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ℓ-adic Representations Associated to Modular Forms over Imaginary Quadratic Fields

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Let π be a regular algebraic cuspidal automorphic representation of GL_2 over an imaginary quadratic number field K, and let ℓ be a prime number. Assuming the central character of π is invariant under the nontrivial automorphism of K, it is shown that there is a continuous irreducible ℓ-adic representation ρ of $\mathrm{Gal}(\overline{K}/K)$ such that $L(s,\rho_v)=L(s,\pi_v)$ whenever v is a prime of K outside an explicit finite set.

1 Introduction

Let K be an imaginary quadratic field with nontrivial automorphism c, and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ with unitary central character ω. If π_∞ has Langlands parameter $W_{\mathbb{C}}=\mathbb{C}^\times\to\mathrm{GL}_2(\mathbb{C})$ given by $z\mapsto\mathrm{diag}(z^{1-k},\overline{z}^{1-k})$ for some integer $k\geq 2$ (i.e., in the sense of Clozel [4], π is any regular algebraic cuspidal automorphic representation up to twist), then by the Langlands philosophy π should give rise (for any prime number ℓ) to a continuous irreducible ℓ-adic representation ρ of the Galois group $\mathrm{Gal}(\overline{K}/K)$ such that the associated L-functions agree. In other words,

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at each prime v of K the Frobenius polynomial of ρ at v agrees with the Hecke polynomial of π at v. Under the assumption that $\omega = \omega^c$ it is possible to relate π to holomorphic Siegel modular forms via theta lifts and deduce (using ℓ-adic cohomology on Siegel threefolds) a weak version of this predicted correspondence. In fact Taylor [18] managed to obtain the above equality of Frobenius and Hecke polynomials for all v outside a zero density set of places, but he had to make some additional technical assumptions. It is our aim here to describe how the results of Friedberg and Hoffstein [7] on the nonvanishing of certain central L-values and those of Laumon [9, 10] and Weissauer [21] on associated Galois representations to Siegel modular forms enable one to remove these technical assumptions and conclude the statement for all v outside an explicit finite set. Precisely, we shall prove the following theorem.

THEOREM 1.1. *Assume that $\omega = \omega^c$. Let S denote the set of places in K which divide ℓ or where $K/\mathbb{Q}$ or π or π^c is ramified. There exists a continuous irreducible representation $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that if v is a prime of K outside S then ρ is unramified at v and the characteristic polynomial of $\rho(\mathrm{Frob}_v)$ agrees with the Hecke polynomial of π at v, that is, $L(s, \rho_v) = L(s, \pi_v)$.* □

REMARK 1.2. Throughout the article we write Frob_v for the geometric Frobenius and we shift-normalize all L-functions such that $s = 1/2$ is their center.

REMARK 1.3. This theorem strengthens Theorem A of [18]. The assumption $\omega = \omega^c$ is inherent to Taylor's method; one would hope to remove this condition by other methods.

REMARK 1.4. By [16, Lemma 6] the Galois representation takes values in $\mathrm{GL}_2(E)$ for a finite extension $E/\mathbb{Q}_\ell$ (see [18, p. 635] for an explicit description).

REMARK 1.5. There are two cases in which Theorem 1.1 is known to hold (cf. [18, Lemmata 1 and 2]):

1. $\pi \otimes \delta \cong \pi$ for some nontrivial quadratic character δ of K. In this case, if L/K denotes the quadratic extension corresponding to δ then there is an algebraic idèle class character ψ of L unequal to its conjugate under the nontrivial element of $\mathrm{Gal}(L/K)$ such that π is the automorphic induction of ψ to K; the conclusion of Theorem 1.1 follows by work of Serre [14].
2. $\pi \otimes \nu \cong (\pi \otimes \nu)^c$ for some finite order character ν of K. In this case, a twist of π is a base change from $\mathbb{Q}$; the conclusion of Theorem 1.1 follows from work of Deligne [5].

