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A hybrid asymptotic formula for the second moment of Rankin–Selberg L-functions

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Abstract

Let g be a fixed modular form of full level, and let $\{f_{j,k}\}$ be a basis of holomorphic cuspidal newforms of even weight k, fixed level and fixed primitive nebentypus. We consider the Rankin–Selberg L-functions $L(\frac{1}{2}+it, f_{j,k}\otimes g)$ and compute their second moment over $t \asymp T$ and $k \asymp K$. For $K^{3/4+\varepsilon} \leqslant T \leqslant K^{5/4-\varepsilon}$, we obtain an asymptotic formula with a power-saving error term. Our result covers the second moment of $L(\frac{1}{2}+it+ir, f_{j,k})L(\frac{1}{2}+it-ir, f_{j,k})$ for any fixed real number r, hence also the fourth moment of $L(\frac{1}{2}+it, f_{j,k})$. For the proof, we develop a precise uniform approximate functional equation with explicit dependence on the archimedean parameters.

1. *Introduction*

Moments of families of L-functions on the critical line play an important role in arithmetic. Often they constitute a crucial input for subconvexity or nonvanishing results that have far-reaching applications. In this article, we consider degree 4 Rankin–Selberg convolutions of a fixed modular form with holomorphic cusp forms of large weight k and their associated L-functions at a point $\frac{1}{2}+it$ high on the critical line. We aim at an asymptotic formula for the second moment with a power-saving error term, where both k and t are large, but possibly in different ranges. A special case treats the fourth moment of L-functions associated with holomorphic cusp forms.

More precisely, we shall consider the following situation. Let $N \geqslant 1$ be an integer and χ an even primitive Dirichlet character modulo N, in particular $N \not\equiv 2 \pmod 4$. For even $k \geqslant 2$, let

$$f_{j,k}(z) = \sum_{m=1}^{\infty} \lambda_{j,k}(m) m^{(k-1)/2} e(mz), \quad 1 \leqslant j \leqslant \theta_k(N,\chi) := \dim_{\mathbb{C}} S_k(N,\chi) \tag{1}$$

be an orthogonal basis, consisting of arithmetically normalized newforms, for the cuspidal space $S_k(N,\chi)$; such a basis exists by the assumption that χ is primitive. It follows from [**4**, Théorème 1] that with the exception of finitely many pairs (N,k), we have (see the end of this introduction for a precise statement)

$$\theta_k(N,\chi) = kN^{1+o(1)}. \tag{2}$$

For a cusp form $f \in S_k(N,\chi)$ we define

$$\rho(f) := \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|^2},$$

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where

$$\|f\|^2 := \int_{\Gamma_0(N)\backslash\mathbb{H}} |f(z)|^2 y^{k-2}\, dx\, dy$$

is the usual Petersson norm. For the Hecke eigenform $f_{j,k}$, it is known by the work of Hoffstein–Lockhart and Iwaniec [**15**, **16**] that

$$\rho(f_{j,k}) = (kN)^{-1+o(1)}. \tag{3}$$

Let g be a fixed automorphic form for the full modular group, which can be

(i) a holomorphic cusp form of some weight $k' < k$ with Hecke eigenvalues $\lambda_g(n)$;
(ii) a Maaß cusp form of weight 0, spectral parameter $r = \sqrt{\lambda - \frac{1}{4}} \geqslant 0$, sign $\epsilon_g \in \{\pm 1\}$, with Hecke eigenvalues $\lambda_g(n)$;
(iii) an Eisenstein series $E_r := E(z, \frac{1}{2} + ir)$ with $r \in \mathbb{R} \setminus \{0\}$, or $E_0 := (\partial/\partial s)E(z,s)|_{s=1/2}$, with Hecke eigenvalues $\lambda_g(n) = \sum_{ab=n}(a/b)^{ir}$.

Let

$$L(s, f_{j,k} \otimes g) = L(2s, \chi) \sum_{n=1}^{\infty} \frac{\lambda_{j,k}(n)\lambda_g(n)}{n^s} \tag{4}$$

be the corresponding Rankin–Selberg L-function. Our assumptions on $f_{j,k}$ and g imply that, for g cuspidal, this is really the usual Rankin–Selberg L-function (see [**26**, Example 2, p. 146]). This L-function is entire (and by a deep result of Ramakrishnan [**32**] it is even cuspidal on GL_4 over $\mathbb{Q}$, but we will not need this). In the case $g = E_r$ we take the right-hand side of (4) as the definition of the left-hand side, and then, by the multiplicative properties of Hecke eigenvalues, we obtain

$$L(s, f_{j,k} \otimes E_r) = L(s + ir, f_{j,k})L(s - ir, f_{j,k}). \tag{5}$$

Let $W_1, W_2 : (0,\infty) \to [0,\infty)$ be fixed smooth functions with nonempty support in $[1,2]$, and let $T, K \geqslant 1$ be two sufficiently large parameters. The aim of this article is to obtain an asymptotic formula for

$$\mathcal{I}(T,K) := \int_0^{\infty} W_1\left(\frac{t}{T}\right) \sum_{k \equiv 0\,(2)} W_2\left(\frac{k-1}{K}\right) \sum_{j=1}^{\theta_k(N,\chi)} \rho(f_{j,k}) \left| L\left(\frac{1}{2} + it, f_{j,k} \otimes g\right)\right|^2 dt. \tag{6}$$

The square root of the analytic conductor of $|L(\frac{1}{2} + it, f_{j,k} \otimes g)|^2$ is approximately given by (cf. (24))

$$\mathcal{C}(t,k) := \frac{N^2}{(2\pi)^4}\left(t^2 + \frac{k^2}{4}\right)^2. \tag{7}$$

For $j \in \mathbb{N}_0$ and $r \in \mathbb{R}$, we shall need the following smooth averages:

$$\begin{aligned} \mathcal{L}_j(T,K) &:= \frac{1}{TK}\int_0^{\infty}\int_0^{\infty} W_1\left(\frac{t}{T}\right) W_2\left(\frac{x}{K}\right) \log^j \mathcal{C}(t,x)\, dt\, dx \asymp \log^j(T+K), \\ \mathcal{M}_{ir}(T,K) &:= \frac{1}{TK}\int_0^{\infty}\int_0^{\infty} W_1\left(\frac{t}{T}\right) W_2\left(\frac{x}{K}\right) \mathcal{C}(t,x)^{ir}\, dt\, dx \ll 1. \end{aligned} \tag{8}$$

The shape of the asymptotic formula depends on the type of g, hence we formulate our main results in three separate theorems. We derive an asymptotic formula that is nontrivial precisely when

$$K^{3/4+\varepsilon} \leqslant T \leqslant K^{5/4-\varepsilon} \tag{9}$$

for some $\varepsilon > 0$.

Theorem 1. *For g cuspidal there are constants $a_0, a_1 \in \mathbb{R}$ depending only on N and g such that*

$$\mathcal{I}(T,K) = TK(a_1\mathcal{L}_1(T,K) + a_0\mathcal{L}_0(T,K)) + O_{N,g,W_1,W_2,\varepsilon}((TK)^{1+\varepsilon}(T^4K^{-5} + T^{-4}K^3)).$$

The leading constant is given by

$$a_1 = \frac{1}{2}L(1, \mathrm{Ad}^2 g)\prod_{p|N}(1-p^{-2}).$$

Theorem 2. *For $g = E_r$ with $r \neq 0$, there are constants $a_0, a_1, a_2 \in \mathbb{R}$ and $b_\pm \in \mathbb{C}$ depending only on N and r such that*

$$\mathcal{I}(T,K) = TK\left(\sum_{j=0}^{2} a_j\mathcal{L}_j(T,K) + b_+\mathcal{M}_{ir}(T,K) + b_-\mathcal{M}_{-ir}(T,K)\right) + O_{N,r,W_1,W_2,\varepsilon}((TK)^{1+\varepsilon}(T^4K^{-5} + T^{-4}K^3)).$$

The leading constant is given by

$$a_2 = \frac{1}{8}|\zeta(1+2ir)|^2\prod_{p|N}(1-p^{-2});$$

moreover,

$$b_\pm = \frac{1}{2}\zeta(1\pm 2ir)^4\prod_{p|N}(1-p^{-2\mp 4ir}).$$

Theorem 3. *For $g = E_0$ there are constants $a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$ depending only on N such that*

$$\mathcal{I}(T,K) = TK\left(\sum_{j=0}^{4} a_j\mathcal{L}_j(T,K)\right) + O_{N,W_1,W_2,\varepsilon}((TK)^{1+\varepsilon}(T^4K^{-5} + T^{-4}K^3)).$$

The leading constant is given by

$$a_4 = \frac{1}{384}\prod_{p|N}(1-p^{-2}).$$

Remark 1. For $T = K$ and fixed $W_{1,2}$, the above asymptotic formulae take a particularly simple shape as $\mathcal{L}_j(K,K)$ is a polynomial in $\log K$ of degree j and $\mathcal{M}_{ir}(K,K)$ is proportional to K^{4ir}.

The present paper was inspired by recent work of Kim–Zhang [**24**] based on a similar result of Duke [**6**], which in turn was motivated by a paper of Sarnak [**34**]. These three papers have in common that they estimate an archimedean family of L-functions of conductor T^8 and size T^2, see [**34**] (or T^3, see [**6**, **24**]), without using approximate functional equations and orthogonality properties of the family. Rather, the argument is based on L^2-techniques, and the idea is to express the moment as the norm of an automorphic form on hyperbolic 3-space [**34**] (or 5-space [**6**]) when restricted to a certain cone. This very elegant and flexible approach gives upper bounds with the correct log-power, but (in its basic form) does not yield an asymptotic formula of the above kind. Moreover, it requires the various archimedean parameters to be of the same size. With more work, these problems could be dealt with essentially by treating the off-diagonal contribution nontrivially as in [**31**]. We also note that there are many other results in the literature on various moments of Rankin–Selberg L-functions, for example, [**13**, **25**, **35**]. Particularly interesting is the recent article [**37**], where an upper bound for the $\mathrm{GL}_3 \times \mathrm{GL}_2$ analogue of the quantity (6) for $T \approx K$ is obtained. The paper [**23**] is also somewhat similar in spirit to the present work. General conjectures for a wide variety of moments of L-functions have been formulated by Conrey, Farmer, Keating, Rubinstein and Snaith [**5**, Conjecture 4.5.1], where the family of Theorems 2 and 3 is considered for fixed t and k. It seems very complicated to integrate the right-hand side of [**5**, (4.5.8)] over t and k.

We use essentially the same type of family as in [**6**, **24**]. The novelty in our work is the derivation of an asymptotic formula with independent parameters T and K; there are not many results of this type in the literature. An upper bound of essentially the right order of magnitude for $\mathcal{I}(T,K)$ in Theorem 1 follows immediately from [**22**, Section 3], even without integration over t. The main point here is really to prove an asymptotic formula with a power-saving error term which turns out to be quite strong in the case $T = K$. Our family has conductor $(T+K)^8$ and size TK^2, and hence in the extreme cases of Theorem 1 (cf. (9)) we are dealing with a family of conductor K^8 and size $K^{11/4}$, or a family of conductor T^8 and size $T^{13/5}$. Theorem 3 states a precise asymptotic formula for the fourth moment of certain modular L-functions (cf. (5) and (6)), improving substantially on [**6**, Theorem 1; **24**, Theorem 2].

Our approach is based on classical techniques, in particular an approximate functional equation, Petersson's formula, Voronoi summation etc. It may be observed, however, that, at least in the range (9), we do not have to treat shifted convolutions sums explicitly, although of course they appear throughout the discussion of the off-diagonal term. In addition, we do not have to use any nontrivial bound for twisted Kloosterman sums.

As an interesting feature, we remark that in the Eisenstein case the off-diagonal term contributes to the main terms in Theorems 2 and 3, and it requires a nontrivial manipulation to recover the exact shape of this off-diagonal contribution. The shape of the asymptotic formula in Theorem 2 is particularly interesting, as it contains oscillating secondary terms $\mathcal{M}_{\pm ir}(T,K)$. This seems to be a new phenomenon of this family. The fact that the off-diagonal term contributes towards the main term seems to be a general feature in moment computations of Rankin–Selberg L-functions with Eisenstein series; see, for instance, [**1**] or [**7**]. It would be desirable to have a conceptual understanding of this empirical phenomenon. It also indicates that there is some intrinsic difficulty in deriving an asymptotic formula for a fourth moment of automorphic L-functions as in Theorems 2 and 3.

Our method could in principle treat more general assumptions on the parameters of $f_{j,k}$ and g, but this would infer technicalities that may obscure the paper rather than be useful in applications. It is possible by standard techniques to remove the harmonic weight $\rho(f_{j,k})$; see, for example, [**19**, Section 26.5]. It should also be possible to replace the family of holomorphic cusp forms of weights $k \asymp K$ by a basis of Maaß forms of spectral parameters $t_j \asymp K$. In this situation, however, the analysis would be somewhat different, and in certain ranges one would encounter effects where the conductor drops; see, for example, [**38**] for a related discussion. Finally and most importantly, it is possible to prove asymptotic formulae in the spirit of Theorem 1 in other ranges, for example, in $K^{\varepsilon} \leqslant T \leqslant K$, and most likely also for T considerably bigger than $K^{5/4}$. This would require, however, a careful treatment of shifted convolution sums, in the latter case of large T with oscillating weight functions. As remarked above, the present approach avoids shifted convolution sums.

In the course of the proof, we use two auxiliary results that may be useful in other situations. These are certainly known to specialists, but we could not find them in the literature. Proposition 1 proves a relatively sophisticated form of the approximate functional equation, while Proposition 2 establishes a Voronoi summation formula for Eisenstein series E_r.

Finally, we remark that [**4**, Théorème 1] can be used to show that $\theta_k(N,\chi) = 0$ for at least one even primitive χ if and only if the pair (N,k) is one of $(1,2)$, $(1,4)$, $(1,6)$, $(1,8)$, $(1,10)$,

$(1,14)$, $(5,2)$, $(7,2)$, $(8,2)$, $(9,2)$, $(11,2)$, $(12,2)$, $(13,2)$, $(15,2)$, $(17,2)$, $(19,2)$, $(21,2)$. In other words, (2) holds for (N,k) outside this explicit set.

Notation and conventions. In an effort to lighten the notational burden, we shall use the following conventions. The symbol ε denotes an arbitrarily small positive constant and the letter A denotes an arbitrarily large positive constant, not necessarily the same at each occurrence. All implied constants may depend on ε, A, W_1, W_2. This allows us to write, for instance, $C^\varepsilon \log C \ll C^\varepsilon$. All implied constants may also depend on N and the parameters of the fixed modular form g; the reader may check that the dependence on these quantities is always polynomial. The phrase 'negligible error' means an error of size $O((T+K)^{-A})$. By (9) this is the same as $O(T^{-A})$ or $O(K^{-A})$.

2. *Preparatory material*

2.1. *A uniform approximate functional equation*

We start with a general result that may be of independent interest. In order to prove an asymptotic formula for $\mathcal{I}(T,K)$, it is very important to have an approximate functional equation with a weight function that is independent of the archimedean parameters. An approximate functional equation of this kind was developed in general in [**12**, Theorem 2.5]. Here we need a slightly stronger variant of this result with a better error term, which forces us to use a somewhat more complicated main term, similarly as in [**14**, Lemma 1]. We shall prove a general result since it requires the same amount of work as the proof of the special case of Rankin–Selberg L-functions we are interested in here.

At this point, we take the opportunity to correct an inaccuracy in our earlier work [**2**]. A uniform approximate functional equation as stated in Proposition 1 should have been used in [**2**, (2.12)]. This would justify, at the cost of an admissible error $O_{\varepsilon,A}(D^{1/2+\varepsilon}T^{-A})$, the tacit assumption that the weight function V in [**2**, (7.2)] is essentially independent of t.

