Beyond the spherical sup-norm problem

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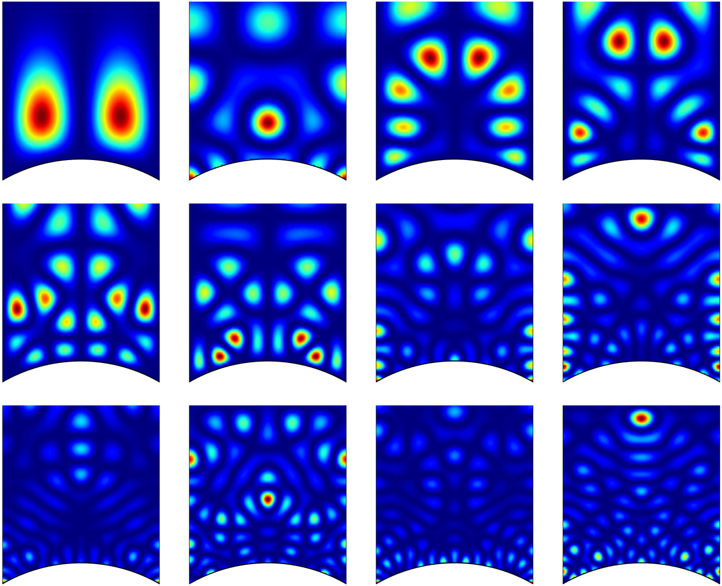
Maass forms on the modular surface



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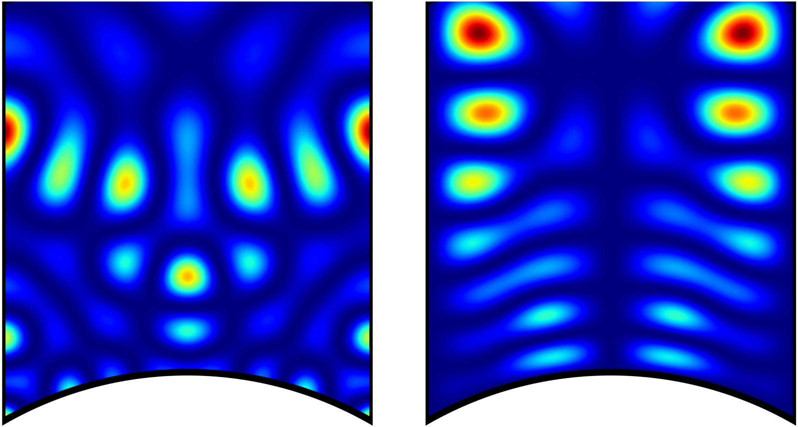
Maass forms with $\lambda \approx 10^3$ on the modular surface



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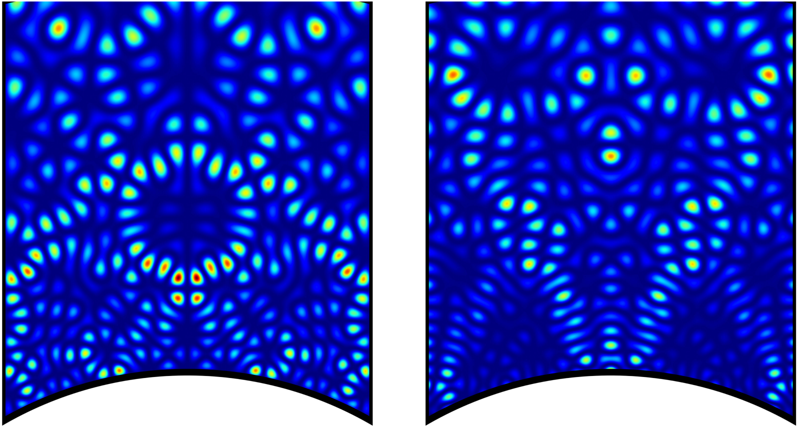
Maass forms with $\lambda \approx 10^4$ on the modular surface



