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ON POWER SUMS OF COMPLEX NUMBERS WHOSE SUM IS 0

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Introduction. Let $z_1, \dots, z_n$ be complex numbers, $s_\nu = z_1^\nu + \dots + z_n^\nu$, and put $M_n = \inf \max\limits_{1 \leqq \nu \leqq n} |s_\nu|$, where the infimum is taken over systems $\min\limits_{1 \leqq j \leqq n} |z_j| = 1$ and $s_1 = 0$. The determination of M_n, which is clearly a minimum by Weierstrass' theorem, is raised by P. Turán in his posthumous book [1]. Simple examples show $M_{2m} \leqq 2$, $M_{3m-1} \leqq 3$, $M_{6m-3} \leqq 3$ ([2]) and by a note of [2] $M_n = O(1)$ for the outstanding case $n = 6m + 1$. M. Szalay has proved the lower bound $1 + \bigl(\log 2 - o(1)\bigr) / \log n$ for M_n ([2]). It is known that $M_2 = 2$, $M_3 = 3$, $M_4 = 2$, $1.9219 < M_5 < 2.2321$ and $1.7936 < M_6 \leqq 1.9968$ ([2]). We shall prove

THEOREM 1. $M_n \leqq 2 + \frac{3\pi^2/2 + o(1)}{n}$.

We also improve the lower bound:

THEOREM 2. $1 + \frac{1 - 55/\log n}{\log n} \leqq M_n$ $\bigl(n \geqq 10^{24}\bigr)$.

In Section 3 we obtain some numerical estimates for M_n $(6 \leqq n \leqq 19)$. The lower bounds are deduced as in [2], the upper ones are gained by direct computing of examples $|z_j| = 1$ $(j = 1, \dots, n)$. A detailed calculation is given for the cases $n = 6, 7$.

1. To prove Theorem 1 we can assume n is odd, since $M_{2m} \leqq 2$ by the result of [2]. Let $n = 2m - 1$ $(m \geqq 2)$ and consider the system

$$z_j = \begin{cases} \alpha e^{\varphi i j} & \text{for } 1 \leqq j \leqq m-1 \\ e^{\varphi i (j+1)} & \text{for } m \leqq j \leqq n \end{cases}$$

where $i^2 = -1$, $\varphi = 2\pi/(n+2)$ and $\alpha \geqq 1$ is to be chosen later. Clearly $\min\limits_{1 \leqq j \leqq n} |z_j| = 1$ and we shall see that $s_1 = 0$ holds for a suitable α. For $1 \leqq \nu \leqq n$ we obtain

$$s_\nu = \alpha^\nu \sum_{j=1}^{m-1} e^{\varphi i j \nu} + \sum_{j=m+1}^{n+1} e^{\varphi i j \nu} =$$

$$= (\alpha^\nu - 1)\sum_{j=1}^{m-1} e^{\varphi ij\nu} + \sum_{\substack{0\leqq j\leqq n+1\\ j\neq 0,m}} e^{\varphi ij\nu} = (\alpha^\nu - 1)\sum_{j=1}^{m-1} e^{\varphi ij\nu} - 1 - e^{\varphi im\nu} =$$

$$= e^{\varphi m\nu i/2}\left[(\alpha^\nu - 1)\frac{\sin\varphi(m-1)\nu/2}{\sin\varphi\nu/2} - 2\cos\varphi m\nu/2\right],$$

i.e.,

$$|s_\nu| = \left|(\alpha^\nu - 1)\frac{\sin\varphi(m-1)\nu/2}{\sin\varphi\nu/2} - 2\cos\varphi m\nu/2\right|.$$

With the notation $\lambda = \frac{\varphi}{4} = \frac{\pi}{2(n+2)}$ we have

$$\frac{\varphi(m-1)}{2} = \frac{\pi}{2} - 3\lambda \quad \text{and} \quad \frac{\varphi m}{2} = \frac{\pi}{2} - \lambda,$$

so

$$|s_\nu| = \left|(\alpha^\nu - 1)\frac{\sin\left(\frac{\pi}{2}\nu - 3\lambda\nu\right)}{\sin 2\lambda\nu} - 2\cos\left(\frac{\pi}{2}\nu - \lambda\nu\right)\right|,$$

which yields

$$|s_\nu| = \left|(\alpha^\nu - 1)\frac{\pm\cos 3\lambda\nu}{\sin 2\lambda\nu} \mp 2\sin\lambda\nu\right| \text{ according as } \nu \equiv 1 \text{ or } 3 \pmod 4,$$

$$|s_\nu| = \left|(\alpha^\nu - 1)\frac{\pm\sin 3\lambda\nu}{\sin 2\lambda\nu} \pm 2\cos\lambda\nu\right| \text{ according as } \nu \equiv 2 \text{ or } 4 \pmod 4.$$

