 1$ of the Dirichlet L-functions $L(s, \chi)$ is the key to the equidistribution of primes in reduced residue classes modulo q:

$$\begin{aligned}
\sum_{p \equiv a \pmod{q}} \frac{\log p}{p^s} &= \sum_p \frac{\log p}{p^s} \left(\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \chi(p) \overline{\chi}(a) \right) \\
&= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \left(\sum_p \frac{\chi(p) \log p}{p^s} \right) \overline{\chi}(a) \\
&\approx \frac{1}{\varphi(q)} \sum_{\chi \bmod q} -\frac{L'}{L}(s, \chi) \overline{\chi}(a) \\
&\approx \frac{1}{\varphi(q)} \sum_{\chi \bmod q} -\frac{L'}{L}(s, \chi_{\mathrm{prim}}) \overline{\chi}(a) \\
&\sim \frac{1}{\varphi(q)} \cdot \frac{1}{s-1}, \qquad s \to 1.
\end{aligned}$$

The left-hand side is meromorphic on $\mathcal{H}_{1/2}$ with simple poles.

Assume that $s_0 \in \mathcal{H}_{1/2}$ is not a zero of the entire function

$$\frac{\zeta_L(s)}{\zeta_{\mathbb{Q}}(s)} = \prod_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} L(s, \chi_{\mathrm{prim}}).$$

Then the point s_0 is not a zero of any of the factors on the right-hand side. We can reformulate this observation as follows.

Proposition

Assume that $s_0 \in \mathcal{H}_{1/2}$ is not a pole of

$$\sum_{p \equiv 1 \pmod q} \frac{\log p}{p^s} - \frac{1}{\varphi(q)} \sum_p \frac{\log p}{p^s} \approx \frac{1}{\varphi(q)} \left(\frac{\zeta_{\mathbb{Q}}'}{\zeta_{\mathbb{Q}}}(s) - \frac{\zeta_L'}{\zeta_L}(s) \right).$$

Then, for $(a, q) = 1$, the point s_0 is not a pole of

$$\sum_{p \equiv a \pmod q} \frac{\log p}{p^s} - \frac{1}{\varphi(q)} \sum_p \frac{\log p}{p^s} \approx \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} -\frac{L'}{L}(s, \chi_{\mathrm{prim}}) \overline{\chi}(a).$$

In particular, by standard Mellin transform techniques, we obtain:

Corollary

Suppose $\sigma \geqslant 1/2$ *is such that for any* $\varepsilon > 0$ *we have*

$$\sum_{\substack{p \leqslant x \\ p \equiv 1 \pmod{q}}} \log p = \frac{1}{\varphi(q)} \sum_{p \leqslant x} \log p + O(x^{\sigma+\varepsilon}).$$

Then for $(a, q) = 1$ *and any* $\varepsilon > 0$ *we have*

$$\sum_{\substack{p \leqslant x \\ p \equiv a \pmod{q}}} \log p = \frac{1}{\varphi(q)} \sum_{p \leqslant x} \log p + O(x^{\sigma+\varepsilon}).$$

Chebotarev's density theorem

Chebotarev (1923) proved a far-reaching generalization of Dirichlet's theorem, originally conjectured by Frobenius (1896).

To fix ideas, let L/K be a Galois extension of number fields with Galois group $G := \mathrm{Gal}(L/K)$. To an unramified prime ideal $\mathfrak{p}$ in K, we associate a conjugacy class $\mathrm{Frob}(\mathfrak{p}) \subset G$ as follows. For any prime divisor $\mathfrak{P} \mid \mathfrak{p}$ in L, there is a unique $\mathrm{Frob}(\mathfrak{P}) \in G$ satisfying

$$x^{\mathrm{Frob}(\mathfrak{P})} \equiv x^{N(\mathfrak{p})} \pmod{\mathfrak{P}}$$

for all integers x in L. The class $\mathrm{Frob}(\mathfrak{p})$ is the set of $\mathrm{Frob}(\mathfrak{P})$'s.

Theorem

For each conjugacy class $C \subset G$, consider the Dirichlet series

$$D_G(s, C) := \sum_{\mathrm{Frob}(\mathfrak{p})=C} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^s}, \qquad s \in \mathcal{H}_1.$$

This function is meromorphic on $\mathcal{H}_{1/2}$ with simple poles, and it has a simple pole at $s = 1$ with residue $|C|/|G|$.

Artin L-functions

In order to prove Chebotarev's density theorem (and more), we shall use the L-functions introduced by Artin (1923). These Artin L-functions are associated to (characters of) Galois representations.