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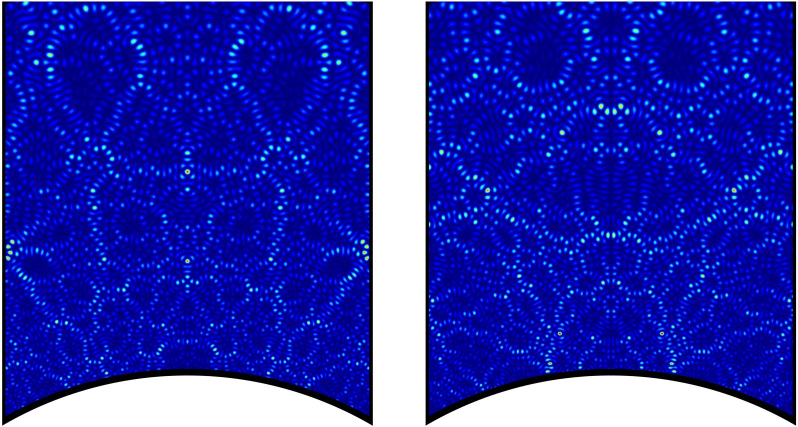
Maass forms with $\lambda \approx 10^5$ on the modular surface



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Classical mechanics vs. quantum mechanics

Let M be a compact orientable Riemannian manifold, and consider a particle moving freely on M with unit speed.

	classical mechanics	**quantum mechanics**
phase space	SM sphere bundle	$L^2(M)$ Hilbert space
moving particle	$f : \mathbb{R} \to SM$ smooth	$\psi : \mathbb{R} \to L^2(M)$ $\|\psi\| = 1$
bounded observable	$a : SM \to \mathbb{R}$ smooth	$\mathrm{Op}(a) : L^2(M) \to L^2(M)$ self-adjoint & bounded
time evolution	$G^t : SM \to SM$ geodesic flow	$U_t : L^2(M) \to L^2(M)$ $U_t = e^{-it\sqrt{\Delta}}$

Solutions of the Schrödinger equation

$$\psi(t) = U_t(\psi(0)) = \sum_{j=0}^{\infty} c_j e^{-it\sqrt{\lambda_j}} \phi_j, \qquad (c_j)_{j=0}^{\infty} \in \ell^2(\mathbb{N})$$

Quantum ergodicity on the modular surface (1 of 2)

Theorem (Shnirelman 1974, Colin de Verdière 1985, Zelditch 1987)

Assume that the geodesic flow on SM is ergodic, and let $\{\phi_j\}$ be an orthonormal basis of $L^2(M)$ satisfying $\Delta\phi_j = \lambda_j\phi_j$. Consider $d\omega_j$ defined via $\langle \mathrm{Op}(a)\phi_j, \phi_j\rangle = \int_{SM} a\, d\omega_j$ for $a \in C^\infty(SM)$. Then $d\omega_j \xrightarrow{*} d\omega$ along a subsequence of λ_j's of density 1.

Proof (sketch).

Assume that $a \in C^\infty(SM)$ has space average $\int_{SM} a\, d\omega = 0$.
Consider also a fixed time average $a^T := \frac{1}{T}\int_0^T a \circ G^t\, dt$.
By Egorov, Cauchy–Schwarz, Weyl, and Birkhoff, we have

$$\frac{1}{N(\lambda,1)}\sum_{\lambda_j \leqslant \lambda}\left|\int_{SM} a\, d\omega_j\right|^2 = \frac{1}{N(\lambda,1)}\sum_{\lambda_j \leqslant \lambda}\left|\int_{SM} a^T\, d\omega_j\right|^2 + o(1)$$

$$\leqslant \frac{1}{N(\lambda,1)}\sum_{\lambda_j \leqslant \lambda}\int_{SM} |a^T|^2\, d\omega_j + o(1) = \int_{SM} |a^T|^2\, d\omega + o(1) < \varepsilon,$$

for $T = T_0(\varepsilon)$ and $\lambda > \lambda_0(\varepsilon)$. Hence the left hand side is $o(1)$. □

Quantum ergodicity on the modular surface (2 of 2)

Theorem (Hopf 1936)

Let $M := \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^2$ be the modular surface. The geodesic flow on the sphere bundle $SM \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ is ergodic.