Thus

$$(1) \qquad |s_\nu| = \begin{cases} \left|(\alpha^\nu - 1)\frac{\cos 3\lambda\nu}{\sin 2\lambda\nu} - 2\sin\lambda\nu\right| & \text{if } \nu \text{ is odd} \\ \left|(\alpha^\nu - 1)\frac{\sin 3\lambda\nu}{\sin 2\lambda\nu} + 2\cos\lambda\nu\right| & \text{if } \nu \text{ is even.} \end{cases}$$

Now define $\nu' = n + 2 - \nu$, then $\lambda\nu = \pi/2 - \lambda\nu'$ and (1) gives

$$(2) \qquad |s_\nu| = \begin{cases} \left|(\alpha^\nu - 1)\frac{\sin 3\lambda\nu'}{\sin 2\lambda\nu'} + 2\cos\lambda\nu'\right| & \text{if } \nu \text{ is odd} \\ \left|(\alpha^\nu - 1)\frac{\cos 3\lambda\nu'}{\sin 2\lambda\nu'} - 2\sin\lambda\nu'\right| & \text{if } \nu \text{ is even.} \end{cases}$$

Denoting $\min\{\nu, \nu'\}$ by κ and noting that ν and ν' are of opposite parity, we get by (1) and (2)

$$(3) \qquad |s_\nu| = \begin{cases} \left|(\alpha^\nu - 1)\frac{\cos 3\lambda\kappa}{\sin 2\lambda\kappa} - 2\sin\lambda\kappa\right| & \text{if } \kappa \text{ is odd} \\ \left|(\alpha^\nu - 1)\frac{\sin 3\lambda\kappa}{\sin 2\lambda\kappa} + 2\cos\lambda\kappa\right| & \text{if } \kappa \text{ is even.} \end{cases}$$

(3) shows that $s_1 = 0$ holds if α is chosen such that $\alpha - 1 = \frac{2\sin\lambda\sin 2\lambda}{\cos 3\lambda}$. Clearly

$$\alpha = 1 + \frac{2\lambda\cdot 2\lambda}{1}\bigl(1+o(1)\bigr) = 1 + \bigl(\pi^2+o(1)\bigr)/n^2 = \\ = \exp\Bigl\{\bigl(\pi^2+o(1)\bigr)/n^2\Bigr\},$$

i.e.,

$$\alpha^n = \exp\Bigl\{\bigl(\pi^2+o(1)\bigr)/n\Bigr\} = 1 + \bigl(\pi^2+o(1)\bigr)/n. \tag{4}$$

Suppose first that κ is even. Since $\kappa < (n+2)/2$, $0 < \lambda\kappa < \pi/4$, we can observe that $\cos\lambda\kappa$, $\sin 2\lambda\kappa$ and $\sin 3\lambda\kappa$ are positive. Thus we can omit the sign of absolute-value in (3) and deduce by (4)

$$|s_\nu| \leqq (\alpha^n - 1)\frac{\sin 3\lambda\kappa}{\sin 2\lambda\kappa} + 2\cos\lambda\kappa < (\alpha^n-1)\frac{3}{2} + 2 = 2 + \frac{3\pi^2/2 + o(1)}{n}.$$

Secondly, assume κ to be odd. If $\kappa = 1$ then $\nu = 1$ and $s_\nu = 0$. Therefore we only have to deal with the case $\kappa \geqq 3$. (3), (4) and $0 < \lambda\kappa < \pi/4$ yield

$$|s_\nu| \leqq \frac{\alpha^n - 1}{\sin 2\lambda\kappa} + 2\sin\lambda\kappa < \frac{\bigl(\pi^2+o(1)\bigr)/n}{\sin 2\lambda\kappa} + 2\sin\lambda\kappa \leqq$$

$$\leqq \frac{\bigl(\pi^2+o(1)\bigr)/n}{\frac{2}{\pi}\cdot 2\lambda\kappa} + 2\lambda\kappa = \frac{\bigl(\pi^2+o(1)\bigr)/n}{\frac{2}{n+2}\kappa} + \frac{\pi}{n+2}\kappa = \frac{\pi^2+o(1)}{2\kappa} + \frac{\pi}{n+2}\kappa.$$