Basic properties

1. *For the trivial character χ_0 of G, we have $L(s, \chi_0) = \zeta_K(s)$.*
2. $L(s, \chi_1 + \chi_2) = L(s, \chi_1)L(s, \chi_2)$.
3. *For a subgroup $H \leqslant G$ and a character ψ of H, we have $L(s, \operatorname{Ind}_H^G \psi) = L(s, \psi)$.*

Corollary

$$\frac{\zeta_L(s)}{\zeta_K(s)} = \prod_{\substack{\chi \in \operatorname{Irr}(G) \\ \chi \neq \chi_0}} L(s, \chi)^{\chi(1)}.$$

Artin (1923) conjectured that the factors on the right-hand side are entire functions. It follows from the celebrated reciprocity law of Artin (1927) that the conjecture is true when G is abelian.

Chebotarev's density theorem via Artin reciprocity (1 of 2)

The definition of $L(s, \chi)$ yields readily that

$$-\frac{L'}{L}(s, \chi) \approx \sum_{\mathfrak{p}} \frac{\chi(\mathrm{Frob}(\mathfrak{p})) \log N(\mathfrak{p})}{N(\mathfrak{p})^s}.$$

Hence if $g_C \in C$ is any element, then we get by Schur orthogonality

$$\begin{aligned} D_G(s, C) &= \sum_{\mathfrak{p}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^s} \left(\frac{|C|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \chi(\mathrm{Frob}(\mathfrak{p})) \overline{\chi}(g_C) \right) \\ &= \frac{|C|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \left(\sum_{\mathfrak{p}} \frac{\chi(\mathrm{Frob}(\mathfrak{p})) \log N(\mathfrak{p})}{N(\mathfrak{p})^s} \right) \overline{\chi}(g_C) \\ &\approx \frac{|C|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} -\frac{L'}{L}(s, \chi) \overline{\chi}(g_C). \end{aligned}$$

We claim that the last sum is meromorphic on $\mathbb{C}$ with simple poles, and it has a simple pole at $s = 1$ with residue 1. By Artin reciprocity, the claim holds when G is abelian. Hence it suffices to show that the sum remains the same when G is replaced by $\langle g_C \rangle$.

Chebotarev's density theorem via Artin reciprocity (2 of 2)

Notation

$$U_G(s,g) := \sum_{\chi\in\mathrm{Irr}(G)} \frac{L'}{L}(s,\chi)\overline{\chi}(g), \qquad s\in\mathcal{H}_1, \quad g\in G.$$

Master relation

For any subgroup $H \leqslant G$, we have $\mathrm{Res}^G_H U_G(s,*) = U_H(s,*)$.

Proof.

Let us fix $s\in\mathcal{H}_1$. For any character χ of G, we have

$$\langle U_G(s,*),\overline{\chi}\rangle_G = \frac{L'}{L}(s,\chi).$$

Hence for any character ψ of H, Frobenius reciprocity gives that

$$\langle \mathrm{Res}^G_H U_G(s,*),\overline{\psi}\rangle_H = \langle U_G(s,*),\mathrm{Ind}^G_H\overline{\psi}\rangle_G =$$
$$= \frac{L'}{L}(s,\mathrm{Ind}^G_H\psi) = \frac{L'}{L}(s,\psi) = \langle U_H(s,*),\overline{\psi}\rangle_H. \qquad \square$$

The meromorphicity of $L'(s,\chi)/L(s,\chi)$

The previous proof used a fundamental idea of Heilbronn (1973).

Corollary

For any character $\chi \in \mathrm{Irr}(G)$, the function $L'(s,\chi)/L(s,\chi)$ is meromorphic on $\mathbb{C}$ with simple poles. Moreover, for $\chi \neq \chi_0$, the point $s = 1$ is not a pole.

Proof.

We have seen that $U_G(s,g)$ is meromorphic on $\mathbb{C}$ with simple poles, and it has a simple pole at $s = 1$ with residue -1. Hence both statements follow upon noting that

$$\frac{L'}{L}(s,\chi) = \langle U_G(s,*), \overline{\chi} \rangle_G. \qquad \square$$

A supplement to Chebotarev's density theorem

Theorem

For each conjugacy class $C \subset G$, consider the Dirichlet series

$$F_G(s, C) := \sum_{\mathrm{Frob}(\mathfrak{p})=C} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^s} - \frac{|C|}{|G|} \sum_{\mathfrak{p}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^s}, \qquad s \in \mathcal{H}_1.$$

This function is meromorphic on $\mathcal{H}_{1/2}$ with simple poles. For any point $s_0 \in \mathcal{H}_{1/2}$, the following statements are equivalent:

- (a) *s_0 is a zero of $\zeta_L(s)/\zeta_K(s)$;*
- (b) *s_0 is a pole of $F_G(s, \{1\})$;*
- (c) *s_0 is a pole of $F_G(s, C)$ for some conjugacy class $C \subset G$;*
- (d) *s_0 is a pole of $L'(s, \chi)/L(s, \chi)$ for some nontrivial $\chi \in \mathrm{Irr}(G)$.*