Proof (sketch).

Assume that $f \in L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}))$ is fixed by the right action of positive diagonal matrices $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$.

Then, for any fixed $b \in \mathbb{R}$ and for $a > 0$ tending to infinity,

$$\begin{aligned}
\left\| \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} f - f \right\| &= \left\| \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f - \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f \right\| \\
&= \left\| \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f - f \right\| \\
&= \left\| \begin{pmatrix} 1 & a^{-1}b \\ & 1 \end{pmatrix} f - f \right\| \to \| f - f \| = 0.
\end{aligned}$$

Hence any upper triangular matrix in $\mathrm{SL}_2(\mathbb{R})$ fixes f. Similarly, any lower triangular matrix in $\mathrm{SL}_2(\mathbb{R})$ fixes f. In the end, the entire group $\mathrm{SL}_2(\mathbb{R})$ fixes f, and so f is constant almost everywhere. □

The spherical sup-norm problem (1 of 2)

Theorem (Iwaniec–Sarnak 1995)

Let ϕ be an L^2-normalized Hecke–Maass form on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^2$ of Laplacian eigenvalue λ. Then for any $\varepsilon > 0$ we have

$$\|\phi\|_\infty \ll_\varepsilon \lambda^{5/24+\varepsilon}.$$

Using that $\mathcal{H}^2 \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, the functions ϕ above can be thought of as functions on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$.

They span the subspace of $L^2_{\mathrm{cusp}}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}))$ fixed by the right action of $\mathrm{SO}_2(\mathbb{R})$. That is, they span the spherical subspace of the cuspidal subspace.

In adelic language, the functions ϕ fixed by T_{-1} live on

$$\mathrm{PGL}_2(\mathbb{Q})\backslash\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})/\mathrm{PO}_2(\mathbb{R})\prod_p \mathrm{PGL}_2(\mathbb{Z}_p).$$

So these “even” functions ϕ are spherical at every place of $\mathbb{Q}$.

The spherical sup-norm problem (2 of 2)

Theorem (Blomer–Harcos–Milićević 2016)

Let ϕ be an L^2-normalized Hecke–Maass form on $\mathrm{SL}_2(\mathbb{Z}[i])\backslash\mathcal{H}^3$ of Laplacian eigenvalue λ. Then for any $\varepsilon > 0$ we have

$$\|\phi\|_\infty \ll_\varepsilon \lambda^{5/12+\varepsilon}.$$

Using that $\mathcal{H}^3 \cong \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{C})$, the functions ϕ above can be thought of as functions on $\mathrm{SL}_2(\mathbb{Z}[i])\backslash\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{C})$.

They span the subspace of $L^2_{\mathrm{cusp}}(\mathrm{SL}_2(\mathbb{Z}[i])\backslash\mathrm{SL}_2(\mathbb{C}))$ fixed by the right action of $\mathrm{SU}_2(\mathbb{C})$. That is, they span the spherical subspace of the cuspidal subspace.

In adelic language, the functions ϕ fixed by T_i live on

$$\mathrm{PGL}_2(\mathbb{Q}(i))\backslash\mathrm{PGL}_2(\mathbb{A}_{\mathbb{Q}(i)})/\mathrm{PU}_2(\mathbb{C})\prod_{\mathfrak{p}}\mathrm{PGL}_2(\mathbb{Z}[i]_{\mathfrak{p}}).$$

So these "even" functions ϕ are spherical at every place of $\mathbb{Q}(i)$.