The proof of Theorem 1.1 can be briefly outlined as follows. The initial strategy is that of Taylor [18]. We assume we are not in a case covered by Remark 1.5. Using the deep results of [8] and [7] we construct a nonzero theta lift on $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ of the twist $\pi \otimes \mu$ for a "dense" set (in the sense of Definition 2.1 below) of quadratic idèle class characters μ of K. The irreducible constituent Π^{μ} of such a lift is generated by a vector-valued holomorphic semi-regular cusp form on the Siegel three-space. Using Hasse invariant forms and the theory of pseudo-representations developed by Wiles [22] and Taylor [16, 17], Taylor had shown that one can associate a four-dimensional representation to Π^{μ} if one could associate four-dimensional Galois representations to regular holomorphic Siegel cusp forms. This is now possible by work of Laumon [9, 10] and Weissauer [21]. We obtain therefore, for each μ in some dense set, a four-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with the same partial L-function as the one associated to Π^{μ}, and we prove that it is induced from some two-dimensional representation ρ^{μ} of $\mathrm{Gal}(\overline{K}/K)$. By exploring global compatibility relations among the various ρ^{μ} we show that they can be replaced by quadratic twists[1] $\rho \otimes \mu$ of a single two-dimensional representation ρ of $\mathrm{Gal}(\overline{K}/K)$, and we verify that this ρ has the required property of Theorem 1.1.

REMARK 1.6. Since the construction of the four-dimensional Galois representation associated to Π^{μ} involves an ℓ-adic limit process one loses information about the geometricity of the Galois representation. For ℓ split in $K/\mathbb{Q}$, Urban [20, Corollaire 2] has proved, however, that if π is ordinary then the Galois representation ρ of Theorem 1.1 is ordinary at $v \mid \ell$.

2 Theta Lifts

Theorem 1 and Proposition 5 of [8] show how to construct nonzero theta lifts on $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ of many quadratic twists of π, conditional on "Conjecture/Theorem 1" [8, p. 403]. The analytic nonvanishing result of [7] implies "Conjecture/Theorem 1". For completeness, we have decided to summarize how [7] implies a strengthening of Theorem 1 of [8].

By assumption, the central character of π factors through the norm map as $\omega = \tilde{\omega} \circ N_{K/\mathbb{Q}}$, where $\tilde{\omega}$ is a character of $\mathbb{Q}$. The ratio of the two characters $\tilde{\omega}$ satisfying this equation is the quadratic character corresponding to $K/\mathbb{Q}$, hence one of them is odd and the other is even. We shall consider the character $\tilde{\omega}$ with $\tilde{\omega}_{\infty}(-1) = (-1)^k$. By Proposition 1 of [8] the pair $(\pi, \tilde{\omega})$ defines a cuspidal automorphic representation of $\mathrm{GO}^{\circ}(\mathbb{A}_{\mathbb{Q}})$, where GO is the group of orthogonal similitudes of a certain quadratic space W_K of sign $(3, 1)$

[1] Here and later we identify finite order idèle class characters with continuous Galois characters.

over $\mathbb{Q}$. [8] also introduces a signature $\delta = (\delta_v)$, a map from the places of $\mathbb{Q}$ to $\{\pm 1\}$ which is 1 at all but finitely many places such that $\delta_v = 1$ whenever $\pi_v \not\cong \pi_v^c$ (here we view π as a representation of $R_{K/\mathbb{Q}}\,\mathrm{GL}_2$, the group obtained from GL_2 by restriction of scalars). By Proposition 2 of [8] the triple $\hat{\pi} := (\pi, \tilde{\omega}, \delta)$ modulo of the action of $\{1, c\}$ can be identified with a cuspidal automorphic representation of $\mathrm{GO}(\mathbb{A}_\mathbb{Q})$ which in turn has a theta lift $\Theta(\hat{\pi})$ to $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$. By Proposition 3 of [8] the lift $\Theta(\hat{\pi})$ is contained in the space of cusp forms, and if Π denotes an irreducible constituent of $\Theta(\hat{\pi})$ then Π_∞ is a holomorphic limit of discrete series representation of weight $(k, 2)$ whenever $\delta_\infty = -1$, while Π_v is an unramified irreducible principal series representation with $L(s, \Pi_v) = L(s, \pi_v)$ whenever $\delta_v = 1$ and v is a rational prime which does not lie under a prime in S. In addition, Π is nonzero assuming there is a character φ of K restricting to $\tilde{\omega}$ on $\mathbb{Q}$ and satisfying the following two properties[2]:

$$\pi_v \cong \pi_v^c \quad \Longrightarrow \quad \delta_v = \tilde{\omega}_v(-1)\varepsilon(\pi_v \otimes \varphi_v^{-1}, 1/2);$$
$$L(\pi \otimes \varphi^{-1}, 1/2) \neq 0.$$

Using these as preliminaries we can deduce from the nonvanishing result [7] that $\pi \otimes \mu$ gives rise to suitable Π^μ on $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ for a dense set of quadratic characters μ of K.