We keep the notation of Harcos [**12**]. Let F be a number field of degree d, and let $\pi = \bigotimes_v \pi_v$ be an isobaric automorphic representation of GL_m over F (cf. [**33**]) with unitary central character and contragradient representation $\tilde{\pi}$. The corresponding L-functions are defined for $\Re s > 1$ by absolutely convergent Dirichlet series as

$$L(s,\pi) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad L(s,\tilde{\pi}) = \sum_{n=1}^{\infty} \frac{\bar{a}_n}{n^s},$$

which extend to meromorphic functions on $\mathbb{C}$ with finitely many poles, and these are connected by a functional equation of the form

$$\mathcal{N}^{s/2} L(s,\pi_\infty) L(s,\pi) = \kappa \mathcal{N}^{(1-s)/2} L(1-s,\tilde{\pi}_\infty) L(1-s,\tilde{\pi}).$$

Here $\mathcal{N}$ is the conductor (a positive integer), κ is the root number (of modulus 1) and

$$L(s,\pi_\infty) = \prod_{j=1}^{md} \pi^{-(s+\mu_j)/2} \Gamma\left(\frac{s+\mu_j}{2}\right), \quad L(s,\tilde{\pi}_\infty) = \prod_{j=1}^{md} \pi^{-(s+\bar{\mu}_j)/2} \Gamma\left(\frac{s+\bar{\mu}_j}{2}\right)$$

for certain $\mu_j \in \mathbb{C}$ that satisfy

$$\Re \mu_j \geqslant \frac{1}{m^2+1} - \frac{1}{2} \tag{10}$$

by a result of Luo–Rudnick–Sarnak [**27**]. We put

$$\eta_j := \frac{1}{4} + \frac{\mu_j}{2}, \quad \eta := \min_{1 \leqslant j \leqslant md} |\eta_j|, \quad \lambda := \frac{L(1/2,\tilde{\pi}_\infty)}{L(1/2,\pi_\infty)},$$

and we define the analytic conductor (at $s = \frac{1}{2}$) as

$$C := \frac{\mathcal{N}}{(2\pi)^{md}} \prod_{j=1}^{md} \left| \frac{1}{2} + \mu_j \right| = \frac{\mathcal{N}}{\pi^{md}} \prod_{j=1}^{md} |\eta_j|.$$

For a multi-index $\boldsymbol{n} \in \mathbb{N}_0^{2md}$ we write $|\boldsymbol{n}| := n(1) + \ldots + n(2md)$ and

$$\boldsymbol{\eta}^{-\boldsymbol{n}} := \prod_{j=1}^{md} \eta_j^{-\boldsymbol{n}(2j-1)} \bar{\eta}_j^{-\boldsymbol{n}(2j)}. \tag{11}$$

By a result of Molteni [**29**, Theorem 4], the coefficients of $L(s, \pi)$ satisfy the uniform bound

$$\sum_{n \leqslant x} |a_n| \ll_\varepsilon x^{1+\varepsilon} C^\varepsilon. \tag{12}$$

To be precise, [**29**, Axiom (A4)] includes $\Re\mu_j \geqslant 0$ but (10) is sufficient for the proof.

Let us assume that $L(s, \pi)$ is entire; then we can state the following result.

PROPOSITION 1. *Let $G_0 : (0, \infty) \to \mathbb{R}$ be a smooth function with functional equation $G_0(x) + G_0(1/x) = 1$ and derivatives decaying faster than any negative power of x as $x \to \infty$. Let $M \in \mathbb{N}$. There are explicitly computable rational constants $c_{\boldsymbol{n},\ell} \in \mathbb{Q}$ depending only on $\boldsymbol{n}, \ell, M, m, d$ such that the following holds for*

$$G(x) := G_0(x) + \sum_{\substack{0<|\boldsymbol{n}|<M \\ 0<\ell<|\boldsymbol{n}|+M}} c_{\boldsymbol{n},\ell} \boldsymbol{\eta}^{-\boldsymbol{n}} \left(x \frac{\partial}{\partial x} \right)^\ell G_0(x). \tag{13}$$

(a) *The function G is smooth and its Mellin transform $\check{G}$ is holomorphic everywhere except for a simple pole at $s = 0$ with formal Laurent expansion*

$$\check{G}(s) \sim \frac{1}{s} + \sum_{j=0}^{\infty} c_j s^j, \quad c_j = \frac{-1}{(j+1)!} \int_0^\infty G'(x)(\log x)^{j+1}\, dx = d_j + O(\eta^{-1}). \tag{14}$$

Moreover, one has

$$\frac{\partial^j}{\partial x^j} G(x) \ll (1+x)^{-A}; \quad \check{G}(s) \ll (1+|s|)^{-A} \text{ for } |\Re s| < \sigma_0 \text{ and } |s| > 1. \tag{15}$$

Here d_j and the implied constants depend at most on j, A, σ_0, M, m, d and the function G_0.

(b) *For any $\varepsilon > 0$ one has*

$$L\left(\frac{1}{2}, \pi\right) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} G\left(\frac{n}{\sqrt{C}}\right) + \kappa\lambda \overline{\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} G\left(\frac{n}{\sqrt{C}}\right)} + O(\eta^{-M} C^{1/4+\varepsilon}), \tag{16}$$

where the implied constant depends at most on ε, M, m, d and the function G_0.

REMARK 2. For self-contragradient representations π, we can strengthen Proposition 1 by imposing the additional symmetry $c_{\bar{\boldsymbol{n}},\ell} = c_{\boldsymbol{n},\ell}$ for the involution $\boldsymbol{n} \mapsto \bar{\boldsymbol{n}}$ defined by $\boldsymbol{\eta}^{-\bar{\boldsymbol{n}}} = \bar{\boldsymbol{\eta}}^{-\boldsymbol{n}}$, that is, $\bar{\boldsymbol{n}}(2j-1) := \boldsymbol{n}(2j)$ and $\bar{\boldsymbol{n}}(2j) := \boldsymbol{n}(2j-1)$. Then G defined by (13) is real-valued and the two main terms in (16) are identical. Indeed, in the self-contragradient situation $\kappa = \lambda = 1$ and the coefficients a_n are real, hence we can replace the original G by its real part without affecting the validity of (14)–(16). This corresponds to replacing the original coefficients $c_{\boldsymbol{n},\ell}$ by $(c_{\boldsymbol{n},\ell} + c_{\bar{\boldsymbol{n}},\ell})/2$, upon noting that the original coefficients are rational.

Proof. We follow closely the proof of Harcos [12, Theorem 2.5]. We can write G_0 as an inverse Mellin transform

$$G_0(x) = \frac{1}{2\pi i}\int_{(\sigma)} x^{-s} H(s)\frac{ds}{s}, \quad \sigma > 0, \tag{17}$$

where $H(s) := s\check{G}_0(s)$ extends to an entire function satisfying

$$H(0) = 1, \quad H(s) = H(-s) = \overline{H(\bar{s})}, \quad H(s) \ll (1+|s|)^{-A} \text{ for } |\Re s| < \sigma_0. \tag{18}$$

As in the Erratum of Harcos [12], we define

$$F(s,\pi_\infty) := \frac{1}{2}C^{-s/2}\mathcal{N}^s \frac{L(1/2+s,\pi_\infty)L(1/2,\tilde{\pi}_\infty)}{L(1/2-s,\tilde{\pi}_\infty)L(1/2,\pi_\infty)} + \frac{1}{2}C^{s/2},$$

and then, by the proof of Harcos [12, Theorem 2.1], we have the exact formula

$$L\left(\frac{1}{2},\pi\right) = \sum_{n=1}^{\infty}\frac{a_n}{\sqrt{n}}W\left(\frac{n}{\sqrt{C}}\right) + \kappa\lambda\overline{\sum_{n=1}^{\infty}\frac{a_n}{\sqrt{n}}W\left(\frac{n}{\sqrt{C}}\right)}$$

with the weight function

$$W(x) = \frac{1}{2\pi i}\int_{(\sigma)} x^{-s}C^{-s/2}F(s,\pi_\infty)H(s)\frac{ds}{s}, \quad \sigma > 0.$$

Now the idea is to approximate W by a function G of the form (cf. (11))

$$G(x) = \frac{1}{2\pi i}\int_{(\sigma)} x^{-s}\left(1 + \sum_{\substack{0<|\boldsymbol{n}|<M\\ 0<\ell<|\boldsymbol{n}|+M}} c_{\boldsymbol{n},\ell}\boldsymbol{\eta}^{-\boldsymbol{n}}(-s)^\ell\right)H(s)\frac{ds}{s}, \quad \sigma > 0,$$

which, by (17), is precisely the function defined in (13). In fact,

$$\check{G}(s) = \left(1 + \sum_{\substack{0<|\boldsymbol{n}|<M\\ 0<\ell<|\boldsymbol{n}|+M}} c_{\boldsymbol{n},\ell}\boldsymbol{\eta}^{-\boldsymbol{n}}(-s)^\ell\right)\check{G}_0(s), \tag{19}$$

hence (14) and (15) are immediate from (18) and $\check{G}_0(s) = H(s)/s$. The rapid decay of W together with (12) implies (16) as soon as the constants $c_{\boldsymbol{n},\ell}$ are chosen such that

$$W(x) = G(x) + O_{M,m,d}(\eta^{-M}).$$

We shall derive this from

$$W(x) - G(x) = \frac{1}{2\pi i}\int_{(0)} x^{-s}\left(C^{-s/2}F(s,\pi_\infty) - 1 - \sum_{\substack{0<|\boldsymbol{n}|<M\\ 0<\ell<|\boldsymbol{n}|+M}} c_{\boldsymbol{n},\ell}\boldsymbol{\eta}^{-\boldsymbol{n}}(-s)^\ell\right)H(s)\frac{ds}{s}$$

by showing that, for explicitly computable rational constants $c_{\boldsymbol{n},\ell} \in \mathbb{Q}$, we have

$$C^{-it/2}F(it,\pi_\infty) = 1 + \sum_{\substack{0<|\boldsymbol{n}|<M\\ 0<\ell<|\boldsymbol{n}|+M}} c_{\boldsymbol{n},\ell}\boldsymbol{\eta}^{-\boldsymbol{n}}(-it)^\ell + O_{M,m,d}(\eta^{-M}(|t|+|t|^{2M})), \quad t \in \mathbb{R}. \tag{20}$$

Note that this strengthens [12, Lemma 4.1].

In proving (20), we can restrict to the range $|t| < \eta$, because, for $|t| \geqslant \eta$, the approximation is trivial for any constants $c_{\boldsymbol{n},\ell}$. Indeed, $\eta \gg_m 1$ by (10), hence we have

$$1 \ll_{M,m} |t|^M \leqslant \eta^{-M}|t|^{2M},$$

and similarly, for $0<|\boldsymbol{n}|<M$ and $0<\ell<|\boldsymbol{n}|+M$, we have

$$c_{\boldsymbol{n},\ell}\boldsymbol{\eta}^{-\boldsymbol{n}}(-it)^\ell \ll_{M,m,d} \eta^{-|\boldsymbol{n}|}|t|^\ell \ll_{M,m} \eta^{-|\boldsymbol{n}|}|t|^{|\boldsymbol{n}|+M} \leqslant \eta^{-M}|t|^{2M}.$$

Let us now assume that $|t|<\eta$; then our starting point is the identity

$$\begin{aligned} C^{-it/2}F(it,\pi_\infty) &= \frac{1}{2}+\frac{1}{2}\prod_{j=1}^{md}\left|\frac{1}{4}+\frac{\mu_j}{2}\right|^{-it}\frac{\Gamma(1/4+\mu_j/2+it/2)\Gamma(1/4+\bar{\mu}_j/2)}{\Gamma(1/4+\bar{\mu}_j/2-it/2)\Gamma(1/4+\mu_j/2)} \\ &= \frac{1}{2}+\frac{1}{2}\exp\left\{2i\Re\sum_{j=1}^{md}\int_0^{t/2}\left(\frac{\Gamma'}{\Gamma}(\eta_j+i\tau)-\log\eta_j\right)d\tau\right\}. \end{aligned}$$

In the integral we have $|i\tau|<\eta/2$ and $|\eta_j+i\tau|>\eta/2$, hence the well-known asymptotic expansion

$$\frac{\Gamma'}{\Gamma}(z)=\log z-\sum_{0<n<M}\frac{B_n}{nz^n}+O_{\sigma,M}(|z|^{-M}),\quad \Re z\geqslant\sigma>0,$$

where $B_n\in\mathbb{Q}$ is the nth Bernoulli number (cf. [**11**, 8.361.8]), yields

$$\int_0^{t/2}\left(\frac{\Gamma'}{\Gamma}(\eta_j+i\tau)-\log\eta_j\right)d\tau=\sum_{\substack{0<n<M\\1\leqslant\ell\leqslant n+1}}b_{n,\ell}\eta_j^{-n}i^{\ell-1}t^\ell+O_M(\eta^{-M}(|t|+|t|^{M+1}))$$

for some constants $b_{n,\ell}\in\mathbb{Q}$. It follows that

$$\begin{aligned} &2i\Re\sum_{j=1}^{md}\int_0^{t/2}\left(\frac{\Gamma'}{\Gamma}(\eta_j+i\tau)-\log\eta_j\right)d\tau \\ &\quad=\sum_{\substack{0<n<M\\1\leqslant\ell\leqslant n+1}}b_{n,\ell}(it)^\ell\sum_{j=1}^{md}(\eta_j^{-n}+(-1)^{\ell-1}\bar{\eta}_j^{-n})+O_{M,m,d}(\eta^{-M}(|t|+|t|^{M+1})). \end{aligned}$$

Finally, we approximate uniformly the exponential function *for imaginary arguments* by its Taylor polynomial of degree $M-1$ to arrive at (20) for some $c_{\boldsymbol{n},\ell}\in\mathbb{Q}$ depending only on $\boldsymbol{n}$, ℓ, M, m, d. Here we use that, for $M\leqslant|\boldsymbol{n}|\leqslant M^2$ and $1\leqslant\ell\leqslant|\boldsymbol{n}|+M$, we have

$$\eta^{-|\boldsymbol{n}|}|t|^\ell\leqslant\eta^{-|\boldsymbol{n}|}(|t|+|t|^{|\boldsymbol{n}|+M})\ll_{M,m}\eta^{-M}(|t|+|t|^{2M}),$$

because $\eta\gg_m 1$ by (10) and $|t|<\eta$. □

We shall fix once and for all a weight function $G_0:(0,\infty)\to\mathbb{R}$ as in Proposition 1, so we will not display the dependence of our statements on this function. We shall fix $M\in\mathbb{N}$ later in the paper. For t real and $k\geqslant 2$ an integer, we shall apply Proposition 1 to the particular L-function

$$\mathfrak{L}_{t,j,k}(s):=L(s+it,f_{j,k}\otimes g)\overline{L(\bar{s}+it,f_{j,k}\otimes g)}.$$

This L-function is associated with a self-contragradient representation π, namely the isobaric sum of $f_{j,k}\otimes g\otimes|\det|^{it}$ and its contragradient. According to Remark 2, we can and we shall assume that G is real-valued, then (16) becomes

$$\left|L\left(\frac{1}{2}+it,f_{j,k}\otimes g\right)\right|^2=2\sum_{n=1}^\infty\frac{a_n}{\sqrt{n}}G\left(\frac{n}{\sqrt{C}}\right)+O(\eta^{-M}C^{1/4+\varepsilon}). \tag{21}$$

We now compute all the necessary data occurring in this situation. We have

$$m=8,\quad d=1,\quad \mathcal{N}=N^4. \tag{22}$$

Using (4), we find that the Dirichlet coefficients of $\mathfrak{L}_{t,j,k}(s)$ are given by

$$a_n = \sum_{d_1^2 d_2^2 m_1 m_2 = n} \chi(d_1)\overline{\chi(d_2)} d_1^{-2it} d_2^{2it} \lambda_{j,k}(m_1)\overline{\lambda_{j,k}(m_2)}\lambda_g(m_1)\overline{\lambda_g(m_2)} m_1^{-it} m_2^{it}, \tag{23}$$

so that (12) is satisfied by previous remarks or by standard bounds. There are constants $\nu_1, \ldots, \nu_4 \in \mathbb{C}$ depending only on g such that the archimedean parameters $\mu_1, \ldots, \mu_8 \in \mathbb{C}$ of $\mathfrak{L}_{t,j,k}(s)$ are given by the numbers $(k-1)/2 + \nu_j + it$ and their complex conjugates. Hence, we have

$$\begin{aligned} \eta_j &= \frac{k}{4} + \frac{\Re\nu_j}{2} \pm \frac{i(\Im\nu_j + t)}{2}, \quad \eta \asymp k + |t|, \\ C = C_{t,k} &= \frac{N^4}{(2\pi)^8} \prod_{j=1}^{4} \left| \frac{k}{2} + \nu_j + it \right|^2 \asymp (k + |t|)^8. \end{aligned} \tag{24}$$

2.2. *Voronoi summation*

The purpose of this section is to compile summation formulae for the Hecke eigenvalues $\lambda_g(n)$ twisted by a finite order additive character. This generalizes Voronoi's original formula for the divisor function (without twist).