Spectral decomposition of the automorphic L^2 space

Γ	G	K
$\mathrm{SL}_2(\mathbb{Z})$	$\mathrm{SL}_2(\mathbb{R})$	$\mathrm{SO}_2(\mathbb{R})$
$\mathrm{SL}_2(\mathbb{Z}[i])$	$\mathrm{SL}_2(\mathbb{C})$	$\mathrm{SU}_2(\mathbb{C})$
$\mathrm{PGL}_2(\mathbb{Q})$	$\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$	$\mathrm{PO}_2(\mathbb{R})\prod_p \mathrm{PGL}_2(\mathbb{Z}_p)$
$\mathrm{PGL}_2(\mathbb{Q}(i))$	$\mathrm{PGL}_2(\mathbb{A}_{\mathbb{Q}(i)})$	$\mathrm{PU}_2(\mathbb{C})\prod_\mathfrak{p} \mathrm{PGL}_2(\mathbb{Z}[i]_\mathfrak{p})$

We have the following Hilbert space decompositions into irreducible G-spaces and their spherical (i.e. K-invariant) subspaces:

$$L^2_{\mathrm{cusp}}(\Gamma\backslash G) = \bigoplus_\pi V_\pi,$$

$$L^2_{\mathrm{cusp}}(\Gamma\backslash G/K) = L^2_{\mathrm{cusp}}(\Gamma\backslash G)^K = \bigoplus_\pi V_\pi^K.$$

For the adelic groups, the above decompositions are unique (multiplicity one), and the spaces V_π consist of Hecke eigenforms. The nonzero subspaces V_π^K are the one-dimensional spaces $\mathbb{C}\phi$, where ϕ is a spherical Hecke–Maass form as before.

The unitary dual of $G = \mathrm{SL}_2(\mathbb{R})$

The unitary dual $\widehat{G}$ was determined by Bargman (1947), perhaps inspired by the work of Wigner (1939). The nontrivial irreducible unitary representations of G are infinite dimensional, and the ones relevant here (the tempered ones) come in 4 families:

- spherical/non-spherical principal series $\pi_{it}^{\pm}$ for $t \in \mathbb{R}_{\geqslant 0}$;
- holomorphic/antiholomorphic discrete series $\pi_k^{\pm}$ for $k \in \mathbb{Z}_{\geqslant 1}$.

These representations can be defined explicitly, e.g. by letting G act on $L^2(\mathbb{R})$ in natural but different ways. We can clearly distinguish between the above 4 types by looking at how they decompose into irreducible K-spaces (we parametrize $\widehat{K}$ by $\mathbb{Z}$):

$$V_{it}^{+} = \bigoplus_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv 0 \bmod 2}} V_{it}^{+,\ell} \qquad V_{it}^{-} = \bigoplus_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv 1 \bmod 2}} V_{it}^{-,\ell}$$

$$V_{k}^{+} = \bigoplus_{\substack{\ell\geqslant k \\ \ell\equiv k \bmod 2}} V_{k}^{+,\ell} \qquad V_{k}^{-} = \bigoplus_{\substack{\ell\leqslant -k \\ \ell\equiv k \bmod 2}} V_{k}^{-,\ell}$$

The summands here are one-dimensional (i.e. isomorphic to $\mathbb{C}$).

The non-spherical sup-norm problem for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$

Inspired by the theory of newforms in the level aspect, the natural non-spherical sup-norm problem would concern a minimal weight vector in π_k^+ (or equivalently a maximal weight vector in π_k^-), which is again unique up to scaling. However, such a vector has weight k Laplacian eigenvalue $\frac{k}{2}\left(1-\frac{k}{2}\right)$, which grows with k. So the weight aspect is not separated from the eigenvalue aspect in this variant of the problem.

To go genuinely beyond the spherical sup-norm problem, we are led to work with $\mathrm{SL}_2(\mathbb{C})$ rather than $\mathrm{SL}_2(\mathbb{R})$.