Moreover,

$$\sum_C \frac{|G|}{|C|} \left| \operatorname*{res}_{s=s_0} F_G(s, C) \right|^2 \leqslant \left(\operatorname*{ord}_{s=s_0} \zeta_L(s) \right)^2 - \left(\operatorname*{ord}_{s=s_0} \zeta_K(s) \right)^2. \qquad (*)$$

The Foote–Murty inequality (1 of 2)

First we prove the key bound $(*)$. Proceeding as in the proof of Chebotarev's density theorem, we see that

$$F_G(s, C) \approx -\frac{|C|}{|G|} V_G(s, g_C),$$

where

Notation

$$V_G(s, g) := \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi \neq \chi_0}} \frac{L'}{L}(s, \chi)\overline{\chi}(g), \qquad s \in \mathcal{H}_1, \quad g \in G.$$

Hence it suffices to prove the following inequality that is essentially due to Foote–Murty (1989):

$$\frac{1}{|G|} \sum_{g \in G} \left| \operatorname*{res}_{s=s_0} V_G(s, g) \right|^2 \leqslant \left(\operatorname*{ord}_{s=s_0} \zeta_L(s) \right)^2 - \left(\operatorname*{ord}_{s=s_0} \zeta_K(s) \right)^2.$$

The Foote–Murty inequality (2 of 2)

Let us work with an arbitrary $s_0 \in \mathbb{C}$. Since

$$V_G(s,g) = U_G(s,g) - \frac{\zeta_K'}{\zeta_K}(s),$$

the bound is clear when $s_0 = 1$:

$$\operatorname*{res}_{s=1} V_G(s,g) = \operatorname*{res}_{s=1} U_G(s,g) + 1 = 0.$$

For $s_0 \neq 1$, we combine the Master relation with Artin reciprocity:

$$\left|\operatorname*{res}_{s=s_0} U_G(s,g)\right| = \left|\operatorname*{res}_{s=s_0} U_{\langle g\rangle}(s,g)\right| \leqslant$$
$$\leqslant \operatorname*{res}_{s=s_0} U_{\langle g\rangle}(s,1) = \operatorname*{res}_{s=s_0} U_{\{1\}}(s,1) = \operatorname*{ord}_{s=s_0} \zeta_L(s).$$

We square this bound and average over G. We get that

$$\frac{1}{|G|}\sum_{g\in G}\left|\operatorname*{res}_{s=s_0} V_G(s,g) + \operatorname*{ord}_{s=s_0}\zeta_K(s)\right|^2 \leqslant \left(\operatorname*{ord}_{s=s_0}\zeta_L(s)\right)^2.$$

This is what we need, since the average of $V_G(s,g)$ over G is zero.

The final equivalences

The Foote–Murty inequality yields in particular that

$$\operatorname*{ord}_{s_0} \zeta_K(s) \leqslant \operatorname*{ord}_{s_0} \zeta_L(s), \qquad s_0 \in \mathbb{C},$$

hence $\zeta_L(s)/\zeta_K(s)$ is an entire function. This is originally due to Aramata (1931), and re-discovered by Brauer (1947).

Now we can prove that the statements (a), (b), (c) are equivalent. If (a) holds, then s_0 is a pole of the logarithmic derivative

$$\frac{\zeta_L'}{\zeta_L}(s) - \frac{\zeta_K'}{\zeta_K}(s) = U_{\{1\}}(s,1) - \frac{\zeta_K'}{\zeta_K}(s) = U_G(s,1) - \frac{\zeta_K'}{\zeta_K}(s) = V_G(s,1),$$

which then implies (b). Now (b) trivially implies (c), while (c) implies (a) by $(*)$. Finally, (c) is equivalent to (d), because the functions $V_G(s,g)$ for $g \in G$ span the same $\mathbb{C}$-vector space as the functions $L'(s,\chi)/L(s,\chi)$ for $\chi \neq \chi_0$.

A supplement to Chebotarev's density theorem (cont.)

In particular, by standard Mellin transform techniques, we obtain:

Corollary

Suppose $\sigma \geqslant 1/2$ is such that for any $\varepsilon > 0$ we have

$$\sum_{\substack{N(\mathfrak{p}) \leqslant x \\ \mathrm{Frob}(\mathfrak{p}) = \{1\}}} \log N(\mathfrak{p}) = \frac{1}{|G|} \sum_{N(\mathfrak{p}) \leqslant x} \log N(\mathfrak{p}) + O(x^{\sigma+\varepsilon}).$$

Then for any conjugacy class $C \subset G$ and any $\varepsilon > 0$ we have

$$\sum_{\substack{N(\mathfrak{p}) \leqslant x \\ \mathrm{Frob}(\mathfrak{p}) = C}} \log N(\mathfrak{p}) = \frac{|C|}{|G|} \sum_{N(\mathfrak{p}) \leqslant x} \log N(\mathfrak{p}) + O(x^{\sigma+\varepsilon}).$$