DEFINITION 2.1. A set $\mathcal{M}$ of quadratic characters of K is *dense* if it has the following property. If $\tilde{\mu}$ is an arbitrary quadratic character of K and M is an arbitrary finite set of rational primes then there is a character $\mu \in \mathcal{M}$ such that $\mu_v = \tilde{\mu}_v$ for all $v \in M$.

DEFINITION 2.2. For a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_K)$ let $S_K(\tilde{\pi})$ denote the set of places in K which divide ℓ or where $K/\mathbb{Q}$ or $\tilde{\pi}$ or $\tilde{\pi}^c$ is ramified, and let $S_\mathbb{Q}(\tilde{\pi})$ denote the set of rational places which lie under some place in $S_K(\tilde{\pi})$.

THEOREM 2.3. *There exists a dense set $\mathcal{M}$ of quadratic characters of K with the following property. For each $\mu \in \mathcal{M}$ there is a signature δ such that the representation $(\pi \otimes \mu, \tilde{\omega}, \delta)$ of $\mathrm{GO}(\mathbb{A}_\mathbb{Q})$ gives rise to a cuspidal automorphic representation Π^μ of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ satisfying:*

- *Π^μ_∞ is a holomorphic limit of discrete series representation of weight $(k, 2)$;*
- *if v is a rational prime outside $S_\mathbb{Q}(\pi \otimes \mu)$ then Π^μ_v is an unramified irreducible principal series representation with $L(s, \Pi^\mu_v) = L(s, (\pi \otimes \mu)_v)$.* □

[2]The nonvanishing condition for the central L-value arises in [8] by evaluating a particular Fourier coefficient of a form in Π. It is possible that this condition can be removed, that is, the local conditions are sufficient. See the remark following the proof of [8, Proposition 3] and also a recent result proved by Takeda [15, Theorem 1.2], building on earlier work of Roberts [11–13].

To prove this let $\tilde{\mu}$ be any quadratic character of K and let M be any finite set of rational places. In the light of the above discussion (i.e. by Propositions 1–3 of [8]) it suffices to show that for $\tilde{\pi} := \pi \otimes \tilde{\mu}$ there exist a quadratic character η of K with $\eta_v = 1$ for all $v \in M$, a signature δ with $\delta_\infty = -1$ and $\delta_v = 1$ for any rational prime $v \notin S_{\mathbb{Q}}(\tilde{\pi} \otimes \eta)$, and a character φ of K with $\varphi|_{\mathbb{Q}} = \tilde{\omega}$, satisfying the following additional properties

$$\delta_v = \begin{cases} \tilde{\omega}_v(-1)\varepsilon(\tilde{\pi}_v \otimes \eta_v \varphi_v^{-1}, 1/2) & \text{if } \tilde{\pi}_v \otimes \eta_v \cong (\tilde{\pi}_v \otimes \eta_v)^c, \\ 1 & \text{if } \tilde{\pi}_v \otimes \eta_v \not\cong (\tilde{\pi}_v \otimes \eta_v)^c; \end{cases} \tag{2.1}$$

$$L(\tilde{\pi} \otimes \eta\varphi^{-1}, 1/2) \neq 0. \tag{2.2}$$

The proofs of Lemma 13 and Proposition 5 in [8] provide us with η and φ satisfying

$$\varepsilon(\pi_\infty \otimes \varphi_\infty^{-1}, 1/2) = -\tilde{\omega}_\infty(-1) \tag{2.3}$$

and

$$\varepsilon(\tilde{\pi} \otimes \eta\varphi^{-1}, 1/2) = 1.$$

Here we used our initial assumptions for π and $\tilde{\omega}$. Theorem A and the first part of Theorem B in [7] show that η can be replaced by another quadratic character satisfying (2.2). Now we define δ according to (2.1). By [8, Lemma 14], $\delta_v = \pm 1$ for all rational primes, and $\delta_v = 1$ for all rational primes $v \notin S_{\mathbb{Q}}(\tilde{\pi} \otimes \eta)$. In addition, $\delta_\infty = -1$ holds by (2.3) combined with $\tilde{\mu}_\infty \eta_\infty = 1$ and $\pi_\infty \cong \pi_\infty^c$.