If π_k^+ occurs in $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$, then *the absolute value* of its minimal weight vector is invariant under K, and as a function on $\mathcal{H}^2$ it agrees with $F(x+iy) := y^{k/2}|f(x+iy)|$, where f is a holomorphic cusp form of weight k and level 1 on $\mathcal{H}^2$. If f is a Hecke eigenform and $\|F\|_2 = 1$, then $\|F\|_\infty \ll_\varepsilon k^{1/4+\varepsilon}$ by a result of Xia (2007). For co-compact $\Gamma \leqslant \mathrm{SL}_2(\mathbb{R})$, a similar result was proved by Khayutin–Steiner (2020), improving on Das–Sengupta (2015).

The unitary dual of $G = \mathrm{SL}_2(\mathbb{C})$

The unitary dual $\widehat{G}$ was determined by Gelfand–Naimark (1947). The nontrivial irreducible unitary representations of G are infinite dimensional, and the ones relevant for the moment (the tempered ones) come in a single family:

- principal series $\pi_{it,p}$ for $t \in \mathbb{R}_{\geqslant 0}$ and $p \in \frac{1}{2}\mathbb{Z}$.

These representations can be defined explicitly, e.g. by letting G act on $L^2(\mathbb{C})$ in a natural way.

It is instructive (and crucial for us) to look at how these representations decompose into irreducible K-spaces (we parametrize $\widehat{K}$ by $\frac{1}{2}\mathbb{Z}_{\geqslant 0}$), and further into one-dimensional subspaces under the action of the diagonal subgroup of K:

$$V_{it,p} = \bigoplus_{\substack{\ell \geqslant |p| \\ \ell \equiv p \bmod 1}} V_{it,p}^{\ell} = \bigoplus_{\substack{\ell \geqslant |p| \\ \ell \equiv p \bmod 1}} \bigoplus_{\substack{|q| \leqslant \ell \\ q \equiv \ell \bmod 1}} V_{it,p}^{\ell,q}.$$

Here $\dim V_{it,p}^{\ell} = 2\ell + 1$ and $\dim V_{it,p}^{\ell,q} = 1$.

New results

Notation

$$\Gamma := \mathrm{SL}_2(\mathbb{Z}[i]), \qquad G := \mathrm{SL}_2(\mathbb{C}), \qquad K := \mathrm{SU}_2(\mathbb{C}).$$

Theorem (Blomer–Harcos–Maga–Milićević 2021)

Let $\ell \geqslant 1$ be an integer, $I \subset \mathbb{R}$ and $\Omega \subset G$ be compact sets. Let $V_\pi \subset L^2_{\mathrm{cusp}}(\Gamma \backslash G)$ be a cuspidal representation such that $\pi \simeq \pi_{it,p}$, where $t \in I$ and $|p| = \ell$. As usual, we assume that V_π consists of Hecke eigenfunctions. Let us choose an orthonormal basis $\{\phi_q : |q| \leqslant \ell\}$ of V_π^ℓ, with $\phi_q \in V_\pi^{\ell,q}$. Then for any $\varepsilon > 0$ we have

$$\sum_{|q| \leqslant \ell} |\phi_q(g)|^2 \ll_{\varepsilon,I,\Omega} \ell^{8/3+\varepsilon}, \qquad g \in \Omega.$$

For the individual summands we have

$$\phi_q(g) \ll_{\varepsilon,I,\Omega} \ell^{26/27+\varepsilon}, \qquad g \in \Omega.$$

Finally, for $q = 0$ (resp. for $q = \pm\ell$ under a technical assumption), we can improve the exponent to $7/8 + \varepsilon$ (resp. to $1/2 + \varepsilon$).