PROPOSITION 2. *Let a and c be coprime positive integers, and let $F : (0, \infty) \to \mathbb{C}$ be a smooth function of compact support. Then*

$$c \sum_{n=1}^{\infty} \lambda_g(n) e\left(n\frac{a}{c}\right) F(n) = \sum_{n=1}^{\infty} \lambda_g(n) \sum_{\pm} e\left(\mp n \frac{\bar{a}}{c}\right) \int_0^{\infty} F(x) J_g^{\pm}\left(\frac{4\pi\sqrt{nx}}{c}\right) dx, \tag{25}$$

where

$$J_g^+(x) := 2\pi i^k J_{k-1}(x), \quad J_g^-(x) := 0$$

if g is a holomorphic cusp form of level 1 and weight k;

$$J_g^+(x) := \frac{-\pi}{\cosh(\pi r)}(Y_{2ir}(x) + Y_{-2ir}(x)), \quad J_g^-(x) := \epsilon_g 4 \cosh(\pi r) K_{2ir}(x) \tag{26}$$

if g is a Maaß cusp form of level 1, weight 0, Laplacian eigenvalue $\frac{1}{4} + r^2$ and sign $\epsilon_g \in \{\pm 1\}$. For $g = E_r$ ($r \in \mathbb{R}$) the same formula holds with $J_g^{\pm}$ as in the Maaß case (with $\epsilon_g = 1$), except that on the right-hand side the following polar term has to be added:

$$\sum_{\pm} \zeta(1 \pm 2ir) \int_0^{\infty} \left(\frac{x}{c^2}\right)^{\pm ir} F(x)\, dx \quad \textit{for } r \neq 0, \tag{27}$$

$$\int_0^{\infty} \left(\log\left(\frac{x}{c^2}\right) + 2\gamma\right) F(x)\, dx \quad \textit{for } r = 0. \tag{28}$$

Proof. For Maaß cusp forms the result is due to Meurman [**28**, Theorem 2]; for holomorphic cusp forms it is due to Jutila [**21**, Section 1.9] and Duke–Iwaniec [**8**, Theorem 4]; for $g = E_0$ it is due to Jutila [**20**, Theorem 4]; for $g = E_r$ ($r \neq 0$) we provide the proof below.

Let $g = E_r$ ($r \neq 0$); then the Hecke eigenvalues can be explicitly described as

$$\lambda_g(n) = \sum_{ab=n} \left(\frac{a}{b}\right)^{ir}.$$

We shall derive the Voronoi formula from analytic properties of the Dirichlet series

$$D(g, x, s) := \sum_{n=1}^{\infty} \lambda_g(n) e(nx) n^{-s}, \quad \Re s > 1,$$

summarized by the following lemma.

LEMMA 1. *For $(a,c)=1$ the Dirichlet series $D(g,a/c,s)$ can be analytically continued to a meromorphic function that is holomorphic in the whole complex plane up to simple poles at $s=1\pm ir$, where the residues equal $\zeta(1\pm 2ir)/c^{1\pm 2ir}$. It satisfies the functional equation*

$$D\left(g,\frac{a}{c},s\right)=\pi^{-1}2^{2s-1}\left(\frac{c}{\pi}\right)^{1-2s}\Gamma(1-s+ir)\Gamma(1-s-ir) \times\left\{\cos(\pi ir)D\left(g,\frac{\bar{a}}{c},1-s\right)-\cos(\pi s)D\left(g,-\frac{\bar{a}}{c},1-s\right)\right\}. \tag{29}$$

REMARK 3. The limiting case $r\to 0$ of this lemma is due to Estermann [**10**] and served for Jutila as the starting point for his Voronoi formula; see [**20**, Lemma 1]. Correspondingly, the limit of (27) under $r\to 0$ equals (28).

Proof of Lemma 1. The result follows from [**30**, Lemma 3.7] upon noting that the Dirichlet series $D(g,a/c,s)$ is a shift of a so-called Estermann zeta-function:

$$D\left(g,\frac{a}{c},s-ir\right)=\sum_{n=1}^{\infty}\sigma_{2ir}(n)e\left(\frac{an}{c}\right)n^{-s}.$$

The proof of Motohashi [**30**, Lemma 3.7] is based on analytic properties of the Hurwitz zeta-function. In the Appendix, we provide an alternate proof of Lemma 1, based on the modularity of $g=E_r$, which displays the similarity with the cuspidal case. □

With Lemma 1 at hand, it is straightforward to deduce (25) with the additional polar term (27) on the right-hand side. The left-hand side of (25) equals

$$c\sum_{n=1}^{\infty}\lambda_g(n)e\left(n\frac{a}{c}\right)F(n)=\frac{c}{2\pi i}\int_{(2)}\check{F}(s)D\left(g,\frac{a}{c},s\right)ds,$$

where $\check{F}$ denotes the Mellin transform of F. We shift the contour to the vertical line at -1 and record the contribution of the residues at $s=1\pm ir$; we obtain

$$c\sum_{n=1}^{\infty}\lambda_g(n)e\left(n\frac{a}{c}\right)F(n)=\sum_{\pm}\frac{\zeta(1\pm 2ir)}{c^{\pm 2ir}}\check{F}(1\pm ir)+\frac{c}{2\pi i}\int_{(-1)}\check{F}(s)D\left(g,\frac{a}{c},s\right)ds.$$

We observe that the sum on the right-hand side equals (27), hence we are left with proving that the remaining term equals the right-hand side of (25). By (29) the term in question can be rewritten as

$$\frac{1}{\pi i}\int_{(-1)}\check{F}(s)\left(\frac{c}{2\pi}\right)^{2-2s}\Gamma(1-s+ir)\Gamma(1-s-ir) \times\left\{\cos(\pi ir)D\left(g,\frac{\bar{a}}{c},1-s\right)-\cos(\pi s)D\left(g,-\frac{\bar{a}}{c},1-s\right)\right\}ds.$$

We apply the change of variable $s\to 1-s/2$ and unfold the series $D(g,\pm a/c,s)$. By absolute convergence, we see that the previous display equals

$$\sum_{n=1}^{\infty}\lambda_g(n)e\left(+n\frac{\bar{a}}{c}\right)\frac{1}{2\pi i}\int_{(2)}\left(\frac{2\pi\sqrt{n}}{c}\right)^{-s}\cos(\pi ir)\,\Gamma\left(\frac{s+2ir}{2}\right)\Gamma\left(\frac{s-2ir}{2}\right)\check{F}\left(1-\frac{s}{2}\right)ds$$
$$-\sum_{n=1}^{\infty}\lambda_g(n)e\left(-n\frac{\bar{a}}{c}\right)\frac{1}{2\pi i}\int_{(2)}\left(\frac{2\pi\sqrt{n}}{c}\right)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{s+2ir}{2}\right)\Gamma\left(\frac{s-2ir}{2}\right)\check{F}\left(1-\frac{s}{2}\right)ds.$$

By [9, 6.8.17 and 6.8.26] and (26), this is the same as

$$\sum_{n=1}^{\infty} \lambda_g(n) \sum_{\pm} e\left(\mp n\frac{\bar{a}}{c}\right) \frac{1}{2\pi i} \int_{(2)} \left(\frac{4\pi\sqrt{n}}{c}\right)^{-s} \check{J}_g^{\pm}(s) \check{F}\left(1-\frac{s}{2}\right) ds,$$

which is precisely the right-hand side of (25), owing to the following identity for $t > 0$:

$$\begin{aligned}\frac{1}{2\pi i}\int_{(2)} t^{-s}\check{J}_g^{\pm}(s)\check{F}\left(1-\frac{s}{2}\right) ds &= \frac{1}{2\pi i}\int_{(2)} t^{-s}\check{J}_g^{\pm}(s)\left\{\int_0^{\infty} F(x)x^{-s/2}\,dx\right\} ds \\ &= \int_0^{\infty} F(x)\left\{\frac{1}{2\pi i}\int_{(2)} \left(t\sqrt{x}\right)^{-s}\check{J}_g^{\pm}(s)\,ds\right\} dx \\ &= \int_0^{\infty} F(x) J_g^{\pm}\left(t\sqrt{x}\right) dx.\end{aligned}$$

The proof of Proposition 2 is complete. □

2.3. *Bessel functions*

We start with some standard bounds; more precise results can be found in [13, Appendix]. It follows from the power series expansion that

$$J_{k-1}(x) \ll \frac{x^{k-1}}{\Gamma(k)}, \quad 0 < x \leqslant 1, \tag{30}$$

uniformly in $k \in \mathbb{N}$. It follows from the asymptotic formula and the power series expansion that

$$J_{k-1}(x),\ Y_{2ir}(x) \ll x^{-1/2}, \quad x > 0 \tag{31}$$

for fixed $k \in \mathbb{N}$ and $r \in \mathbb{R}$. Finally, we recall the bound

$$K_{2ir}(x) \ll e^{-x}, \quad x \geqslant 1, \tag{32}$$

for fixed $r \in \mathbb{R}$, which follows again from the asymptotic formula.

For a smooth function $h : (0,\infty) \to \mathbb{C}$ of compact support, let

$$\widehat{h}(t) := \int_0^{\infty} h(x)e(xt)\,dx, \quad \tilde{h}(t) := \frac{1}{\sqrt{2\pi}}\int_0^{\infty} h(\sqrt{x})e\left(\frac{xt}{2\pi}\right)\frac{dx}{\sqrt{x}}.$$

Then the following summation formula holds.

Lemma 2. *Let $K > 0$, $\xi > 0$ and $h : (0,\infty) \to \mathbb{R}$ be a smooth function of compact support. Then*

$$\begin{aligned}\sum_{k \equiv 0\,(2)} i^{-k} h\left(\frac{k-1}{K}\right) J_{k-1}(\xi) = &-\frac{K}{2\sqrt{\xi}}\Im\left(e\left(\frac{\xi}{2\pi}-\frac{1}{8}\right)\tilde{h}\left(\frac{K^2}{2\xi}\right)\right) \\ &+ O\left(\frac{\xi}{K^4}\int_{-\infty}^{\infty} |\widehat{h}(t)t^4|\,dt\right)\end{aligned}$$

with an absolute implied constant.

Remark 4. For odd weights one has the similar formula

$$\begin{aligned}\sum_{k \equiv 1\,(2)} i^{-k} h\left(\frac{k-1}{K}\right) J_{k-1}(\xi) = &-\frac{iK}{2\sqrt{\xi}}\Re\left(e\left(\frac{\xi}{2\pi}-\frac{1}{8}\right)\tilde{h}\left(\frac{K^2}{2\xi}\right)\right) \\ &+ O\left(\frac{\xi}{K^4}\int_{-\infty}^{\infty} |\widehat{h}(t)t^4|\,dt\right).\end{aligned}$$

REMARK 5. It is well known that an individual Bessel function $J_{k-1}(x)$ is hard to control in the range $k \ll x \ll k^2$. The lemma shows, however, that, on average over k, its values cancel almost completely until $x \approx k^2$, when the asymptotic behaviour becomes stable. For further discussion of the error term see [**17**, p. 87].

Proof. This is based on [**17**, Lemma 5.8] and can be found in [**23**, Lemma 2.3]. □

Finally, we state a result that is useful for estimating the Hankel-type transform occurring in the Voronoi formula (25).

LEMMA 3. *Let $F:(0,\infty)\to\mathbb{C}$ be a smooth function of compact support. For $s\in\mathbb{C}$ let B_s denote either of the Bessel functions J_s, Y_s or K_s. Then, for $\alpha>0$ and $j\in\mathbb{N}$, we have*

$$\int_0^\infty F(x)B_s(\alpha\sqrt{x})\,dx = \pm\left(\frac{2}{\alpha}\right)^j\int_0^\infty \frac{\partial^j}{\partial x^j}(F(x)x^{-s/2})x^{(s+j)/2}B_{s+j}(\alpha\sqrt{x})\,dx.$$

Proof. The Bessel functions B_s satisfy the recurrence relation $(x^sB_s(x))' = \pm x^sB_{s-1}$ which translates to

$$(\alpha\sqrt{x})^sB_s(\alpha\sqrt{x}) = \pm\frac{2}{\alpha^2}\frac{\partial}{\partial x}((\alpha\sqrt{x})^{s+1}B_{s+1}(\alpha\sqrt{x})).$$

Using this identity and applying integration by parts j times, we obtain

$$\begin{aligned}\int_0^\infty F(x)B_s(\alpha\sqrt{x})\,dx &= \pm\left(\frac{2}{\alpha^2}\right)^j\int_0^\infty F(x)(\alpha\sqrt{x})^{-s}\frac{\partial^j}{\partial x^j}((\alpha\sqrt{x})^{s+j}B_{s+j}(\alpha\sqrt{x}))\,dx\\ &= \pm\left(\frac{2}{\alpha^2}\right)^j\int_0^\infty \frac{\partial^j}{\partial x^j}(F(x)(\alpha\sqrt{x})^{-s})(\alpha\sqrt{x})^{s+j}B_{s+j}(\alpha\sqrt{x})\,dx\\ &= \pm\left(\frac{2}{\alpha}\right)^j\int_0^\infty \frac{\partial^j}{\partial x^j}(F(x)x^{-s/2})x^{(s+j)/2}B_{s+j}(\alpha\sqrt{x})\,dx.\end{aligned}$$

□

2.4. *Fourier coefficients of cusp forms*

For a Dirichlet character χ modulo N and a positive integer c divisible by N, let

$$S_\chi(m,n,c) := \sideset{}{^*}\sum_{d\,(\mathrm{mod}\,c)}\chi(d)\,e\left(\frac{m\bar{d}+nd}{c}\right)$$

be the twisted Kloosterman sum. We formulate Petersson's formula for the basis of holomorphic cusp forms specified in (1).

LEMMA 4. *For $k,m,n\in\mathbb{N}$ and $k\geqslant 3$, we have*

$$\sum_{j=1}^{\theta_k(N,\chi)}\rho(f_{j,k})\lambda_{j,k}(m)\overline{\lambda_{j,k}(n)} = \delta_{m,n} + 2\pi i^{-k}\sum_{N\mid c}\frac{S_\chi(m,n,c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Proof. This follows from [**19**, Proposition 14.5]. □

We shall often use the following standard result.