3 Four-dimensional Galois Representations of $\mathbb{Q}$

In the previous section we constructed, for each quadratic character μ of K in some dense set $\mathcal{M}$, a cuspidal automorphic representation Π^μ of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ such that Π^μ_∞ is a holomorphic limit of discrete series representation of weight $(k, 2)$. It has the property

$$L^{S(\mu)}(s, \Pi^\mu) = L^{S(\mu)}(s, I_K^{\mathbb{Q}}(\pi \otimes \mu)),$$

where $S(\mu)$ abbreviates $S_{\mathbb{Q}}(\pi \otimes \mu)$, $L^{S(\mu)}$ denotes the product of local L-factors outside $S(\mu)$, and $I_K^{\mathbb{Q}}$ stands for automorphic induction. Note that $S(\mu)$ includes all the rational primes where Π^μ is ramified.

In this section we shall construct, for each $\mu \in \mathcal{M}$, a continuous semisimple representation

$$\tau^{\mu} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\overline{\mathbb{Q}}_{\ell})$$

such that

$$L^{S(\mu)}(s, \tau^{\mu}) = L^{S(\mu)}(s, \Pi^{\mu}).$$

In other words, we shall show that $\pi \otimes \mu$ is associated to a Galois representation over $\mathbb{Q}$. We shall rely on the following deep result of Weissauer [21] (see also the closely related work of Laumon [9, 10]):

THEOREM 3.1. *Let Π be an irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ such that Π_{∞} belongs to the holomorphic discrete series of weight (k_1, k_2) with $k_1 \geq k_2 \geq 3$. Let S denote the union of $\{\ell\}$ and the set of rational primes where Π is ramified. There exists a continuous semisimple representation*

$$\tau : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\overline{\mathbb{Q}}_{\ell})$$

such that if v is a rational prime outside S then τ is unramified at v and the characteristic polynomial of $\tau(\mathrm{Frob}_v)$ agrees with the Hecke polynomial of Π at v. In other words,

$$L^S(s, \tau) = L^S(s, \Pi),$$

where L^S denotes the product of local L-factors outside S. □

This theorem was not available for Taylor in [18]; instead, he utilized the weaker yet powerful results of [17] to conclude $L^S(s, \tau) = L^S(s, \Pi)$ for some exceptional set S of zero Dirichlet density. While the above theorem is not directly applicable to the representations Π^{μ} we can combine it with Taylor's method of pseudo-representations [16] to achieve our goal. Our situation is analogous to associating two-dimensional Galois representations to elliptic cusp forms of weight 1 which was accomplished by Deligne–Serre in the classical paper [6] by a technique involving lifting the weight and then applying a "horizontal" family of congruences.

Let f be a Hecke eigenform belonging to Π^{μ}: it is a vector-valued holomorphic semi-regular Siegel cusp form of weight $(k, 2)$ and some level N (i.e. Π^{μ} is ramified exactly at the primes dividing N). Let $\mathscr{H}_0^N(\mathbb{Z})$ be the $\mathbb{Z}$-algebra generated by the Hecke operators corresponding to primes not dividing N and denote by $\overline{\mathbb{T}}_{(k_1,k_2)}(N)$ the image of $\mathscr{H}_0^N(\mathbb{Z})$ in the space of holomorphic Siegel cusp forms of weight (k_1, k_2) and level N (see [16, p. 315] for precise definitions). It is known that $\overline{\mathbb{T}}_{(k_1,k_2)}(N) \otimes \mathbb{Q}$ is a semisimple $\mathbb{Q}$-algebra. In particular, we have a homomorphism $\lambda_f : \overline{\mathbb{T}}_{(k,2)}(N) \to \mathcal{O}_f$ such that $T(f) = \lambda_f(T)f$ for all $T \in \mathscr{H}_0^N(\mathbb{Z})$, where $\mathcal{O}_f$ is the ring of integers of some (minimally chosen) number field E_f.

Using the cup product of f with the ℓ^n-th power of the "Hasse Invariant" form exhibited by Blasius and Ramakrishnan[3] [3, Proposition 3.6], Taylor [16, Proposition 3] constructs a "vertical" family of morphisms[4]

$$\lambda_{nf} : \overline{\mathbb{T}}_{(k,2)+m\ell^n(\ell-1,\ell-1)}(N) \to \mathcal{O}_f/\ell^{n+1}$$

such that

$$\lambda_{nf}(T) = \lambda_f(T) \mod \ell^{n+1}, \qquad T \in \mathscr{H}_0^N(\mathbb{Z}).$$

Here $m = m(\ell)$ is a positive integer and n is an arbitrary positive integer. Together with Theorem 3.1 this allows us to apply the theory of pseudo-representations, as in Example 1 in [16, Section 1.3], to piece together the required Galois representation τ^μ corresponding to Π^μ.