Bounding the sup-norm via an automorphic kernel (1 of 2)

Following Selberg (1956), consider a rapidly decaying continuous function $f \in L^1(G)$, and its action on $L^2(\Gamma\backslash G)$ given by

$$(R(f)\psi)(g) := \int_G f(h)\psi(gh)\,\mathrm{d}h = \int_{\Gamma\backslash G}\left(\sum_{\gamma\in\Gamma} f(g^{-1}\gamma h)\right)\psi(h)\,\mathrm{d}h.$$

Assume that $R(f)$ is a positive operator, and $\pi(f)$ acts by a scalar $c(\pi,\ell)$ on V_π^ℓ. Then $R(f)$ preserves the orthogonal decomposition $V_\pi^\ell \oplus V_\pi^{\ell,\perp}$. Moreover, $R(f)$ composed with the projection to V_π^ℓ has a simple kernel just like $R(f)$:

$$(R(f)_\pi^\ell\psi)(g) = \int_{\Gamma\backslash G}\left(c(\pi,\ell)\sum_{|q|\leqslant\ell}\phi_q(g)\overline{\phi_q(h)}\right)\psi(h)\,\mathrm{d}h.$$

Bounding the sup-norm via an automorphic kernel (2 of 2)

By a simple approximation argument, on the diagonal $g = h$, the kernel of $R(f)^\ell_\pi$ is upper bounded by the kernel of $R(f)$:

$$c(\pi,\ell) \sum_{|q|\leqslant\ell} |\phi_q(g)|^2 \leqslant \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g), \qquad g \in G.$$

To make this work in practice, we consider all functions $f \in L^2(G)$ for which every $\pi_{it,p}(f)$ acts by a scalar on the component $V^\ell_{it,p}$, and by zero on the other components $V^m_{it,p}$. These functions form a Hilbert subspace $\mathcal{H}(\tau_\ell) \subset L^2(G)$ defined by the conditions

- $f(g) = f(kgk^{-1})$ for almost every $g \in G$ and $k \in K$;
- $f = \overline{\chi}_\ell \star f \star \overline{\chi}_\ell$, where χ_ℓ is $2\ell + 1$ times the character of τ_ℓ.

For $f \in L^1(G) \cap \mathcal{H}(\tau_\ell)$ the scalar $c(\pi,\ell)$ exists, and for $\pi \cong \pi_{it,p}$ it equals $\hat{f}(it,p)/(2\ell+1)$, where

$$\widehat{f}(it,p) := \mathrm{tr}\big(\pi_{it,p}(f)\big) = \int_G f(g)\,\varphi^\ell_{it,p}(g)\,\mathrm{d}g,$$

$$\varphi^\ell_{it,p}(g) := \mathrm{tr}\big(\pi_{it,p}(\overline{\chi}_\ell)\pi_{it,p}(g)\pi_{it,p}(\overline{\chi}_\ell)\big).$$

Generalized spherical transform

The theory of Gelfand–Naimark (1947 & 1950) yields the Hilbert space isomorphism $\mathcal{H}(\tau_\ell) \cong L^2(\widehat{G}(\tau_\ell))$ with the Plancherel identity

$$\int_G |f(g)|^2 \, \mathrm{d}g = \frac{1}{2\ell+1} \sum_{\substack{|p|\leqslant \ell \\ p\equiv \ell \bmod 1}} \int_0^\infty |\widehat{f}(it,p)|^2 \, (t^2+p^2) \, \mathrm{d}t.$$

In practice we define $f(g)$ in terms of its generalized spherical transform $\widehat{f}(it,p)$ using the inversion formula

$$f(g) = \frac{1}{2\ell+1} \sum_{\substack{|p|\leqslant \ell \\ p\equiv \ell \bmod 1}} \int_0^\infty \widehat{f}(it,p) \, \varphi^\ell_{it,p}(g^{-1}) \, (t^2+p^2) \, \mathrm{d}t.$$

We need to ensure that $f(g)$ is continuous, rapidly decaying, and of reasonable size. For this we need to understand the spherical trace function $\varphi^\ell_{it,p}(g)$ in some detail.