LEMMA 5. *We have the uniform bound*

$$\sum_{m\leqslant M} |\lambda_g(m)|^2 \ll_{g,\varepsilon} M^{1+\varepsilon}. \tag{33}$$

Proof. See [19, (1.80) and (14.56)] for stronger bounds. □

3. *The diagonal term*

We shall assume (9) for the rest of the paper, since otherwise the asymptotic formula is trivial. We substitute the approximate functional equation (21) in the special case (22)–(24) into (6) and choose $M := 3$. We note that both $G = G_{t,k}$ and $C = C_{t,k}$ depend (mildly) on t and k, but in the support of W_1 and W_2 we have $C_{t,k} \asymp \tilde{C} := (T+K)^8$. By (2) and (3) (or directly by Lemma 4), the error term contributes at most

$$\ll TK\tilde{C}^{1/4+\varepsilon}(T+K)^{-M} \ll (TK)^{1+\varepsilon}(T+K)^{-1}. \tag{34}$$

After inserting the main term, the j-sum in (6) equals

$$2\sum_n G\left(\frac{n}{\sqrt{C}}\right)\frac{1}{\sqrt{n}} \sum_{d_1^2d_2^2m_1m_2=n} \frac{\chi(d_1)}{d_1^{2it}}\frac{\overline{\chi(d_2)}}{d_2^{-2it}}\frac{\lambda_g(m_1)}{m_1^{it}}\frac{\overline{\lambda_g(m_2)}}{m_2^{-it}} \sum_{j=1}^{\theta_k(N,\chi)} \rho(f_{j,k})\lambda_{j,k}(m_1)\overline{\lambda_{j,k}(m_2)}. \tag{35}$$

By Lemma 4, the innermost sum equals

$$\delta_{m_1,m_2} + 2\pi i^{-k}\sum_{N\mid c}\frac{S_\chi(m_1,m_2,c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{m_1m_2}}{c}\right). \tag{36}$$

The δ-term contributes

$$2\int_0^\infty W_1\left(\frac{t}{T}\right)\sum_{k\equiv 0\,(2)} W_2\left(\frac{k-1}{K}\right)\sum_{d_1,d_2,m}\frac{\chi(d_1)}{d_1^{1+2it}}\frac{\overline{\chi(d_2)}}{d_2^{1-2it}}\frac{|\lambda_g(m)|^2}{m}G\left(\frac{d_1^2d_2^2m^2}{\sqrt{C}}\right)dt.$$

Let $\check{G}$ denote the Mellin transform of G and similarly for other functions. By Li [26, p. 145] we have

$$\sum_{m=1}^\infty \frac{|\lambda_g(m)|^2}{m^s} = \frac{L(s,g\otimes\tilde{g})}{\zeta(2s)},$$

when g is cuspidal. If $g = E_r$ is an Eisenstein series, then we take the preceding display as the definition for $L(s,g\otimes\tilde{g})$, so that $L(s,E_r\otimes\widetilde{E_r}) = \zeta(s)^2\zeta(s+2ir)\zeta(s-2ir)$. With this notation, the contribution of the δ-term in Petersson's formula equals

$$2\int_0^\infty W_1\left(\frac{t}{T}\right)\sum_{k\equiv 0\,(2)} W_2\left(\frac{k-1}{K}\right) \times \frac{1}{2\pi i}\int_{(1)} L(1+2it+2s,\chi)L(1-2it+2s,\bar{\chi})\frac{L(1+2s,g\otimes\tilde{g})}{\zeta(2+4s)}C^{s/2}\check{G}(s)\,ds\,dt. \tag{37}$$

We shift the contour to the line $\Re s = -\frac{1}{4}+\varepsilon$; then the new integral contributes (cf. (15))

$$\ll (TK)^{1+\varepsilon}(T+K)^{-1}. \tag{38}$$

There is pole at $s=0$ whose order v depends on g: if g is a cusp form, then $v=2$; if $g = E_r$ with $r\neq 0$, then $v=3$; while for $g=E_0$ we have $v=5$. The residue of the pole is given by a linear combination (with coefficients depending at most on N and g) of

$$\int_0^\infty W_1\left(\frac{t}{T}\right)\sum_{k\equiv 0\,(2)} W_2\left(\frac{k-1}{K}\right) L^{(j_1)}(1+2it,\chi)L^{(j_2)}(1-2it,\bar\chi)\log^{j_3}\mathcal{C}(t,k)\,dt$$
$$+O\left((TK)^{1+\varepsilon}(T+K)^{-1}\right) \tag{39}$$

for $j_1+j_2+j_3\leqslant v-1$. The error term comes from approximating the square root of $C=C_{t,k}$ in (24) by (7) and inserting (14). The constant in front of the leading term $(j_1,j_2,j_3)=(0,0,v-1)$ equals (cf. (14))

$$\frac{2}{(v-1)!\zeta(2)}\lim_{s\to 0}(s^{v-1}L(1+2s,g\otimes\tilde g))=\begin{cases}\dfrac{L(1,\mathrm{Ad}^2g)}{\zeta(2)}, & g \text{ cuspidal},\\ \dfrac{|\zeta(1+2ir)|^2}{4\zeta(2)}, & g=E_r,\ r\neq 0,\\ \dfrac{1}{192\zeta(2)}, & g=E_0.\end{cases} \tag{40}$$

For χ trivial (that is, N=1) there are also poles at $s=\pm it$, but their contribution is (cf. (15))

$$\ll (TK)^{1+\varepsilon}T^{-A}. \tag{41}$$

If $g=E_r$ with $r\neq 0$, then there are two additional simple poles in (37) at $\pm ir$ with residue

$$2\int_0^\infty W_1\left(\frac{t}{T}\right)\sum_{k\equiv 0\,(2)} W_2\left(\frac{k-1}{K}\right)$$
$$\times\frac{L(1+2it\pm 2ir,\chi)L(1-2it\pm 2ir,\bar\chi)\zeta(1\pm 2ir)^2\zeta(1\pm 4ir)}{2\zeta(2\pm 4ir)}C^{\pm ir/2}\check G(\pm ir)\,dt. \tag{42}$$

Here again we can approximate $C^{1/2}$ by (7) and $\check G(\pm ir)$ by $\check G_0(\pm ir)$ at the cost of an error term $O((TK)^{1+\varepsilon}(T+K)^{-1})$; cf. (19) and (24).

We continue with the analysis of (39). Applying Poisson summation, it is straightforward to see that the main term in (39) equals, for any $A>0$,

$$\frac{1}{2}\int_0^\infty\int_0^\infty W_1\left(\frac{t}{T}\right)W_2\left(\frac{x}{K}\right)L^{(j_1)}(1+2it,\chi)L^{(j_2)}(1-2it,\bar\chi)\log^{j_3}\mathcal{C}(t,x)\,dt\,dx$$
$$+O\left((TK)^{1+\varepsilon}K^{-A}\right). \tag{43}$$

Now we use the simple approximation

$$L^{(j)}(1+2it,\chi)=\sum_{m\leqslant M}\frac{\chi(m)(-\log m)^j}{m^{1+2it}}+O_j\left(\frac{(\log M)^j}{M}\right),\quad M\geqslant\frac{e^{3j/2}}{3}N|t|,$$

which follows in a standard fashion from van der Corput's lemma (see [**36**, Lemma 4.10 and Theorem 4.11]). We substitute this into (43) with $M:=e^6NT$ (note that $j_1,j_2\leqslant 4$ and $|t|\leqslant 2T$). Then the t-integral equals, uniformly for $x\asymp K$,

$$\sum_{m,n\leqslant e^6NT}\frac{\chi(m)\overline{\chi(n)}(-\log m)^{j_1}(-\log n)^{j_2}}{mn}\int_0^\infty W_1\left(\frac{t}{T}\right)\log^{j_3}\mathcal{C}(t,x)\left(\frac{n}{m}\right)^{2it}dt+O((TK)^\varepsilon).$$

The diagonal and error terms together contribute (cf. (8))

$$\frac{\zeta_{(N)}^{(j_1+j_2)}(2)}{2}TK\mathcal{L}_{j_3}(T,K)+O(K(TK)^\varepsilon), \tag{44}$$

where (N) indicates the removal of the Euler factors at primes dividing N. Let us now estimate the off-diagonal term. Partial integration shows, for any $m \neq n$ and $x \asymp K$,

$$\int_0^\infty W_1\left(\frac{t}{T}\right) \log^{j_3} \mathcal{C}(t,x) \left(\frac{n}{m}\right)^{2it} dt \ll \frac{(TK)^\varepsilon}{|\log(n/m)|},$$

hence the off-diagonal term contributes at most

$$\begin{aligned} &\ll K(TK)^\varepsilon \sum_{\substack{m,n \leqslant e^6 NT \\ m \neq n}} \frac{1}{mn|\log(n/m)|} \\ &\ll K(TK)^\varepsilon \sum_{m<n \leqslant e^6 NT} \frac{1}{n \min(m, n-m)} \\ &\ll K(TK)^\varepsilon \sum_{m',n' \leqslant e^6 NT} \frac{1}{m'n'} \ll K(TK)^\varepsilon. \end{aligned} \tag{45}$$

The same argument shows that (42) equals, up to an error term already present in (39),

$$TK\mathcal{M}_{\pm ir}(T,K) \frac{\zeta_{(N)}(2 \pm 4ir)\zeta(1 \pm 2ir)^2\zeta(1 \pm 4ir)}{2\zeta(2 \pm 4ir)} \check{G}_0(\pm ir). \tag{46}$$

Remark 6. We shall see in Subsection 4.6 that the term (46) will be cancelled by some portion of the off-diagonal term. We note that we could have avoided the computation of (42) and (46) in the case $g = E_r$ with $r \neq 0$ by choosing G_0 as in (17) and (18) with the additional feature that $H(s)$ is divisible by $s^2 + r^2$. Then $\check{G}_0(\pm ir) = 0$, and hence, by (19), also $\check{G}(\pm ir) = 0$, and the residues vanish. This trick was probably used for the first time in [**3**].

The various error terms in (34), (38), (39), (41), (43)–(45) are admissible in Theorems 1–3. The main term in (44) furnishes the main term of Theorem 1 and part of the main terms in Theorems 2 and 3. We note that Subsection 4.6 discusses an additional contribution to the main terms of Theorems 2 and 3. The leading constant in the various cases of g follows from (40) and (44).

4. *The off-diagonal term*

4.1. *Averaging over k and t*

We return to (35) which is the j-sum in (6), and substitute now the Kloosterman term in (36) for the innermost sum. This gives a total contribution of

$$\begin{aligned} &4\pi \int_0^\infty W_1\left(\frac{t}{T}\right) \sum_{k \equiv 0\,(2)} W_2\left(\frac{k-1}{K}\right) i^{-k} \sum_n G\left(\frac{n}{\sqrt{C}}\right) \frac{1}{\sqrt{n}} \\ &\times \sum_{d_1^2 d_2^2 m_1 m_2 = n} \frac{\chi(d_1)}{d_1^{2it}} \frac{\overline{\chi(d_2)}}{d_2^{-2it}} \frac{\lambda_g(m_1)}{m_1^{it}} \frac{\overline{\lambda_g(m_2)}}{m_2^{-it}} \sum_{N|c} \frac{S_\chi(m_1,m_2,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{m_1m_2}}{c}\right) dt. \end{aligned} \tag{47}$$

Recall that both $G = G_{t,k}$ and $C = C_{t,k}$, defined in (13) and (24), depend (mildly) on t and k, but we have $C_{t,k} \asymp \tilde{C} := (T+K)^8$. Moreover, G is real-valued as we have assumed according to Remark 2. First, we observe that (47) is absolutely convergent, due to the rapid decay of the Bessel J-function near 0 for large k; see (30). More precisely, we can truncate the multiple sum at

$$n \leqslant \tilde{C}^{1/2+\varepsilon}, \quad c \leqslant \sqrt{m_1 m_2}\tilde{C}^\varepsilon$$

at the cost of a negligible error.

Next we write

$$W_1\left(\frac{t}{T}\right)W_2\left(\frac{k-1}{K}\right)G_{t,k}\left(\frac{n}{\sqrt{C_{t,k}}}\right) =: \Omega_1\left(\frac{t}{T},\frac{k-1}{K},\frac{n}{\sqrt{\tilde{C}}}\right),$$

where $\Omega_1 : (0,\infty)^3 \to \mathbb{R}$ is a smooth function depending on K, T, N and g with the following properties: it is compactly supported in the first two variables and rapidly decaying in the third variable, uniformly in K and T. Let

$$\tilde{\Omega}_1(x,y,z) := W_1(x)W_2(y)G_0\left(\frac{(2\pi)^4 z\sqrt{\tilde{C}}}{N^2\left((xT)^2+((1/2)yK)^2\right)^2}\right). \tag{48}$$

Then, by (11), (13) and (24), the partial derivatives satisfy, uniformly in $\boldsymbol{x}\in(0,\infty)^3$,

$$\boldsymbol{x}^{\boldsymbol{k}}\left(\Omega_1^{(\boldsymbol{j})}(\boldsymbol{x})-\tilde{\Omega}_1^{(\boldsymbol{j})}(\boldsymbol{x})\right) \ll_{\boldsymbol{j},\boldsymbol{k}} (T+K)^{-1}, \quad \boldsymbol{j},\boldsymbol{k}\in\mathbb{N}_0^3.$$

In order to sum over k with the help of Lemma 2, we define

$$\Omega_2(x,u,z) := \int_0^\infty \Omega_1(x,y,z)\, e^{iy^2u}\, dy.$$

Then $\Omega_2 : (0,\infty)\times\mathbb{R}\times(0,\infty)\to\mathbb{C}$ is compactly supported in the first variable. Integration by parts shows that it is rapidly decaying in the other two variables, and, moreover, for

$$\tilde{\Omega}_2(x,u,z) := \int_0^\infty \tilde{\Omega}_1(x,y,z)\, e^{iy^2u}\, dy \tag{49}$$

we have, uniformly in $\boldsymbol{x}\in(0,\infty)\times\mathbb{R}\times(0,\infty)$,

$$\boldsymbol{x}^{\boldsymbol{k}}\left(\Omega_2^{(\boldsymbol{j})}(\boldsymbol{x})-\tilde{\Omega}_2^{(\boldsymbol{j})}(\boldsymbol{x})\right) \ll_{\boldsymbol{j},\boldsymbol{k}} (T+K)^{-1}, \quad \boldsymbol{j},\boldsymbol{k}\in\mathbb{N}_0^3.$$

Now applying Lemma 2 with $\xi := 4\pi\sqrt{m_1m_2}/c$ to (47), we obtain the main term

$$K\sum_{\pm}\frac{1\pm i}{2}\int_0^\infty \sum_{d_1,d_2,m_1,m_2}\sum_{N\mid c}\Omega_2\left(\frac{t}{T},\frac{\pm K^2c}{8\pi\sqrt{m_1m_2}},\frac{d_1^2d_2^2m_1m_2}{\sqrt{\tilde{C}}}\right)$$
$$\times\frac{\chi(d_1)}{d_1^{1+2it}}\frac{\overline{\chi(d_2)}}{d_2^{1-2it}}\frac{\lambda_g(m_1)}{m_1^{3/4+it}}\frac{\overline{\lambda_g(m_2)}}{m_2^{3/4-it}}e\left(\pm\frac{2\sqrt{m_1m_2}}{c}\right)\frac{S_\chi(m_1,m_2,c)}{c^{1/2}}\,dt,$$

while the error term in Lemma 2 infers a total error of

$$\begin{aligned}
&\ll \tilde{C}^\varepsilon\frac{T}{K^4}\sum_{m_1m_2\leqslant\tilde{C}^{1/2+\varepsilon}}|\lambda_g(m_1)\lambda_g(m_2)|\sum_{\substack{c\leqslant\tilde{C}^{1/4+\varepsilon}\\ N\mid c}}\frac{|S_\chi(m_1,m_2,c)|}{c^2}\\
&\ll \tilde{C}^\varepsilon\frac{T}{K^4}\sum_{mn\leqslant\tilde{C}^{1/2+\varepsilon}}|\lambda_g(m)|^2 \ll (TK)^\varepsilon\frac{T(T^4+K^4)}{K^4}\\
&\ll (TK)^{1+\varepsilon}\left(\frac{T^4}{K^5}+\frac{K^3}{T^4}\right).
\end{aligned} \tag{50}$$

Here it is enough to use the trivial bound $|S_\chi(m_1,m_2,c)|\leqslant c$ and (33).