This gives for $3 \leqq \kappa \leqq (n+2)/12$ and all large enough n

$$|s_\nu| \leqq \frac{\pi^2+o(1)}{6} + \frac{\pi}{12} < \frac{10}{6} + \frac{4}{12} = 2,$$

and for $(n+2)/12 < \kappa$ and all large enough n

$$|s_\nu| \leqq \frac{\pi^2+o(1)}{2(n+2)/12} + \frac{\pi}{2} < \frac{4}{2} = 2.$$

The results gained in the even and in the odd cases imply Theorem 1. □

REMARK. One would expect that a similar good or perhaps a better estimate can be obtained for M_n by the system

$$z_j = \begin{cases} \alpha + e^{\varphi i j} & \text{for } 1 \leqq j \leqq m-1 \\ e^{\varphi i (j+1)} & \text{for } m \leqq j \leqq n \end{cases}$$

where the complex α is chosen such that $s_1 = 0$. However, this is not so, since — by similar calculation as above — for any fixed and even ν

$$|s_\nu| = 2 + \frac{2\nu + o(1)}{n} \quad \text{as} \quad n \to \infty.$$

2. To prove Theorem 2 we use the fact (see [2]) that the unique positive root R_n of the polynomial (of degree $[n/2]$)

$$F_n(x) = -1 + \sum_{2j_2+\ldots+nj_n=n} \prod_{2\leqq\nu\leqq n} \frac{1}{j_\nu!}\left(\frac{x}{\nu}\right)^{j_\nu}$$

furnishes a lower bound for M_n (the j_ν's are nonnegative integers). Using the formula

$$\sum_{j_1+\ldots+nj_n=n} \prod_{1\leqq\nu\leqq n} \frac{1}{j_\nu!}\left(\frac{x}{\nu}\right)^{j_\nu} = \binom{x+n-1}{n}$$

(see [3]) we easily get for any positive integer $k < n/2$

$$F_n(x) = -1 + \sum_{r=0}^{2k-1} \binom{x+n-r-1}{n-r} \frac{(-x)^r}{r!} +$$

$$+ \frac{x^{2k}}{(2k-1)!} \sum_{j_1+\ldots+nj_n=n-2k} \frac{1}{j_1+2k} \prod_{1\leqq\nu\leqq n} \frac{1}{j_\nu!}\left(\frac{x}{\nu}\right)^{j_\nu} \leqq$$

$$\leqq -1 + \sum_{r=0}^{2k} \binom{x+n-r-1}{n-r} \frac{(-x)^r}{r!}.$$

Now let $n \geqq 10^{24}$, $\varepsilon = \frac{55}{\log n}$, $1 < x \leqq 1 + \frac{1-\varepsilon}{\log n}$ and $k = \left[\frac{n}{3}\right]$, then

$$F_n(x) \leqq -1 + \binom{x+n-2k-1}{n-2k} \sum_{r=0}^{2k} \frac{\binom{x+n-r-1}{n-r}}{\binom{x+n-2k-1}{n-2k}} \frac{(-x)^r}{r!}.$$

We have

$$\binom{x+n-2k-1}{n-2k} < \exp\left(\sum_{j=1}^{n-2k} \frac{x-1}{j}\right) <$$

$$< \exp\left\{(1-\varepsilon)\frac{\log(n-2k)+1}{\log n}\right\} < e^{1-\varepsilon},$$

and for any $0 \leqq r \leqq 2k$

$$1 \leqq \frac{\binom{x+n-r-1}{n-r}}{\binom{x+n-2k-1}{n-2k}} \leqq \frac{\binom{x+n-1}{n}}{\binom{x+n-2k-1}{n-2k}} < \exp\left(\sum_{j=n-2k+1}^{n} \frac{x-1}{j}\right) <$$

$$< \exp\left(\frac{1}{\log n} \sum_{j=n-2k+1}^{n} \frac{1}{j}\right) < \exp\left\{\frac{\log n - \log(n-2k)}{\log n}\right\} < \exp\left(\frac{2}{\log n}\right),$$

i.e.,

$$\left|\frac{\binom{x+n-r-1}{n-r}}{\binom{x+n-2k-1}{n-2k}} - 1\right| < \exp\left(\frac{2}{\log n}\right) - 1 < \frac{2}{\log n} \exp\left(\frac{2}{\log n}\right).$$