Spherical trace function

The spherical trace function has an integral representation over K involving the diagonal matrix coefficients of τ_ℓ. As a result, it extends holomorphically to $(\nu, p, g) \in \mathbb{C} \times \frac{1}{2}\mathbb{Z} \times G$, and it satisfies a soft general bound that we skip for simplicity. In particular,

$$\varphi^{\ell}_{\nu,\ell}(g) = (2\ell + 1) \int_K \kappa_\ell(k^{-1} g k) \, \mathrm{d}k,$$

where

$$\kappa_\ell\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \bar{a}^{2\ell} \left(|a|^2 + |c|^2\right)^{\nu - \ell - 1}.$$

The spherical trace function also has remarkable symmetries:

$$\varphi^{\ell}_{\nu,p}(g) = \overline{\varphi^{\ell}_{-\overline{\nu},p}(g)} = \varphi^{\ell}_{\nu,p}(g^{-1}),$$

$$\varphi^{\ell}_{\nu,p}(g) = \varphi^{\ell}_{p,\nu}(g), \quad \nu \equiv p \ (\mathrm{mod}\ 1), \quad |\nu|, |p| \leqslant \ell.$$

These properties become transparent by analytically continuing the representations $\pi_{it,p}$ to (non-unitary Frechét) representation $\pi_{\nu,p}$.

Paley–Wiener space and Schwartz space

Now we see that if $f \in L^1(G) \cap \mathcal{H}(\tau_\ell)$ decays rapidly, then its transform $\widehat{f}$ extends holomorphically to $\mathbb{C} \times \frac{1}{2}\mathbb{Z}$ such that

$$\widehat{f}(\nu, p) = \widehat{f}(p, \nu), \quad \nu \equiv p \pmod 1, \quad |\nu|, |p| \leqslant \ell. \qquad (*)$$

This means that ν and p are not independent as we thought!

Theorem (Wang 1974)

For $f \in \mathcal{H}(\tau_\ell)$ and $R > 0$, the following conditions are equivalent.

1. *f is smooth, and $f(k_1 a_h k_2) = 0$ for $|h| > R$ and $k_1, k_2 \in K$.*
2. *$\widehat{f}$ extends holomorphically to $\mathbb{C} \times \frac{1}{2}\mathbb{Z}$ such that $(*)$ holds true, and we also have $\widehat{f}(\nu, p) \ll_C (1 + |\nu|)^{-C} e^{R|\Re\nu|}$.*

Theorem (Blomer–Harcos–Maga–Milićević 2021)

For $f \in \mathcal{H}(\tau_\ell)$, the following conditions are equivalent.

1. *f is smooth, and $\frac{\partial^m}{\partial h^m} f(k_1 a_h k_2) \ll_{m,A} e^{-A|h|}$ for $k_1, k_2 \in K$.*
2. *$\widehat{f}$ extends holomorphically to $\mathbb{C} \times \frac{1}{2}\mathbb{Z}$ such that $(*)$ holds true, and we also have $\widehat{f}(\nu, p) \ll_{B,C} (1 + |\nu|)^{-C}$ for $|\Re\nu| \leqslant B$.*

Choice of test function

We ended up using the function $f \in \mathcal{H}(\tau_\ell)$ whose transform equals

$$\widehat{f}(\nu, p) = \begin{cases} e^{(p^2-\ell^2+\nu^2)/2}, & \nu \in \mathbb{C}, \quad p \in \frac{1}{2}\mathbb{Z}, \quad |p| \leqslant \ell; \\ 0, & \nu \in \mathbb{C}, \quad p \in \frac{1}{2}\mathbb{Z}, \quad |p| > \ell. \end{cases}$$

This provides a positive operator $R(f)$ on $L^2(\Gamma \backslash G)$ such that

$$c(\pi, \ell) \gg 1/\ell \qquad \text{and} \qquad f(g) \ll \ell^2 e^{-\log^2 \|g\|}.$$

However, this only yields the baseline bound

$$\sum_{|q| \leqslant \ell} |\phi_q(g)|^2 \ll \ell^3.$$

In order to improve on this, we need to amplify π by Hecke operators. This idea was introduced by Iwaniec–Sarnak (1995).