In order to integrate over t, we define

$$\Omega_3(w,u,z) := \int_0^\infty \Omega_2(x,u,z)\, e^{iwx}\, dx.$$

Then $\Omega_3 : \mathbb{R}^2\times(0,\infty)\to\mathbb{C}$ is rapidly decaying in all three variables. Putting

$$\tilde{\Omega}_3(w,u,z) := \int_0^\infty \tilde{\Omega}_2(x,u,z)\, e^{iwx}\, dx, \tag{51}$$

we see as above, that uniformly in $\boldsymbol{x} \in \mathbb{R}^2 \times (0,\infty)$,

$$\boldsymbol{x}^{\boldsymbol{k}} \left(\Omega_3^{(\boldsymbol{j})}(\boldsymbol{x}) - \tilde{\Omega}_3^{(\boldsymbol{j})}(\boldsymbol{x}) \right) \ll_{\boldsymbol{j},\boldsymbol{k}} (T+K)^{-1}, \quad \boldsymbol{j},\boldsymbol{k} \in \mathbb{N}_0^3. \tag{52}$$

In addition, we factor out $\delta = (d_1, d_2)$ to arrive at

$$TK \sum_{\pm} \frac{1 \pm i}{2} \sum_{\substack{(\delta,N)=1 \\ (d_1,d_2)=1}} \sum_{m_1,m_2} \sum_{N|c} \Omega_3 \left(T \log \frac{d_2^2 m_2}{d_1^2 m_1}, \frac{\pm K^2 c}{8\pi \sqrt{m_1 m_2}}, \frac{\delta^4 d_1^2 d_2^2 m_1 m_2}{\sqrt{\tilde{C}}} \right)$$
$$\times \frac{\chi(d_1)\overline{\chi(d_2)}}{\delta^2 d_1 d_2} \frac{\lambda_g(m_1)}{m_1^{3/4}} \frac{\overline{\lambda_g(m_2)}}{m_2^{3/4}} e\left(\pm \frac{2\sqrt{m_1 m_2}}{c} \right) \frac{S_\chi(m_1, m_2, c)}{c^{1/2}}.$$

It is convenient to introduce dyadic decompositions. Let $\omega : (0,\infty) \to [0,\infty)$ be a smooth function supported on $[\frac{1}{2}, 2]$ such that

$$\sum_{j=0}^{\infty} \omega(x/2^j) = 1, \quad x \geqslant 1. \tag{53}$$

Then we can recast the preceding expression as

$$TK \sum_{\pm} \frac{1 \pm i}{2} \sum_{\substack{M_1,M_2,G \geqslant 1 \\ \text{powers of } 2}} \sum_{\substack{(\delta,N)=1 \\ (d_1,d_2)=1}} \sum_{m_1,m_2} \sum_{N|c} \Omega_3 \left(T \log \frac{d_2^2 m_2}{d_1^2 m_1}, \frac{\pm K^2 c}{8\pi \sqrt{m_1 m_2}}, \frac{\delta^4 d_1^2 d_2^2 m_1 m_2}{\sqrt{\tilde{C}}} \right)$$
$$\times \omega\left(\frac{m_1}{M_1}\right) \omega\left(\frac{m_2}{M_2}\right) \omega\left(\frac{c}{G}\right) \frac{\chi(d_1)\overline{\chi(d_2)}}{\delta^2 d_1 d_2} \frac{\lambda_g(m_1)}{m_1^{3/4}} \frac{\overline{\lambda_g(m_2)}}{m_2^{3/4}} e\left(\pm \frac{2\sqrt{m_1 m_2}}{c} \right) \frac{S_\chi(m_1, m_2, c)}{c^{1/2}}. \tag{54}$$

By the rapid decay of Ω_3 in all three variables, we can restrict the summation to

$$M_1 M_2 \leqslant \frac{\tilde{C}^{1/2+\varepsilon}}{\delta^4 d_1^2 d_2^2}, \quad G \leqslant \frac{\tilde{C}^{\varepsilon} (M_1 M_2)^{1/2}}{K^2}. \tag{55}$$

Moreover, we can assume

$$d_2^2 m_2 = d_1^2 m_1 \left(1 + O(T^{\varepsilon - 1}) \right). \tag{56}$$

We will keep this in mind for later and, in particular, often use

$$d_2^2 M_2 \asymp d_1^2 M_1. \tag{57}$$

4.2. *Interlude on character sums*

For later purposes, we need to transform the expression

$$S_\chi(m_1, m_2, c) e\left(\pm \frac{2\sqrt{m_1 m_2}}{c} \right)$$

in (54). We proceed similarly as in [**37**, Section 6]. We start by writing

$$e\left(\pm \frac{2\sqrt{m_1 m_2}}{c} \right) = e\left(\pm \frac{d_1^2 m_1 + d_2^2 m_2}{d_1 d_2 c} \right) e\left(\mp \frac{(\sqrt{d_1^2 m_1} - \sqrt{d_2^2 m_2})^2}{d_1 d_2 c} \right).$$

We infer

$$\begin{aligned} S_\chi(m_1,m_2,c)&e\left(\pm\frac{d_1^2m_1+d_2^2m_2}{d_1d_2c}\right)\\ &=\frac{\phi(c)}{\phi(d_1d_2c)}S_\chi(d_1d_2m_1,d_1d_2m_2,d_1d_2c)e\left(\pm\frac{d_1^2m_1+d_2^2m_2}{d_1d_2c}\right)\\ &=\frac{\phi(c)}{\phi(d_1d_2c)}\sideset{}{^*}\sum_{d\,(d_1d_2c)}\chi(d)e\left(m_1\frac{d_2\bar{d}\pm d_1}{d_2c}+m_2\frac{d_1d\pm d_2}{d_1c}\right). \end{aligned}\tag{58}$$

We write

$$(d_1d\pm d_2,c)=g,\quad c=gh,\quad d_1d\pm d_2=gf.$$

Note that $(gf,d_1d_2)=1$ by $(d_1,d_2)=1$. Let us fix a decomposition $c=gh$ with $(g,d_1d_2)=1$. It is straightforward to check, using again $(d_1,d_2)=1$, that if d runs through $(\mathbb{Z}/d_1d_2c\mathbb{Z})^\times$ with $(d_1d\pm d_2,c)=g$ then f runs through $(\mathbb{Z}/d_1^2d_2h\mathbb{Z})^\times$ with $(gf\mp d_2,d_1^2d_2h)=d_1$, and this is a bijection between the relevant residue classes. With this notation, we have

$$d_2\bar{d}\pm d_1\equiv\pm gf\overline{(gf\mp d_2)/d_1}\pmod{d_1d_2c}.$$

Hence, we can recast (58) as

$$\frac{\phi(c)}{\phi(d_1d_2c)}\sum_{\substack{gh=c\\(g,d_1d_2)=1}}\ \sideset{}{^*}\sum_{\substack{f\,(d_1^2d_2h)\\(gf\mp d_2,d_1^2d_2h)=d_1}}\chi\left(\frac{gf\mp d_2}{d_1}\right)e\left(\pm m_1\frac{f\overline{(gf\mp d_2)/d_1}}{d_2h}+m_2\frac{f}{d_1h}\right).$$

Using the change of variable $f\mapsto\bar{f}$, we can summarize the preceding discussion as

$$\begin{aligned} S_\chi(m_1,m_2,c)e\left(\pm\frac{2\sqrt{m_1m_2}}{c}\right)&=e\left(\mp\frac{(\sqrt{d_1^2m_1}-\sqrt{d_2^2m_2})^2}{d_1d_2c}\right)\frac{\phi(c)}{\phi(d_1d_2c)}\\ \times\sum_{\substack{gh=c\\(g,d_1d_2)=1}}\ \sideset{}{^*}\sum_{\substack{f\,(d_1^2d_2h)\\(g\bar{f}\mp d_2,d_1^2d_2h)=d_1}}&\chi\left(\frac{g\bar{f}\mp d_2}{d_1}\right)e\left(\pm m_1\frac{\overline{(g\mp d_2f)/d_1}}{d_2h}+m_2\frac{\bar{f}}{d_1h}\right). \end{aligned}\tag{59}$$

For later purposes, we prove the following two lemmas.

LEMMA 6. *Let d_1,d_2,g,h,m_1,m_2 be positive integers such that gh is divisible by N and d_1,d_2,g are pairwise coprime. Then*

$$\left|\sideset{}{^*}\sum_{\substack{f\,(d_1^2d_2h)\\(g\mp d_2f,d_1^2d_2h)=d_1}}\chi\left(\frac{g\bar{f}\mp d_2}{d_1}\right)e\left((m_1-m_2)\frac{f}{d_1h}\right)\right|\leqslant Nd_1d_2\tau(h)^2(h,(m_1-m_2)d_1d_2).$$

Proof. We use first that χ is a character modulo N, a divisor of gh. Writing

$$h=h_1h_2,\quad h_1:=\frac{N}{(N,g)},$$

we see that the character sum in question equals

$$\sideset{}{^*}\sum_{\substack{f_1\,(d_1^2d_2h_1)\\(g\mp d_2f_1,d_1^2d_2h_1)=d_1}}\chi\left(\frac{g\bar{f}_1\mp d_2}{d_1}\right)\sideset{}{^*}\sum_{\substack{f\,(d_1^2d_2h)\\(g\mp d_2f,d_1^2d_2h)=d_1\\f\equiv f_1\,(d_1^2d_2h_1)}}e\left((m_1-m_2)\frac{f}{d_1h}\right).$$

With the notation

$$f =: f_1 + kd_1^2 d_2 h_1, \quad g_1 := \frac{g \mp d_2 f_1}{d_1},$$

this becomes

$$\sideset{}{^*}\sum_{\substack{f_1\,(d_1^2 d_2 h_1)\\ (g\mp d_2 f_1, d_1^2 d_2 h_1)=d_1}} \chi\left(\frac{g\bar{f}_1 \mp d_2}{d_1}\right) e\left((m_1-m_2)\frac{f_1}{d_1 h}\right) \sum_{\substack{k\,(h_2)\\ (f_1+kd_1^2 d_2 h_1, h_2)=1\\ (g_1 \mp k d_1 d_2^2 h_1, h_2)=1}} e\left((m_1-m_2)d_1 d_2 \frac{k}{h_2}\right).$$

The k-sum equals

$$\sum_{\substack{\ell_1 | h_2\\ \ell_2 | h_2}} \mu(\ell_1)\mu(\ell_2) \sum_{\substack{k\,(h_2)\\ kd_1^2 d_2 h_1 \equiv -f_1\,(\ell_1)\\ kd_1 d_2^2 h_1 \equiv \pm g_1\,(\ell_2)}} e\left((m_1-m_2)d_1 d_2 \frac{k}{h_2}\right).$$

It is straightforward to evaluate the inner sum explicitly, but for our purposes it suffices to record that it is at most $(h_2, (m_1-m_2)d_1 d_2)$, which concludes the proof. □

Lemma 7. *For $r \in \mathbb{R}$ and $\Re s > \frac{1}{2}$, let*

$$\Xi_r^{\pm}(s) := \sum_{\substack{(\delta,N)=1\\ (d_1,d_2)=1}} \frac{\chi(d_1)\overline{\chi(d_2)}}{\delta^{\lambda+\mu}(d_1 d_2)^{\lambda}} \sum_{\substack{gh\equiv 0\,(N)\\ (g,d_1 d_2)=1}} \frac{1}{g^{\mu}h^{\lambda}} \frac{\phi(gh)}{\phi(d_1 d_2 gh)} \sideset{}{^*}\sum_{\substack{f\,(d_1^2 d_2 h)\\ (gf\mp d_2, d_1^2 d_2 h)=d_1}} \chi\left(\frac{gf\mp d_2}{d_1}\right), \tag{60}$$

where χ is an even primitive Dirichlet character modulo N and

$$\lambda := 2 + 2s + 2ir, \quad \mu := 2s - 2ir.$$

Then

$$\Xi_r^{\pm}(s) = \zeta(2s-2ir)\zeta(1+2s+2ir)N^{-2s+2ir}\prod_{p|N}(1-p^{-2-4ir}). \tag{61}$$

Proof. By the remarks below (58), the innermost sum in (60) can be rewritten as

$$\sideset{}{^*}\sum_{\substack{f\,(d_1^2 d_2 h)\\ (gf\mp d_2, d_1^2 d_2 h)=d_1}} \chi\left(\frac{gf\mp d_2}{d_1}\right) = \sideset{}{^*}\sum_{\substack{d\,(d_1 d_2 gh)\\ (d_1 d \pm d_2, gh)=g}} \chi(d).$$

On the right-hand side, the condition on d only depends on $d \bmod gh$, therefore

$$\frac{\phi(gh)}{\phi(d_1 d_2 gh)} \sideset{}{^*}\sum_{\substack{f\,(d_1^2 d_2 h)\\ (gf\mp d_2, d_1^2 d_2 h)=d_1}} \chi\left(\frac{gf\mp d_2}{d_1}\right) = \sideset{}{^*}\sum_{\substack{d\,(gh)\\ (d_1 d\pm d_2, gh)=g}} \chi(d).$$

With the notation

$$S(\chi, g, h, d_1, d_2) := \chi(d_1)\overline{\chi(d_2)} \sideset{}{^*}\sum_{\substack{d\,(gh)\\ (d_1 d\pm d_2, gh)=g}} \chi(d),$$

we can now write

$$\Xi_r^{\pm}(s) = \sum_{(\delta,N)=1} \frac{1}{\delta^{\lambda+\mu}} \sum_{\substack{(d_1 d_2,N)=1\\ (d_1,d_2)=1}} \frac{1}{(d_1 d_2)^{\lambda}} \sum_{\substack{gh\equiv 0\,(N)\\ (g,d_1 d_2)=1}} \frac{1}{g^{\mu}h^{\lambda}} S(\chi, g, h, d_1, d_2).$$

The character sum $S(\chi, g, h, d_1, d_2)$ is multiplicative in χ, g, h in the following sense. If $g = g_1 g_2$ and $h = h_1 h_2$ are any decompositions such that $(g_1 h_1, g_2 h_2) = 1$, and correspondingly $\chi = \chi_1 \chi_2$ where χ_i is a Dirichlet character modulo $g_i h_i$, then

$$S(\chi, g, h, d_1, d_2) = S(\chi_1, g_1, h_1, d_1, d_2) S(\chi_2, g_2, h_2, d_1, d_2).$$

Indeed, this follows easily upon writing the summation variable $d \bmod gh$ as

$$d = \tilde{d}_1 e_1 + \tilde{d}_2 e_2, \quad \tilde{d}_i \bmod g_i h_i,$$

where e_1 and e_2 are fixed integers such that

$$\begin{aligned} e_1 &\equiv 1 \pmod{g_1 h_1}, \quad e_1 \equiv 0 \pmod{g_2 h_2}, \\ e_2 &\equiv 0 \pmod{g_1 h_1}, \quad e_2 \equiv 1 \pmod{g_2 h_2}. \end{aligned}$$

In other words, we have a decomposition over the primes

$$S(\chi, g, h, d_1, d_2) = \prod_p S(\chi_p, g_p, h_p, d_1, d_2),$$

where the subscript p denotes the p-part.

Let us fix p for a moment and use the notation

$$g_p = p^\gamma, \quad h_p = p^\delta, \quad N_p = p^\nu.$$

Note that χ_p is a primitive Dirichlet character modulo p^ν.

Case 1. If $\nu = 0$ (that is, $p \nmid N$), then χ_p is trivial, so that

$$S(\chi_p, g_p, h_p, d_1, d_2) = \sideset{}{^*}\sum_{\substack{d\,(p^{\gamma+\delta}) \\ (d_1 d \pm d_2, p^{\gamma+\delta}) = p^\gamma}} 1 = \begin{cases} 1, & \delta = 0, \\ p^\delta - p^{\delta-1}, & \delta > 0 \text{ and } p \mid d_1 d_2 p^\gamma, \\ p^\delta - 2p^{\delta-1}, & \delta > 0 \text{ and } p \nmid d_1 d_2 p^\gamma. \end{cases}$$

For $\delta = 0$ the right-hand side follows by observing that there is a unique d satisfying the condition, since $\gamma > 0$ implies $p \nmid d_1 d_2$. For $\delta > 0$ and $p \mid d_1 d_2$, the right-hand side follows by observing that $\gamma = 0$ and exactly one of d_1 and d_2 is divisible by p, hence the condition on d is automatically satisfied. For $\delta > 0$ and $\gamma > 0$, the right-hand side follows by observing that $p \nmid d_1 d_2$, so that the numbers d satisfying the condition are in bijection with the reduced residues modulo p^δ. For $\delta > 0$, $\gamma = 0$ and $p \nmid d_1 d_2$, the right-hand side follows by observing that the condition on d is $(d_1 d \pm d_2, p) = 1$ and there are precisely $p^{\delta-1}$ reduced residue classes modulo p^δ that do not have this property.

Case 2. If $\nu > 0$ (that is, $p \mid N$), then $p \nmid d_1 d_2$, $\gamma + \delta \geqslant \nu$ and χ_p induces a nontrivial character modulo $p^{\gamma+\delta}$ of conductor p^ν, so that

$$\begin{aligned} \chi_p(\mp 1) S(\chi_p, g_p, h_p, d_1, d_2) &= \sideset{}{^*}\sum_{\substack{d\,(p^{\gamma+\delta}) \\ (d_1 d \pm d_2, p^{\gamma+\delta}) = p^\gamma}} \chi_p(\mp \bar{d}_2 d_1 d) = \sideset{}{^*}\sum_{\substack{d\,(p^{\gamma+\delta}) \\ (d-1, p^{\gamma+\delta}) = p^\gamma}} \chi_p(d) \\ &= \begin{cases} 1, & \delta = 0 \text{ and } \gamma \geqslant \nu, \\ p^\delta - p^{\delta-1}, & \delta > 0 \text{ and } \gamma \geqslant \nu, \\ -p^{\delta-1}, & \delta > 0 \text{ and } \gamma = \nu - 1, \\ 0, & \delta > 0 \text{ and } \gamma < \nu - 1. \end{cases} \end{aligned}$$

For $\delta = 0$ and $\gamma \geqslant \nu$, the right-hand side follows by observing that there is a unique d satisfying the condition in the second sum, and this d is congruent to 1 modulo p^ν. For $\delta > 0$ and $\gamma \geqslant \nu$, the right-hand side follows by observing that the d's satisfying the condition in the second sum are in bijection with the reduced residues modulo p^δ, and all these numbers d are congruent to 1 modulo p^ν. For $\delta > 0$ and $\gamma = \nu - 1$, the right-hand side follows by observing that the condition on d only depends on $d \bmod p^\nu$, and hence, in this case,

$$\sideset{}{^*}\sum_{\substack{d\,(p^{\gamma+\delta})\\(d-1,p^{\gamma+\delta})=p^\gamma}} \chi_p(d) = p^{\delta-1} \sideset{}{^*}\sum_{\substack{d\,(p^{\nu})\\(d-1,p^{\nu})=p^{\nu-1}}} \chi_p(d),$$

and the sum on the right-hand side equals -1. Indeed, for $\nu = 1$ this is obvious, while, for $\nu > 1$, it follows from the fact that $u \mapsto \chi_p(p^{\nu-1}u+1)$ is a nontrivial additive character modulo p. Finally, for $\delta > 0$ and $\gamma < \nu - 1$, the right-hand side follows by observing that the second sum does not change when we multiply it with a complex number of the form $\chi_p(p^{\nu-1}u+1) \neq 1$.