Thus

$$F_n(x) + 1 < e^{1-\varepsilon}\left\{\sum_{j=0}^{2k} \frac{(-x)^j}{j!} + \frac{2}{\log n} \exp\left(\frac{2}{\log n}\right) \sum_{j=0}^{2k} \frac{x^j}{j!}\right\} <$$

$$< e^{1-\varepsilon}\left\{e^{-x} + \frac{x^{2k+1}}{(2k+1)!} + \frac{2}{\log n} \exp\left(\frac{2}{\log n}\right) e^x\right\} <$$

$$< e^{1-\varepsilon}\left\{\frac{1}{e} + \frac{2^{2k+1}}{\left((2k+1)/3\right)^{2k+1}} + \frac{2}{\log n} \exp\left(1 + \frac{3}{\log n}\right)\right\} <$$

$$< e^{1-\varepsilon}\left(\frac{1}{e} + \frac{6}{2k+1} + \frac{6}{\log n}\right) <$$

$$< e^{1-\varepsilon}\left(\frac{1}{e} + \frac{7}{\log n}\right) < e^{-\varepsilon} + \frac{20}{\log n} < e^{-\varepsilon} + \frac{\varepsilon}{e} < e^{-\varepsilon} + \varepsilon e^{-\varepsilon},$$

since $n \geqq 10^{24}$ implies $\varepsilon < 1$. Finally

$$F_n(x) < -1 + e^{-\varepsilon}(1+\varepsilon) < -1 + e^{-\varepsilon} e^{\varepsilon} = 0.$$

Now $0 \leqq F_n(M_n)$, hence

$$1 + \frac{1 - 55/\log n}{\log n} = 1 + \frac{1-\varepsilon}{\log n} < M_n \quad (n \geqq 10^{24}). \quad \square$$

REMARK. A similar argument leads to the result

$$R_n = 1 + \frac{1 - o(1)}{\log n}.$$

3. Finally we consider systems of type

$$\left.\begin{aligned} z_j &= \exp(\varphi_j i) && (1 \leqq j \leqq m) \\ z_{j+m} &= \exp(-\varphi_j i) && (1 \leqq j \leqq m) \end{aligned}\right\} \quad \text{if} \quad n = 2m,$$

$$\left.\begin{aligned} z_1 &= 1 \\ z_{j+1} &= \exp(\varphi_j i) && (1 \leqq j \leqq m-1) \\ z_{j+m} &= \exp(-\varphi_j i) && (1 \leqq j \leqq m-1) \end{aligned}\right\} \quad \text{if} \quad n = 2m-1$$

where the φ_j are real numbers. If $s_\nu = z_1^\nu + \ldots + z_n^\nu$ and $s_1 = 0$ then $\max\limits_{1 \leqq \nu \leqq n} |s_\nu|$ provides an upper estimate for M_n.

First we deal with the case $n = 6$ in detail. If

$$(z - z_1)\ldots(z - z_6) = z^6 + a_1 z^5 + \ldots + a_5 z + a_6 \tag{5}$$

then $a_6 = 1$, $a_2 = a_4 = \alpha$ and $a_3 = \beta$ with some real α and β and the condition $s_1 = 0$ implies $a_1 = a_5 = 0$. It is easy to verify that the numbers $\lambda_j = 2\cos\varphi_j$ $(j = 1, 2, 3)$, which lie in the interval $[-2; 2]$, are real roots of the equation

$$\lambda^3 + (\alpha - 3)\lambda + \beta = 0. \tag{6}$$

Conversely, if we choose the real α and β such that (6) has three roots in $[-2; 2]$ and define a_6 to be 1, $a_1 = a_5$ to be 0, $a_2 = a_4$ to be α and a_3 to be β then the numbers $z_1, \ldots, z_6$ determined by (5) lie on the unit circle $|z| = 1$ and they satisfy $s_1 = 0$.