Amplificiation (1 of 2)

Using a standard amplifier, we can bound the sum of $|\phi_q(g)|^2$ by

$$L^{-2+\varepsilon}\ell^2 \sum_{\substack{\gamma\in\mathrm{M}_2(\mathbb{Z}[i]) \\ n=\det\gamma\neq 0 \\ \|g^{-1}\tilde{\gamma}g\|\leqslant\ell^\varepsilon}} \frac{|x_n|}{|n|} \sup_{\nu\in i\mathbb{R}} |\varphi^{\ell}_{\nu,\ell}(g^{-1}\tilde{\gamma}g)| + L^{2+\varepsilon}\ell^{-48},$$

where $\tilde{\gamma}$ abbreviates $\gamma/\sqrt{\det\gamma}$ for any choice of $\sqrt{\det\gamma}$, and

- $x_1 \ll L$;
- $x_n \ll 1$ when n equals $l_1 l_2$ or $l_1^2 l_2^2$ for two split Gaussian primes l_1, l_2 of length about $\sqrt{L}$ and angle in $(0, \pi/4)$;
- $x_n = 0$ otherwise.

Theorem (Blomer–Harcos–Maga–Milićević 2021)

Let $\ell \geqslant 1$ be an integer, and let $h = \begin{pmatrix} z & u \\ & z^{-1} \end{pmatrix} \in G$ be upper triangular. Then for any $\nu \in i\mathbb{R}$, $k \in K$, $\varepsilon > 0$, we have

$$\varphi^{\ell}_{\nu,\ell}(k^{-1}hk) \ll_\varepsilon \min\left(\ell, \frac{\ell^\varepsilon\|h\|^6}{|z^2-1|^2}, \frac{\ell^{1/2+\varepsilon}\|h\|^3}{|u|}\right).$$

Amplificiation (2 of 2)

We defined f in terms of the spherical trace function $\varphi^{\ell}_{\nu,p}$ using the inversion formula. Replacing $\varphi^{\ell}_{\nu,p}$ by

$$\varphi^{\ell,q}_{\nu,p}(h) := \frac{1}{2\pi}\int_0^{2\pi} \varphi^{\ell}_{\nu,p}\big(h\,\mathrm{diag}(e^{i\varrho}, e^{-i\varrho})\big)\, e^{-2qi\varrho}\,\mathrm{d}\varrho$$

in this definition has the effect of picking a single ϕ_q:

$$|\phi_q(g)|^2 \leqslant L^{-2+\varepsilon}\ell^2 \sum_{\substack{\gamma\in\mathrm{M}_2(\mathbb{Z}[i]) \\ n=\det\gamma\neq 0 \\ \|g^{-1}\tilde{\gamma}g\|\leqslant\ell^{\varepsilon}}} \frac{|x_n|}{|n|} \sup_{\nu\in i\mathbb{R}} |\varphi^{\ell,q}_{\nu,\ell}(g^{-1}\tilde{\gamma}g)| + L^{2+\varepsilon}\ell^{-48}.$$

Theorem (Blomer–Harcos–Maga–Milićević 2021)

Let $\ell, q \in \mathbb{Z}$ be such that $\ell \geqslant \max(1, |q|)$. Let $\nu \in i\mathbb{R}$ and $h \in G$. Then for any $\varepsilon > 0$ and $\Lambda > 0$, we have

$$\varphi^{\ell,q}_{\nu,\ell}(h) \ll_{\varepsilon,\Lambda} \ell^{\varepsilon}\min\left(1, \frac{\|h\|}{\sqrt{\ell}\,\mathrm{dist}(h,K)^2\,\mathrm{dist}(h,\mathcal{D})}\right) + \ell^{-\Lambda}.$$