Our findings imply that

$$S(\chi_p, g_p, h_p, d_1, d_2) = S(\chi_p, g_p, h_p, d_{1,p}, d_{2,p}),$$

hence we have an Euler product decomposition

$$\Xi_r^\pm(s) = \prod_p \Xi_{r,p}^\pm(s),$$

where, for $p \nmid N$,

$$\Xi_{r,p}^\pm(s) := \frac{1}{1-p^{-\lambda-\mu}} \sum_{\substack{\alpha,\beta \geqslant 0\\ \min(\alpha,\beta)=0}} \frac{1}{p^{(\alpha+\beta)\lambda}} \sum_{\substack{\gamma,\delta \geqslant 0\\ \min(\gamma,\alpha+\beta)=0}} \frac{1}{p^{\gamma\mu+\delta\lambda}} S(\chi_p, p^\gamma, p^\delta, p^\alpha, p^\beta),$$

while, for $p \mid N$,

$$\Xi_{r,p}^\pm(s) := \chi_p(\mp 1) \sum_{\substack{\gamma,\delta \geqslant 0\\ \gamma+\delta \geqslant \nu}} \frac{1}{p^{\gamma\mu+\delta\lambda}} S(\chi_p, p^\gamma, p^\delta, 1, 1).$$

Now we insert the above calculated values of $S(\chi_p, p^\gamma, p^\delta, p^\alpha, p^\beta)$.

Case 1. If $p \nmid N$, then we obtain

$$\begin{aligned}(1-p^{-\lambda-\mu})\Xi_{r,p}^\pm(s) &= \sum_{\gamma=0}^\infty \frac{1}{p^{\gamma\mu}} + \frac{p-2}{p} \sum_{\delta=1}^\infty \frac{1}{p^{\delta(\lambda-1)}} + \frac{p-1}{p} \sum_{\gamma=1}^\infty \sum_{\delta=1}^\infty \frac{1}{p^{\gamma\mu+\delta(\lambda-1)}} \\ &\quad + \left(\sum_{\alpha=1}^\infty \frac{1}{p^{\alpha\lambda}} + \sum_{\beta=1}^\infty \frac{1}{p^{\beta\lambda}} \right) \left(1 + \frac{p-1}{p} \sum_{\delta=1}^\infty \frac{1}{p^{\delta(\lambda-1)}} \right).\end{aligned}$$

Indeed, the first line contains the contribution of $\alpha = \beta = 0$, and the second line contains the rest (where γ must be zero). We sum all the geometric series:

$$\begin{aligned}&(1-p^{-\lambda-\mu})\Xi_{r,p}^\pm(s) \\ &\quad = \frac{1}{1-p^{-\mu}} + \frac{p-2}{p^\lambda(1-p^{1-\lambda})} + \frac{p-1}{p^{\lambda+\mu}(1-p^{1-\lambda})(1-p^{-\mu})} \\ &\qquad + \frac{2}{p^\lambda(1-p^{-\lambda})} \left(1 + \frac{p-1}{p^\lambda(1-p^{1-\lambda})} \right) \\ &\quad = \frac{(1-p^{1-\lambda}) + p^{-\lambda}(p-2)(1-p^{-\mu}) + p^{-\lambda-\mu}(p-1)}{(1-p^{1-\lambda})(1-p^{-\mu})} + \frac{2p^{-\lambda}}{1-p^{1-\lambda}} \\ &\quad = \frac{1-2p^{-\lambda}+p^{-\lambda-\mu}}{(1-p^{1-\lambda})(1-p^{-\mu})} + \frac{2p^{-\lambda}-2p^{-\lambda-\mu}}{(1-p^{1-\lambda})(1-p^{-\mu})} \\ &\quad = \frac{1-p^{-\lambda-\mu}}{(1-p^{1-\lambda})(1-p^{-\mu})}.\end{aligned}$$

Hence, for $p \nmid N$ we have

$$\Xi_{r,p}^\pm(s) = \frac{1}{(1-p^{1-\lambda})(1-p^{-\mu})}.$$

Case 2. If $p \mid N$, then we obtain, for $p^\nu \parallel N$,

$$\chi_p(\mp 1)\Xi^{\pm}_{r,p}(s) = \sum_{\gamma=\nu}^{\infty} \frac{1}{p^{\gamma\mu}} - \frac{1}{p^{1+(\nu-1)\mu}} \sum_{\delta=1}^{\infty} \frac{1}{p^{\delta(\lambda-1)}} + \frac{p-1}{p} \sum_{\gamma=\nu}^{\infty}\sum_{\delta=1}^{\infty} \frac{1}{p^{\gamma\mu+\delta(\lambda-1)}}.$$

We sum all the geometric series:

$$\begin{aligned}\chi_p(\mp 1)\Xi^{\pm}_{r,p}(s) &= \frac{1}{p^{\nu\mu}(1-p^{-\mu})} - \frac{1}{p^{\lambda+(\nu-1)\mu}(1-p^{1-\lambda})} + \frac{p-1}{p^{\lambda+\nu\mu}(1-p^{1-\lambda})(1-p^{-\mu})}\\ &= \frac{(1-p^{1-\lambda}) - p^{-\lambda+\mu}(1-p^{-\mu}) + p^{-\lambda}(p-1)}{p^{\nu\mu}(1-p^{1-\lambda})(1-p^{-\mu})}\\ &= \frac{1-p^{-\lambda+\mu}}{p^{\nu\mu}(1-p^{1-\lambda})(1-p^{-\mu})}.\end{aligned}$$

Hence, for $p \mid N$ we have

$$\Xi^{\pm}_{r,p}(s) = \chi_p(\mp 1)p^{-\nu\mu}\frac{1-p^{-\lambda+\mu}}{(1-p^{1-\lambda})(1-p^{-\mu})}.$$

Collecting the above results, we arrive at (61). □

4.3. *Applying Voronoi summation*

We substitute (59) back into (54) getting

$$\begin{aligned} TK \sum_{\substack{M_1,M_2,G\geqslant 1\\ \text{powers of } 2}} \sum_{\pm} \frac{1\pm i}{2} \sum_{\substack{(\delta,N)=1\\ (d_1,d_2)=1}} \frac{\chi(d_1)\overline{\chi(d_2)}}{\delta^2 d_1 d_2} \sum_{N\mid c} \frac{\phi(c)}{\phi(d_1d_2c)}\frac{\omega(c/G)}{c^{1/2}}&\\ \times \sum_{\substack{gh=c\\ (g,d_1d_2)=1}} \sideset{}{^*}\sum_{\substack{f\ (d_1^2d_2h)\\ (g\bar f \mp d_2, d_1^2d_2h)=d_1}} \chi\left(\frac{g\bar f\mp d_2}{d_1}\right)\mathcal{S}^{M_1,M_2}_{d_1,d_2,c}(f,g,h),& \qquad (62)\end{aligned}$$

where

$$\begin{aligned}\mathcal{S}^{M_1,M_2}_{d_1,d_2,c}(f,g,h) &:= \sum_{m_1,m_2} \lambda_g(m_1)\overline{\lambda_g(m_2)}e\left(\pm m_1 \frac{\overline{(g\mp d_2 f)/d_1}}{d_2h} + m_2\frac{\bar f}{d_1h}\right)F^{M_1,M_2}_{d_1,d_2,c}(m_1,m_2),\\ F^{M_1,M_2}_{d_1,d_2,c}(x,y) &:= \frac{\omega(x/M_1)\omega(y/M_2)}{(xy)^{3/4}}\Omega_3\left(T\log\frac{d_2^2y}{d_1^2x}, \frac{\pm K^2c}{8\pi\sqrt{xy}}, \frac{\delta^4d_1^2d_2^2xy}{\sqrt{\tilde C}}\right)\\ &\quad\times e\left(\mp\frac{(\sqrt{d_1^2x}-\sqrt{d_2^2y})^2}{d_1d_2c}\right).\end{aligned}$$

By applying Proposition 2 for the summation variables m_1 and m_2, we see that

$$d_1d_2h^2\mathcal{S}^{M_1,M_2}_{d_1,d_2,c}(f,g,h)$$

is a sum of terms (suppressing M_1 and M_2 from the notation for simplicity)

$$\sum_{m_1,m_2} \lambda_g(m_1)\overline{\lambda_g(m_2)}e\left(\mp m_1\frac{g}{d_1d_2h} \mp (m_2\mp m_1)\frac{f}{d_1h}\right)F^{\pm,\pm}_{d_1,d_2,c}(m_1,m_2) \qquad (63)$$

with

$$F^{\pm,\pm}_{d_1,d_2,c}(m_1,m_2) := \int_0^\infty\int_0^\infty F^{M_1,M_2}_{d_1,d_2,c}(x,y)J_g^{\pm}\left(\frac{4\pi\sqrt{m_1x}}{d_2h}\right)J_g^{\pm}\left(\frac{4\pi\sqrt{m_2y}}{d_1h}\right)dx\,dy. \qquad (64)$$

If $g = E_r$ is an Eisenstein series, then there are three additional polar terms

$$\sum_{m_1} \lambda_g(m_1) e\left(\mp m_1 \frac{g \mp d_2 f}{d_1 d_2 h}\right) \int_0^\infty F^{\pm,0}_{d_1,d_2,c}(m_1, y) P^{\pm}_{r,d_1 h}(y)\, dy, \tag{65}$$

$$\sum_{m_2} \overline{\lambda(m_2)} e\left(\mp m_2 \frac{f}{d_1 h}\right) \int_0^\infty F^{0,\pm}_{d_1,d_2,c}(x, m_2) P^{\pm}_{r,d_2 h}(x)\, dx, \tag{66}$$

$$\int_0^\infty \int_0^\infty F^{M_1,M_2}_{d_1,d_2,c}(x, y) P^{\pm}_{r,d_2 h}(x) P^{\pm}_{r,d_1 h}(y)\, dx\, dy, \tag{67}$$

with

$$F^{\pm,0}_{d_1,d_2,c}(m_1, y) := \int_0^\infty F^{M_1,M_2}_{d_1,d_2,c}(x, y) J^{\pm}_g\left(\frac{4\pi\sqrt{m_1 x}}{d_2 h}\right) dx, \tag{68}$$

$$F^{0,\pm}_{d_1,d_2,c}(x, m_2) := \int_0^\infty F^{M_1,M_2}_{d_1,d_2,c}(x, y) J^{\pm}_g\left(\frac{4\pi\sqrt{m_2 y}}{d_1 h}\right) dy, \tag{69}$$

$$P^{\pm}_{r,c}(t) := \begin{cases} \zeta(1 \pm 2ir)(t/c^2)^{\pm ir}, & \text{for } r \neq 0, \\ \log(t/c^2) + 2\gamma, & \text{for } r = 0. \end{cases} \tag{70}$$

We proceed to analyse the four terms (63), (65)–(67). It will turn out that the first three terms are small, but (67) contributes to the main term.

4.4. *The contribution of* (63)

We observe first that, in (64), the arguments of the Bessel functions are large:

$$\frac{4\pi\sqrt{m_1 x}}{d_2 h} \gg \frac{\sqrt{M_1}}{d_2 G} \gg \frac{\sqrt{M_1} K^2}{\tilde{C}^\varepsilon d_2 \sqrt{M_1 M_2}} \asymp \frac{K^2}{\tilde{C}^\varepsilon (d_1 d_2)^{1/2} (M_1 M_2)^{1/4}} \gg \frac{K^2}{\tilde{C}^\varepsilon (T + K)}$$

by (55), (57) and the fact that $h \leqslant c$. A similar estimate holds for $4\pi\sqrt{m_2 y}/(d_1 h)$. In view of (9) this is large. In particular, by the rapid decay of the Bessel K-function (32), among $F^{\pm,\pm}_{d_1,d_2,c}(m_1, m_2)$ we only need to consider $F^{+,+}_{d_1,d_2,c}(m_1, m_2)$ as the contribution of the other three expressions is negligible (or zero).

We show now that the sum (63) can be truncated efficiently. Fix any $y \in [(\frac{1}{2})M_2, (\frac{5}{2})M_2]$ in the integral defining $F^{+,+}_{d_1,d_2,c}(m_1, m_2)$. By (56), we can restrict the x-integration to

$$d_1^2 x = d_2^2 y (1 + O(T^{\varepsilon - 1})) \tag{71}$$

at the cost of a negligible error. In this range, we have

$$\frac{\partial^j}{\partial x^j} F_{d_1,d_2,c}(x, y) \ll_j \tilde{C}^\varepsilon \left(\frac{T}{M_1} + \frac{d_1}{T d_2 c}\right)^j,$$

so that, by Lemma 3, the integral is negligible unless

$$\left(\frac{T}{M_1} + \frac{d_1}{T d_2 c}\right) \frac{\sqrt{M_1} d_2 h}{\sqrt{m_1}} \geqslant \tilde{C}^{-\varepsilon},$$

that is,

$$m_1 \leqslant M_1^* := \tilde{C}^\varepsilon d_2^2 \left(\frac{(Th)^2}{M_1} + \frac{M_2}{(Tg)^2}\right). \tag{72}$$

Here we used (57) and the fact that $gh = c$. Similarly, we can assume

$$m_2 \leqslant M_2^* := \tilde{C}^\varepsilon d_1^2 \left(\frac{(Th)^2}{M_2} + \frac{M_1}{(Tg)^2} \right). \tag{73}$$

For convenience we observe, by (57),

$$M^* := \max(M_1^*, M_2^*) \asymp \tilde{C}^\varepsilon d_1 d_2 \left(\frac{(Th)^2}{(M_1 M_2)^{1/2}} + \frac{(M_1 M_2)^{1/2}}{(Tg)^2} \right). \tag{74}$$

To summarize,

$$\begin{aligned} \mathcal{S}_{d_1,d_2,c}^{M_1,M_2}(f,g,h) = \frac{1}{d_1 d_2 h^2} \sum_{m_1 \leqslant M^*} \lambda_g(m_1) e\left(\mp \frac{m_1 g}{d_1 d_2 h} \right) \sum_{m_2 \leqslant M^*} \overline{\lambda_g(m_2)} e\left((m_1 - m_2) \frac{f}{d_1 h} \right) \\ \times \int_0^\infty \int_0^\infty F_{d_1,d_2,c}^{M_1,M_2}(x,y) J_g^+ \left(\frac{4\pi\sqrt{m_1 x}}{d_2 h} \right) J_g^+ \left(\frac{4\pi\sqrt{m_2 y}}{d_1 h} \right) dx\, dy, \end{aligned} \tag{75}$$

up to negligible error terms and the contribution of the three polar terms (65)–(67) that we discuss in a moment.