Calculating the power sums in terms of α and β by the Newton–Girard formulae we get

$$s_2 = -2\alpha,\ s_3 = -3\beta,\ s_4 = 2\alpha^2 - 4\alpha,\ s_5 = 5\alpha\beta,\ s_6 = 3\beta^2 - 2\alpha^3 + 6\alpha^2 - 6.$$

It seems to be convenient to put $\alpha = 1 - \varepsilon$ and $\beta = \frac{2}{5}(1 + \varepsilon)$, where $0 \leqq \varepsilon \leqq \frac{2}{5}$, since then

$$\max_{1 \leqq \nu \leqq 5} |s_\nu| = 2(1 - \varepsilon^2)$$

and

$$|s_6| = 2(1 - \varepsilon^2) - \frac{2}{25}(25\varepsilon^3 - 19\varepsilon^2 - 63\varepsilon + 6).$$

It can be checked that $25\varepsilon^3 - 19\varepsilon^2 - 63\varepsilon + 6$ has the only real root $\varepsilon = 0.092951\ldots$ in the interval $[0; 2/5]$ and this ε determines an $\alpha =$

$= 0.907048\ldots$ and a $\beta = 0.437180\ldots$ such that (6) has three roots in $[-2;\, 2]$. Thus the inequality

$$M_6 \leqq 2(1-\varepsilon^2) = 1.982720\ldots$$

follows. □

Secondly, let $n = 7$. If $z_1 = 1$ and

$$(7) \qquad (z-z_1)(z-z_2)\ldots(z-z_7) = z^7 + a_1 z^6 + \ldots + a_6 z + a_7$$

then $a_7 = -1$, $a_2 = -\alpha$, $a_5 = \alpha$ and $a_3 = -\beta$, $a_4 = \beta$ with some real α and β and the condition $s_1 = 0$ implies $a_1 = a_6 = 0$. It is easy to verify that the numbers $\lambda_j = 2\cos\varphi_j$ $(j = 1,2,3)$, which lie in the interval $[-2;\, 2]$, are real roots of the equation

$$(8) \qquad \lambda^3 + \lambda^2 - (\alpha+2)\lambda - (\alpha+\beta+1) = 0.$$

Conversely, if we choose the real α and β such that (8) has three roots in $[-2;\, 2]$ and define a_7 to be -1, $a_1 = a_6$ to be 0, a_2 as $-\alpha$, a_5 as α, a_3 as $-\beta$ and finally a_4 as β then the numbers $z_1 = 1, z_2, \ldots, z_7$ determined by (7) lie on the unit circle $|z| = 1$ and they satisfy $s_1 = 0$.

It is convenient to put $\beta = 2\alpha/3$. Calculating the power sums in terms of α by the Newton–Girard formulae we get

$$s_2 = s_3 = 2\alpha, \quad s_4 = 2\alpha^2 - \frac{8}{3}\alpha, \quad s_5 = \frac{10}{3}\alpha^2 - 5\alpha,$$
$$s_6 = 2\alpha^3 - \frac{8}{3}\alpha^2, \quad s_7 = 7\left(1 - \frac{13}{9}\alpha^2 + \frac{2}{3}\alpha^3\right),$$

which yields that for $9/10 \leqq \alpha < 1$

$$\max_{1\leqq\nu\leqq6} |s_\nu| = 2\alpha \quad \text{and} \quad |s_7| = 7\left(1 - \frac{13}{9}\alpha^2 + \frac{2}{3}\alpha^3\right).$$

It can be checked that $2\alpha = 7\left(1 - \frac{13}{9}\alpha^2 + \frac{2}{3}\alpha^3\right)$ has the only real root $\alpha = 0.947181\ldots$ in the interval $[0;\, 9/10]$ and this α determines a $\beta = 0.631454\ldots$ such that (8) has three roots in $[-2;\, 2]$. Thus the inequality

$$M_7 \leqq 2\alpha = 1.894363\ldots$$

holds. □

Further on we indicate for comparison the lower bounds R_n of Section 2. The upper bounds are derived from systems described above. We have the following inequalities for M_n $(6 \leqq n \leqq 19)$:

$$1.793610\ldots \leqq M_6 \leqq 1.982720\ldots$$

$$1.719907\ldots \leqq M_7 \leqq 1.894363\ldots$$

$$1.662581\ldots \leqq M_8 \leqq 1.999796\ldots \quad \begin{cases} \varphi_1 = 33.987585\ldots^\circ \\ \varphi_2 = 73.303745\ldots^\circ \\ \varphi_3 = 109.097547\ldots^\circ \\ \varphi_4 = 142.118198\ldots^\circ \end{cases}$$

$$1.618555\ldots \leqq M_9 \leqq 1.790782\ldots \quad \begin{cases} \varphi_1 = 38.430487\ldots^\circ \\ \varphi_2 = 67.220086\ldots^\circ \\ \varphi_3 = 134.114614\ldots^\circ \\ \varphi_4 = 167.022740\ldots^\circ \end{cases}$$