If we substitute this back into (62), then the summation over f produces the exponential sum

$$\sideset{}{^*}\sum_{\substack{f\ (d_1^2 d_2 h) \\ (g \mp d_2 f, d_1^2 d_2 h) = d_1}} \chi\left(\frac{g\bar{f} \mp d_2}{d_1} \right) e\left((m_1 - m_2) \frac{f}{d_1 h} \right).$$

We estimate the double integral in (75) using (31), (55) and (71) as

$$\ll (TM_1M_2)^{\varepsilon - 1} \min\left(\frac{d_1^2 M_1^2}{d_2^2}, \frac{d_2^2 M_2^2}{d_1^2} \right) \frac{(d_1 d_2)^{1/2} h}{(m_1 m_2)^{1/4}} \ll \tilde{C}^\varepsilon \frac{(d_1 d_2)^{1/2} h}{T (m_1 m_2)^{1/4}}.$$

By Lemma 6 and the support of ω, the contribution of (63) to (62) is

$$\begin{aligned} &\ll TK\tilde{C}^\varepsilon \sum_{\delta^2 d_1 d_2 \leqslant \tilde{C}^{1/4+\varepsilon}} \frac{1}{\delta^2 (d_1 d_2)^{3/2}} \sum_{\substack{gh \leqslant 3G \\ m_1, m_2 \leqslant M^*}} \frac{(h, (m_1 - m_2) d_1 d_2)}{(gh)^{1/2} hT} \frac{|\lambda_g(m_1) \lambda_g(m_2)|}{(m_1 m_2)^{1/4}} \\ &\ll TK\tilde{C}^\varepsilon \sum_{\delta^2 d_1 d_2 \leqslant \tilde{C}^{1/4+\varepsilon}} \frac{1}{\delta^2 (d_1 d_2)^{3/2}} \sum_{gh \leqslant 3G} \frac{1}{(gh)^{1/2} hT} \sum_{\ell \mid h} \ell \sum_{\substack{m_1, m_2 \leqslant M^* \\ \ell \mid (m_1 - m_2) d_1 d_2}} \frac{|\lambda_g(m_1) \lambda_g(m_2)|}{(m_1 m_2)^{1/4}}. \end{aligned}$$

By (33) the innermost sum is

$$\leqslant \sum_{\substack{m_1, m_2 \leqslant M^* \\ \ell \mid (m_1 - m_2) d_1 d_2}} \frac{1}{2} \left(\frac{|\lambda_g(m_1)|^2}{m_1^{1/2}} + \frac{|\lambda_g(m_2)|^2}{m_2^{1/2}} \right) \ll (M^*)^{1/2+\varepsilon} \left(1 + \frac{M^* (\ell, d_1 d_2)}{\ell} \right),$$

and hence in the end the contribution of (63) to (62) is, using also (55) and (74),

$$
\begin{aligned}
&\ll TK\tilde{C}^{\varepsilon} \sum_{\delta^2 d_1 d_2 \leqslant \tilde{C}^{1/4+\varepsilon}} \frac{1}{\delta^2 (d_1 d_2)^{3/2}} \sum_{gh \leqslant 3G} \frac{1}{(gh)^{1/2} hT} \left(h(M^*)^{1/2} + (h, d_1 d_2)(M^*)^{3/2} \right) \\
&\ll TK\tilde{C}^{\varepsilon} \sum_{\delta^2 d_1 d_2 \leqslant \tilde{C}^{1/4+\varepsilon}} \frac{1}{\delta^2} \left(\frac{G^{3/2}}{(d_1 d_2)(M_1 M_2)^{1/4}} + \frac{G^{1/2}(M_1 M_2)^{1/4}}{(d_1 d_2) T^2} \right. \\
&\quad \left. + \frac{T^2 G^{5/2}}{(M_1 M_2)^{3/4}} + \frac{(M_1 M_2)^{3/4}}{T^4} \right) \\
&\ll TK\tilde{C}^{\varepsilon} \sum_{\delta^2 d_1 d_2 \leqslant \tilde{C}^{1/4+\varepsilon}} \frac{1}{\delta^2} \left(\frac{(M_1 M_2)^{1/2}}{(d_1 d_2) K^3} + \frac{(M_1 M_2)^{1/2}}{(d_1 d_2) T^2 K} + \frac{T^2 (M_1 M_2)^{1/2}}{K^5} + \frac{(M_1 M_2)^{3/4}}{T^4} \right) \\
&\ll TK\tilde{C}^{\varepsilon} \sum_{\delta^2 d_1 d_2 \leqslant \tilde{C}^{1/4+\varepsilon}} \frac{1}{\delta^2 d_1 d_2} \left(\frac{\tilde{C}^{1/4}}{K^3} + \frac{\tilde{C}^{1/4}}{T^2 K} + \frac{T^2 \tilde{C}^{1/4}}{K^5} + \frac{\tilde{C}^{3/8}}{T^4} \right) \\
&\ll (TK)^{1+\varepsilon} \left(\frac{T^4}{K^5} + \frac{K^3}{T^4} \right).
\end{aligned} \tag{76}
$$

4.5. *The contribution of* (65) *and* (66)

We show that the integrals (68) and (69) are negligible in the ranges (72) and (73). Starting with the first, we see by the rapid decay of Ω_3 in the definition of $F_{d_1,d_2,c}(x,y)$ that the contribution of $|d_1^2 x - d_2^2 y| \geqslant T^{\varepsilon-1} d_1^2 M_1$ is negligible. Introducing $w := (\sqrt{d_1^2 x} - \sqrt{d_2^2 y})^2$ and applying several integrations by parts with respect to w shows that the contribution of $\sqrt{w d_1^2 x} \geqslant d_1 d_2 c T^{1+\varepsilon}$ is also negligible. Using $|d_1^2 x - d_2^2 y| \ll \sqrt{w d_1^2 x}$ we infer that we can restrict the integration to

$$
|d_1^2 x - d_2^2 y| \leqslant Z := \tilde{C}^{\varepsilon} \min\left(\frac{d_1^2 M_1}{T}, d_1 d_2 c T \right)
$$

at the cost of a negligible error. In other words, writing $z := d_1^2 x - d_2^2 y$ we can approximate the above integral by

$$
-\frac{1}{d_2^2} \int_{-Z}^{Z} \int_0^{\infty} F_{d_1,d_2,c}\left(x, \frac{d_1^2 x - z}{d_2^2} \right) P^{\pm}_{r,d_1 h}\left(\frac{d_1^2 x - z}{d_2^2} \right) J_g^{\pm}\left(\frac{4\pi\sqrt{m_1 x}}{d_2 h} \right) dx\, dz
$$

with negligible error. For $|z| \leqslant Z$ we have

$$
\begin{aligned}
\frac{\partial^j}{\partial x^j} \left\{ F_{d_1,d_2,c}\left(x, \frac{d_1^2 x - z}{d_2^2} \right) P^{\pm}_{r,d_1 h}\left(\frac{d_1^2 x - z}{d_2^2} \right) \right\} &\ll_j \tilde{C}^{\varepsilon} \left(\frac{K|z|}{d_1^2 M_1^2} + \frac{1}{M_1} + \frac{z^2}{(d_1^2 M_1^2)(d_1 d_2 c)} \right)^j \\
&\ll_j \tilde{C}^{\varepsilon} M_1^{-j},
\end{aligned}
$$

whence, by Lemma 3 and (55), the last integral is

$$
\begin{aligned}
&\ll_j \tilde{C}^{\varepsilon} Z \left(\frac{d_2 h}{\sqrt{M_1}} \right)^j \ll_j \tilde{C}^{\varepsilon} Z \left(\frac{(d_1 d_1)^{1/2} G}{(M_1 M_2)^{1/4}} \right)^j \ll_j \tilde{C}^{\varepsilon} Z \left(\frac{(d_1 d_1)^{1/2} (M_1 M_2)^{1/4}}{K^2} \right)^j \\
&\ll_j \tilde{C}^{\varepsilon} Z \left(\frac{\tilde{C}^{1/8}}{K^2} \right)^j \ll \tilde{C}^{\varepsilon} Z \left(\frac{T+K}{K^2} \right)^j.
\end{aligned}
$$

Choosing $j \in \mathbb{N}$ sufficiently large and using (9), (55), (72), we conclude that the polar term (65) is negligible. In the same way, we see that the polar term (66) is negligible.

4.6. *The final polar term*

It remains to estimate the contribution of the polar term (67). This requires some nontrivial manipulation. First, we remove the dyadic decompositions by summing over M_1, M_2, G using (53), but we keep in mind that the decay of Ω_3 allows us to restrict to $\delta, d_1, d_2, c \leqslant K^\varepsilon$, up to a negligible error. This gives

$$\begin{aligned} TK \sum_{\pm} \frac{1 \pm i}{2} & \sum_{\substack{(\delta, N)=1 \\ (d_1, d_2)=1}} \frac{\chi(d_1)\overline{\chi(d_2)}}{\delta^2 d_1 d_2} \sum_{N \mid c} \frac{\phi(c)}{\phi(d_1 d_2 c)} \frac{1}{c^{1/2}} \\ \times & \sum_{\substack{gh=c \\ (g, d_1 d_2)=1}} \sideset{}{^*}\sum_{\substack{f\ (d_1^2 d_2 h) \\ (g\bar{f} \mp d_2, d_1^2 d_2 h)=d_1}} \frac{1}{d_1 d_2 h^2} \chi\left(\frac{g\bar{f} \mp d_2}{d_1}\right) \\ \times & \int_0^\infty \int_0^\infty \Omega_3\left(T \log\frac{d_2^2 y}{d_1^2 x}, \frac{\pm K^2 c}{8\pi\sqrt{xy}}, \frac{\delta^4 d_1^2 d_2^2 xy}{\sqrt{\tilde{C}}}\right) e\left(\mp\frac{(\sqrt{d_1^2 x} - \sqrt{d_2^2 y})^2}{d_1 d_2 c}\right) \\ \times & \sum_{\pm,\pm} P^{\pm}_{r, d_2 h}(x) P^{\pm}_{r, d_1 h}(y) \frac{dx\, dy}{(xy)^{3/4}}, \end{aligned} \tag{77}$$

up to a negligible error coming from the fact that $\sum_{j \geqslant 0} \omega(x/2^j)$ is not necessarily 1 for $x < 1$. In order to simplify the notation, we shall only consider the $(+,+)$-term in the last sum and drop the superscripts at P.

We observe that the δ, d_1, d_2-sum is rapidly converging due to the decay properties of Ω_3. A trivial estimation shows that the double integral is $O(\tilde{C}^\varepsilon)$. Now we replace Ω_3 by $\tilde{\Omega}_3$, which introduces by (52) an admissible error of

$$O((TK)^{1+\varepsilon}(T+K)^{-1}). \tag{78}$$

Next we make a change of variables

$$z := xy, \quad w := \frac{d_2^2 y}{d_1^2 x} - 1.$$

By Taylor's formula,

$$T \log\frac{d_2^2 y}{d_1^2 x} = Tw + O(Tw^2) = Tw + O(T^{\varepsilon - 1})$$

in the range where $\tilde{\Omega}_3$ is not negligible. Again, by Taylor's formula,

$$\left(\sqrt{d_1^2 x} - \sqrt{d_2^2 y}\right)^2 = \frac{1}{4} d_1 d_2 \sqrt{z} w^2 (1 + O(w)),$$

hence

$$e\left(\mp\frac{(\sqrt{d_1^2 x} - \sqrt{d_2^2 y})^2}{d_1 d_2 c}\right) = e\left(\mp\frac{\sqrt{z} w^2}{4c}\right)(1 + O(T^{\varepsilon - 1}))$$

in the range where $\tilde{\Omega}_3$ is not negligible. Therefore, the double integral equals

$$\begin{aligned}&\int_{-1}^{\infty}\int_{0}^{\infty}\tilde{\Omega}_3\left(Tw,\frac{\pm K^2c}{8\pi\sqrt{z}},\frac{\delta^4d_1^2d_2^2z}{\sqrt{\tilde{C}}}\right)e\left(\mp\frac{\sqrt{z}w^2}{4c}\right)\\&\qquad\times P_{r,d_2h}\left(\frac{d_2\sqrt{z}}{d_1\sqrt{1+w}}\right)P_{r,d_1h}\left(\frac{d_1\sqrt{z(1+w)}}{d_2}\right)\frac{dw\,dz}{2(1+w)z^{3/4}}+O(T^{\varepsilon-1})\\&=\int_{-\infty}^{\infty}\int_{0}^{\infty}\tilde{\Omega}_3\left(Tw,\frac{\pm K^2c}{8\pi\sqrt{z}},\frac{\delta^4d_1^2d_2^2z}{\sqrt{\tilde{C}}}\right)e\left(\mp\frac{\sqrt{z}w^2}{4c}\right)\\&\qquad\times P_{r,d_2h}\left(\frac{d_2\sqrt{z}}{d_1}\right)P_{r,d_1h}\left(\frac{d_1\sqrt{z}}{d_2}\right)dw\,\frac{dz}{2z^{3/4}}+O(T^{\varepsilon-1}).\end{aligned}\tag{79}$$

We rewrite the last w-integral using the definition (51):

$$I:=\int_{-\infty}^{\infty}\int_{0}^{\infty}\tilde{\Omega}_2\left(x,\frac{\pm K^2c}{8\pi\sqrt{z}},\frac{\delta^4d_1^2d_2^2z}{\sqrt{\tilde{C}}}\right)\exp\left(iTwx\mp\frac{i\pi\sqrt{z}w^2}{2c}\right)dx\,dw.\tag{80}$$

This double integral is not absolutely convergent, so we consider

$$I_\varepsilon:=\int_{-\infty}^{\infty}\int_{0}^{\infty}\tilde{\Omega}_2\left(x,\frac{\pm K^2c}{8\pi\sqrt{z}},\frac{\delta^4d_1^2d_2^2z}{\sqrt{\tilde{C}}}\right)\exp\left(iTwx-\frac{e^{\pm i(\pi/2-\varepsilon)}\pi\sqrt{z}w^2}{2c}\right)dx\,dw\tag{81}$$

for $\varepsilon>0$. The x-integral in (80) and (81) is a Fourier transform decaying rapidly in w, and therefore, by cutting the w-integral at larger and larger parameters, we see that $\limsup_{\varepsilon\to0+}|I-I_\varepsilon|$ is smaller than any positive number, that is, $I=\lim_{\varepsilon\to0+}I_\varepsilon$. As the double integral (81) is absolutely convergent, we can change the order of integration there and compute the w-integral using [**11**, 3.323.2]:

$$I_\varepsilon=e^{\mp i(\pi/4-\varepsilon/2)}\left(\frac{2c}{\sqrt{z}}\right)^{1/2}\int_{0}^{\infty}\tilde{\Omega}_2\left(x,\frac{\pm K^2c}{8\pi\sqrt{z}},\frac{\delta^4d_1^2d_2^2z}{\sqrt{\tilde{C}}}\right)\exp\left(-\frac{e^{\mp i(\pi/2-\varepsilon)}cT^2x^2}{2\pi\sqrt{z}}\right)dx.$$

Here the integrand is rapidly decaying in x, and hence, by a limsup argument as before, we see that

$$I=\lim_{\varepsilon\to0+}I_\varepsilon=e^{\mp i(\pi/4)}\left(\frac{2c}{\sqrt{z}}\right)^{1/2}\int_{0}^{\infty}\tilde{\Omega}_2\left(x,\frac{\pm K^2c}{8\pi\sqrt{z}},\frac{\delta^4d_1^2d_2^2z}{\sqrt{\tilde{C}}}\right)\exp\left(\frac{\pm icT^2x^2}{2\pi\sqrt{z}}\right)dx.$$

Using also the definition (49), we can summarize that the w-integral in (79) equals

$$(1\mp i)\left(\frac{c}{\sqrt{z}}\right)^{1/2}\int_{0}^{\infty}\int_{0}^{\infty}\tilde{\Omega}_1\left(x,y,\frac{\delta^4d_1^2d_2^2z}{\sqrt{\tilde{C}}}\right)\exp\left(\pm\frac{ic}{2\pi\sqrt{z}}\left(x^2T^2+\frac{y^2K^2}{4}\right)\right)dy\,dx.$$

We integrate this over z and substitute it back into (77), getting

$$\begin{aligned}&TK\sum_{\pm}\sum_{\substack{(\delta,N)=1\\(d_1,d_2)=1}}\frac{\chi(d_1)\overline{\chi(d_2)}}{(\delta d_1d_2)^2}\sum_{N|c}\frac{\phi(c)}{\phi(d_1d_2c)}\sum_{\substack{gh=c\\(g,d_1d_2)=1}}\frac{1}{h^2}\sideset{}{^*}\sum_{\substack{f\ (d_1^2d_2h)\\(g\bar{f}\mp d_2,d_1^2d_2h)=d_1}}\chi\left(\frac{g\bar{f}\mp d_2}{d_1}\right)\\&\times\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\tilde{\Omega}_1\left(x,y,\frac{\delta^4d_1^2d_2^2z}{\sqrt{\tilde{C}}}\right)\exp\left(\pm\frac{ic}{2\pi\sqrt{z}}\left(x^2T^2+\frac{y^2K^2}{4}\right)\right)\\&\times P_{r,d_2h}\left(\frac{d_2\sqrt{z}}{d_1}\right)P_{r,d_1h}\left(\frac{d_1\sqrt{z}}{d_2}\right)dy\,dx\frac{dz}{2z}.\end{aligned}$$

Let us consider the case $r \neq 0$ and insert the + case of (70). Then we can recast the preceding display as

$$\zeta(1+2ir)^2 TK \sum_{\pm} \sum_{\substack{(\delta,N)=1\\(d_1,d_2)=1}} \frac{\chi(d_1)\overline{\chi(d_2)}}{(\delta d_1 d_2)^2} \sum_{N\mid c} \frac{\phi(c)}{\phi(d_1 d_2 c)}$$
$$\times \sum_{\substack{gh=c\\(g,d_1d_2)=1}} \frac{1}{h^2} \sideset{}{^*}\sum_{\substack{f\,(d_1^2 d_2 h)\\(g\bar f \mp d_2, d_1^2 d_2 h)=d_1}} \chi\left(\frac{g\bar f \mp d_2}{d_1}\right)$$
$$\times \int_0^\infty\int_0^\infty\int_0^\infty \tilde\Omega_1\left(x,y,\frac{\delta^4 d_1^2 d_2^2 z}{\sqrt{\tilde C}}\right) \exp\left(\pm \frac{ic}{2\pi\sqrt{z}}\left(x^2T^2+\frac{y^2K^2}{4}\right)\right)$$
$$\times \left(\frac{\sqrt{z}}{d_1 d_2 h^2}\right)^{2ir} dy\, dx\, \frac{dz}{2z}. \tag{82}$$

We make a change of variables

$$\frac{c}{\pi\sqrt{z}} = v, \quad z = \frac{c^2}{\pi^2 v^2},$$

and write the triple integral as

$$\int_0^\infty\int_0^\infty\int_0^\infty \tilde\Omega_1\left(x,y,\frac{\delta^4 d_1^2 d_2^2 c^2}{\pi^2 v^2\sqrt{\tilde C}}\right) \exp\left(\pm\frac{1}{2} iv\left((xT)^2+\left(\frac{1}{2}yK\right)^2\right)\right)$$
$$\times \left(\frac{c}{\pi v d_1 d_2 h^2}\right)^{2ir} dy\, dx \frac{dv}{v}.$$

Finally, we insert the definition (48) and arrive at

$$\int_0^\infty\int_0^\infty\int_0^\infty W_1(x)W_2(y)\frac{1}{2\pi i}\int_{(2)} \check G_0(s)\left(\frac{4\pi\delta^2 d_1 d_2 c}{vN\left((xT)^2+((1/2)yK)^2\right)}\right)^{-2s} ds$$
$$\times \exp\left(\pm\frac{1}{2} iv\left((xT)^2+\left(\frac{1}{2}yK\right)^2\right)\right)\left(\frac{c}{\pi v d_1 d_2 h^2}\right)^{2ir} dy\, dx\, \frac{dv}{v}$$

by Mellin inversion. We compute the v-integral by Gradshteyn and Ryzhik [**11**, 3.381.4]; the change of the order of integration can be justified similarly as before. Reorganizing, we obtain

$$\left(\frac{c}{2\pi d_1 d_2 h^2}\right)^{2ir} \int_0^\infty\int_0^\infty W_1(x)W_2(y)\left((xT)^2+\left(\frac{1}{2}yK\right)^2\right)^{2ir} dy\, dx$$
$$\times \frac{1}{2\pi i}\int_{(2)} \check G_0(s)\left(\frac{N}{2\pi\delta^2 d_1 d_2 c}\right)^{2s}\Gamma(2s-2ir)\exp(\pm i\pi(s-ir))\, ds,$$

where, by (8), the x,y-integral is simply

$$\mathcal{M}_{ir}(T,K)\left(\frac{(2\pi)^4}{N^2}\right)^{ir}.$$

We substitute this into (82), recall the definition (60) and recast (82) as

$$\zeta(1+2ir)^2 TK\mathcal{M}_{ir}(T,K)\frac{1}{2\pi i}\int_{(2)} \check G_0(s)\Xi_r(s)N^{2s-2ir}2(2\pi)^{-2s+2ir}\Gamma(2s-2ir)$$
$$\times\cos(\pi(s-ir))\, ds,$$

where $\Xi_r(s) = \Xi_r^{\pm}(s)$ denotes the function in (61).

To summarize, the contribution of (67) equals, up to an admissible error,

$$\zeta(1+2ir)^2\prod_{p\mid N}(1-p^{-2-4ir})TK\mathcal{M}_{ir}(T,K)\frac{1}{2\pi i}\int_{(2)}\check{G}_0(s)Z_r(s)\,ds \tag{83}$$

with the kernel

$$Z_r(s):=2(2\pi)^{-2s+2ir}\Gamma(2s-2ir)\cos(\pi(s-ir))\zeta(2s-2ir)\zeta(1+2s+2ir).$$

Using [**11**, 8.334.2 and 8.335.1], the kernel equals

$$Z_r(s)=\pi^{1/2-2s+2ir}\frac{\Gamma(s-ir)}{\Gamma(1/2-s+ir)}\zeta(2s-2ir)\zeta(1+2s+2ir),$$

and hence, by the functional equation for the Riemann zeta function, we obtain

$$Z_r(s)=\zeta(1-2s+2ir)\zeta(1+2s+2ir).$$

Let us assume $r\neq 0$; then the integrand in (83) is an odd function of s that is holomorphic except for a simple pole at $s=0$ as well as possible poles at $s=\pm ir$. It follows that the s-integral equals half the sum of its residues, that is, (83) equals

$$\begin{aligned}&\frac{1}{2}\zeta(1+2ir)^4\prod_{p\mid N}(1-p^{-2-4ir})TK\mathcal{M}_{ir}(T,K)\\&-\frac{1}{2}\zeta(1+2ir)^2\zeta(1+4ir)\prod_{p\mid N}(1-p^{-2-4ir})\check{G}_0(ir)TK\mathcal{M}_{ir}(T,K).\end{aligned} \tag{84}$$

Here we also used that $\check{G}_0(-ir)=-\check{G}_0(ir)$. The $(-,-)$ case in the last sum of (77) gives a similar contribution to $\mathcal{M}_{-ir}(T,K)$, so that the second line of (84) together with the corresponding part of the $(-,-)$ case precisely cancels (46). The $(+,-)$ and $(-,+)$ cases contribute to $\mathcal{L}_0$.

The case $r=0$ can be treated similarly and gives a linear combination of $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$ in Theorem 3.

This completes the discussion of the off-diagonal term. The various error terms (50), (76), (78), (79) encountered so far are admissible for Theorems 1–3, while the first line of (84) and its counterpart with ir replaced by $-ir$ have the desired shape for Theorem 2. The proofs are complete.

Appendix

Here, we deduce Lemma 1 from the modularity of $g=E_r$ $(r\neq 0)$, in order to emphasize the analogy with the cuspidal case. We follow closely [**13**, Section 2.4].

By Iwaniec [**18**, (3.29)], we have the Fourier decomposition

$$\begin{aligned}\theta\left(\frac{1}{2}+ir\right)g(x+iy)&=\theta\left(\frac{1}{2}+ir\right)y^{1/2+ir}+\theta\left(\frac{1}{2}-ir\right)y^{1/2-ir}\\&\quad+4\sqrt{y}\sum_{n=1}^{\infty}\lambda_g(n)K_{ir}(2\pi ny)\cos(2\pi nx),\end{aligned}$$

where

$$\theta(z):=\pi^{-z}\Gamma(z)\zeta(2z)\quad\text{and}\quad\lambda_g(n):=\sum_{ab=n}\left(\frac{a}{b}\right)^{ir}. \tag{A.1}$$

For convenience we introduce

$$D^{\pm1}(g,x,s):=\tfrac{1}{2}D(g,x,s)\pm\tfrac{1}{2}D(g,-x,s),$$

that is,

$$D^{+1}(g,x,s)=\sum_{n=1}^{\infty}\lambda_g(n)\cos(2\pi nx)n^{-s},$$
$$D^{-1}(g,x,s)=\sum_{n=1}^{\infty}\lambda_g(n)i\sin(2\pi nx)n^{-s}.$$

It will be more pleasant to work with the Maass shift (cf. [7, (4.3)])

$$\tilde{g}(x+iy):=-y\left(i\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\theta\left(\frac{1}{2}+ir\right)g(x+iy),$$

which is a weight 2 Eisenstein series of Laplacian eigenvalue $\frac{1}{4}+r^2$. By Duke, Friedlander and Iwaniec [7, Section 4], we have the Fourier decomposition

$$\tilde{g}(x+iy)=\tilde{g}_{\mathrm{const}}(y)+\tilde{g}_{\mathrm{ser}}(x+iy),$$

where

$$\tilde{g}_{\mathrm{const}}(y):=-\left(\frac{1}{2}+ir\right)\theta\left(\frac{1}{2}+ir\right)y^{1/2+ir}-\left(\frac{1}{2}-ir\right)\theta\left(\frac{1}{2}-ir\right)y^{1/2-ir},\tag{A.2}$$

$$\tilde{g}_{\mathrm{ser}}(x+iy):=\sum_{n=1}^{\infty}\frac{\lambda_g(n)}{\sqrt{n}}\left\{V_{2,ir}^{+1}(\pi ny)\cos(2\pi nx)+V_{2,ir}^{-1}(\pi ny)i\sin(2\pi nx)\right\},\tag{A.3}$$

and $V_{2,ir}^{\pm 1}$ is as in [7, (8.27)]:

$$V_{2,ir}^{\pm 1}(y):=W_{1,ir}(4y)\mp(\tfrac{1}{4}+r^2)W_{-1,ir}(4y).$$

Utilizing the functional equation

$$\tilde{g}\left(\frac{a}{c}+\frac{iy}{c}\right)=-\tilde{g}\left(-\frac{\bar{a}}{c}+\frac{i}{cy}\right),\quad y>0,\tag{A.4}$$

which is a consequence of modularity, we see by standard estimates that

$$\tilde{g}_{\mathrm{ser}}\left(\frac{a}{c}+\frac{iy}{c}\right)\ll_{g,c}\min(y^{-1/2},y\,e^{-2\pi y}).\tag{A.5}$$

Therefore, taking Mellin transforms, we obtain, by (A.3),

$$\int_0^{\infty}\tilde{g}_{\mathrm{ser}}\left(\frac{a}{c}+\frac{iy}{c}\right)y^{s-1/2}\frac{dy}{y}=\frac{1}{\sqrt{c}}\left(\frac{c}{\pi}\right)^s F\left(g,\frac{a}{c},s\right),\quad \Re s>1,\tag{A.6}$$

where

$$F(g,x,s):=\Phi_2^{+1}(s,ir)D^{+1}(g,x,s)+\Phi_2^{-1}(s,ir)D^{-1}(g,x,s)\tag{A.7}$$

and

$$\Phi_2^{\pm 1}(s,ir):=\sqrt{\pi}\int_0^{\infty}V_{2,ir}^{\pm 1}(y)y^{s-1/2}\frac{dy}{y}.$$

The last function is the same as [7, (8.25)] except that we deleted the 4 from the denominator. The reason is that we would like to apply [7, Lemma 8.2] but that lemma requires this correction, because the initial identity [7, (8.30)] is missing a factor $\frac{1}{4}$ on the right-hand side. By Duke, Friedlander and Iwaniec [7, Lemma 8.2],

$$\Phi_2^{+1}(s,ir)=\left(s-\frac{1}{2}\right)\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right),\tag{A.8}$$

$$\Phi_2^{-1}(s,ir)=2\Gamma\left(\frac{s+ir+1}{2}\right)\Gamma\left(\frac{s-ir+1}{2}\right).\tag{A.9}$$

We shall derive the analytic properties of $D(g,x,s)$ from those of $F(g,x,s)$. Splitting the integral in (A.6) and applying (A.4) in the form

$$\tilde{g}_{\text{ser}}\left(\frac{a}{c}+\frac{iy}{c}\right)=-\tilde{g}_{\text{ser}}\left(-\frac{\bar{a}}{c}+\frac{i}{cy}\right)-\tilde{g}_{\text{const}}\left(\frac{1}{cy}\right)-\tilde{g}_{\text{const}}\left(\frac{y}{c}\right),\quad 1>y>0,$$

(A.6) becomes

$$\left(\frac{c}{\pi}\right)^s F\left(g,\frac{a}{c},s\right)=\sqrt{c}\int_1^\infty\left\{\tilde{g}_{\text{ser}}\left(\frac{a}{c}+\frac{iy}{c}\right)y^{s-\frac{1}{2}}-\tilde{g}_{\text{ser}}\left(-\frac{\bar{a}}{c}+\frac{iy}{c}\right)y^{1/2-s}\right\}\frac{dy}{y} + P(ir,c,s), \tag{A.10}$$

where

$$P(ir,c,s):=-\sqrt{c}\int_1^\infty\left\{\tilde{g}_{\text{const}}\left(\frac{1}{cy}\right)+\tilde{g}_{\text{const}}\left(\frac{y}{c}\right)\right\}y^{1/2-s}\frac{dy}{y}. \tag{A.11}$$

By (A.5) the first integral in (A.10) is absolutely convergent for any $s\in\mathbb{C}$, hence it defines an entire function. Moreover, it becomes its own *negative* under the substitution $a/c\to-\bar{a}/c$ and $s\to 1-s$: we say it is *symmetric* for short. The second integral (A.11) can be calculated explicitly using (A.2):

$$P(ir,c,s)=\sum_{\pm}c^{\mp ir}\left(\frac{1}{2}\pm ir\right)\theta\left(\frac{1}{2}\pm ir\right)\left(\frac{1}{s\pm ir}-\frac{1}{1-s\pm ir}\right),\quad \Re s>1. \tag{A.12}$$

The function $P(ir,c,s)$ is meromorphic and symmetric in $s\in\mathbb{C}$, hence $(c/\pi)^sF(g,a/c,s)$ is also symmetric and differs from $P(ir,c,s)$ by an entire function. If we apply this conclusion for $-a/c$ in place of a/c and combine (A.7) with $D^{\pm1}(g,-x,s)=\pm D^{\pm1}(g,x,s)$, then we see that the following functions are entire and symmetric:

$$\left(\frac{c}{\pi}\right)^s\Phi_2^{+1}(s,ir)D^{+1}\left(g,\frac{a}{c},s\right)-P(ir,c,s)\quad\text{and}\quad\left(\frac{c}{\pi}\right)^s\Phi_2^{-1}(s,ir)D^{-1}\left(g,\frac{a}{c},s\right). \tag{A.13}$$

Now we can argue exactly as on [**13**, p. 597] (with $k=2$) to see that $D(g,x,s)$ satisfies the functional equation (29); cf. [**13**, (15)]. Here we note that the functions $\Psi_{k,it}^{\pm}$ on [**13**, p. 597] should have been halved, and therefore, in [**13**, (19)–(20)], the factors 2^{2s} are really 2^{2s-1}.

We also need to analyse the poles of $D(g,a/c,s)$ for which we go back to (A.13). We see, by (A.9), that $D^{-1}(g,a/c,s)$ is entire. By (A.8) and (A.12), the poles of $D^{+1}(g,a/c,s)$ are at $s=1\pm ir$; they are simple with residues (cf. (A.1))

$$\frac{1}{\Phi_2^{+1}(1\pm ir,ir)}\left(\frac{\pi}{c}\right)^{1\pm ir}c^{\mp ir}\left(\frac{1}{2}\pm ir\right)\theta\left(\frac{1}{2}\pm ir\right)=\frac{\zeta(1\pm 2ir)}{c^{1\pm 2ir}}.$$

We note that the potential poles at $s=\pm ir$ coming from the poles of $P(ir,c,s)$ are cancelled by the poles of $\Phi_2^{+1}(s,ir)$ there, while the potential pole at $s=\frac{1}{2}$ coming from the zero of $\Phi_2^{+1}(s,ir)$ is cancelled by the zero of $P(ir,c,s)$ there. This concludes our proof of Lemma 1.

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