$$1.583255\ldots \leqq M_{10} \leqq 1.973688\ldots \quad \begin{cases} \varphi_1 = 32.074778\ldots^\circ \\ \varphi_2 = 57.616740\ldots^\circ \\ \varphi_3 = 91.887543\ldots^\circ \\ \varphi_4 = 117.618984\ldots^\circ \\ \varphi_5 = 152.425330\ldots^\circ \end{cases}$$

$$1.554267\ldots \leqq M_{11} \leqq 2.119011\ldots \quad \begin{cases} \varphi_1 = 44.038349\ldots^\circ \\ \varphi_2 = 70.364417\ldots^\circ \\ \varphi_3 = 96.533487\ldots^\circ \\ \varphi_4 = 125.493345\ldots^\circ \\ \varphi_5 = 149.374899\ldots^\circ \end{cases}$$

$$1.529965\ldots \leqq M_{12} \leqq 1.998574\ldots \quad \begin{cases} \varphi_1 = 26.280566\ldots^\circ \\ \varphi_2 = 52.020428\ldots^\circ \\ \varphi_3 = 76.396810\ldots^\circ \\ \varphi_4 = 102.127160\ldots^\circ \\ \varphi_5 = 129.177757\ldots^\circ \\ \varphi_6 = 154.877600\ldots^\circ \end{cases}$$

$$1.509245\ldots \leqq M_{13} \leqq 2.126728\ldots \qquad \begin{cases} \varphi_1 = 19.191938\ldots^\circ \\ \varphi_2 = 39.803824\ldots^\circ \\ \varphi_3 = 88.675687\ldots^\circ \\ \varphi_4 = 117.904497\ldots^\circ \\ \varphi_5 = 142.222650\ldots^\circ \\ \varphi_6 = 167.789878\ldots^\circ \end{cases}$$

$$1.491331\ldots \leqq M_{14} \leqq 1.828905\ldots \qquad \begin{cases} \varphi_1 = 9.069892\ldots^\circ \\ \varphi_2 = 31.062936\ldots^\circ \\ \varphi_3 = 52.885792\ldots^\circ \\ \varphi_4 = 98.672340\ldots^\circ \\ \varphi_5 = 121.786322\ldots^\circ \\ \varphi_6 = 142.291313\ldots^\circ \\ \varphi_7 = 168.191278\ldots^\circ \end{cases}$$

$$1.475659\ldots \leqq M_{15} \leqq 1.967363\ldots \qquad \begin{cases} \varphi_1 = 21.810490\ldots^\circ \\ \varphi_2 = 40.279274\ldots^\circ \\ \varphi_3 = 61.823098\ldots^\circ \\ \varphi_4 = 105.380928\ldots^\circ \\ \varphi_5 = 125.044087\ldots^\circ \\ \varphi_6 = 146.843332\ldots^\circ \\ \varphi_7 = 170.714120\ldots^\circ \end{cases}$$

$$1.4618007\ldots \leqq M_{16} \leqq 2$$

$$1.449458\ldots \leqq M_{17} \leqq 1.948290\ldots \qquad \begin{cases} \varphi_1 = 19.331397\ldots^\circ \\ \varphi_2 = 39.866743\ldots^\circ \\ \varphi_3 = 58.023924\ldots^\circ \\ \varphi_4 = 75.955699\ldots^\circ \\ \varphi_5 = 114.414529\ldots^\circ \\ \varphi_6 = 133.645456\ldots^\circ \\ \varphi_7 = 153.498174\ldots^\circ \\ \varphi_8 = 170.045331\ldots^\circ \end{cases}$$

$$1.438363\ldots \leqq M_{18} \leqq 2$$

$$1.428328\ldots \leqq M_{19} \leqq 1.888063\ldots \quad \begin{cases} \varphi_1 = \ \ 18.409141\ldots^\circ \\ \varphi_2 = \ \ 37.223032\ldots^\circ \\ \varphi_3 = \ \ 52.895537\ldots^\circ \\ \varphi_4 = \ \ 70.133067\ldots^\circ \\ \varphi_5 = \ \ 88.133122\ldots^\circ \\ \varphi_6 = 123.294802\ldots^\circ \\ \varphi_7 = 139.974654\ldots^\circ \\ \varphi_8 = 156.231133\ldots^\circ \\ \varphi_9 = 172.269165\ldots^\circ \end{cases}$$

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