4 Two-dimensional Galois representations of K

We have exhibited, using previous notation, continuous semisimple representations

$$\tau^\mu : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell), \qquad \mu \in \mathcal{M},$$

satisfying

$$L^{S(\mu)}(s, \tau^\mu) = L^{S(\mu)}(s, I_K^{\mathbb{Q}}(\pi \otimes \mu)). \tag{4.1}$$

Note that $S(\mu)$ includes all the rational primes where Π^μ is ramified. Here $I_K^{\mathbb{Q}}(\pi \otimes \mu)$ is a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ in the light of our initial assumption that $\pi \otimes \mu \not\cong (\pi \otimes \mu)^c$ (cf. Remark 1.5 and the proof of [1, Theorem 6.2 in Section 3.6]),

[3]Ramakrishnan has pointed out to us that there is a mistake in [3], but it does not affect the part we are using.
[4]We assume here that $N \geq 3$ to ensure that the corresponding Siegel modular variety is "nice" as in [16, §3.2–3.3], otherwise we replace N by $N\ell^2$, say.

therefore τ^μ is expected to be irreducible by [19, Conjecture 3.3] and the Chebotarev density theorem.

If χ denotes the quadratic character of $\mathbb{Q}$ corresponding to K, then it is known that (cf. proof of [1, Theorem 6.2 in Section 3.6])

$$I_K^{\mathbb{Q}}(\pi \otimes \mu) \otimes \chi \cong I_K^{\mathbb{Q}}(\pi \otimes \mu).$$

For our purposes here it suffices, however, to regard $I_K^{\mathbb{Q}}(\pi \otimes \mu)$ a purely formal object, i.e. a set of Langlands parameters satisfying [1, (6.1)–(6.2) in Section 3.6] outside $S(\mu)$, for such an object clearly satisfies

$$L^{S(\mu)}(s, I_K^{\mathbb{Q}}(\pi \otimes \mu) \otimes \chi) = L^{S(\mu)}(s, I_K^{\mathbb{Q}}(\pi \otimes \mu)).$$

At any rate, by (4.1) and the Chebotarev density theorem we can infer

$$\tau^\mu \otimes \chi \cong \tau^\mu. \tag{4.2}$$

This implies the following lemma.

Lemma 4.1. *For each $\mu \in \mathcal{M}$ there is a continuous semisimple representation*

$$\rho^\mu : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

such that

$$\tau^\mu \cong \mathrm{Ind}_K^{\mathbb{Q}}(\rho^\mu). \qquad \square$$

Proof. Suppose first that τ^μ is irreducible over $\mathbb{Q}$. Then $\mathrm{Hom}(\chi, \tau^\mu \otimes (\tau^\mu)^\vee) \neq 0$ by (4.2). Since $\tau^\mu \otimes (\tau^\mu)^\vee = \mathrm{End}(\tau^\mu)$, we see by Schur's Lemma that $\tau^\mu|_K$ is reducible as $\chi|_K$ is trivial. Let ρ^μ be an irreducible component of $\tau^\mu|_K$ of minimal dimension (i.e. at most two). By Frobenius reciprocity $\mathrm{Hom}(\mathrm{Ind}_K^{\mathbb{Q}}(\rho^\mu), \tau^\mu) \neq 0$, hence in fact $\tau^\mu \cong \mathrm{Ind}_K^{\mathbb{Q}}(\rho^\mu)$ since τ^μ is irreducible of dimension four and $\mathrm{Ind}_K^{\mathbb{Q}}(\rho^\mu)$ is of dimension at most four.

Suppose now that τ^μ is reducible over $\mathbb{Q}$. If λ (resp. β) is a one-dimensional (resp. two-dimensional) representation occurring in τ^μ, then (4.2) shows that $\lambda\chi$ (resp. $\beta \otimes \chi$) also occurs in τ^μ. Hence there are four cases to consider:

1. $\tau^\mu \cong \beta \oplus (\beta \otimes \chi)$. Then $\tau^\mu \cong \mathrm{Ind}_K^{\mathbb{Q}}(\beta|_K)$.
2. $\tau^\mu \cong \beta \oplus \gamma$, where both β and γ are χ-invariant. Then $\beta \cong \mathrm{Ind}_K^{\mathbb{Q}}(\kappa)$ and $\gamma \cong \mathrm{Ind}_K^{\mathbb{Q}}(\nu)$ for some one-dimensional κ and ν, so that $\tau^\mu \cong \mathrm{Ind}_K^{\mathbb{Q}}(\kappa \oplus \nu)$.
3. $\tau^\mu \cong \beta \oplus \lambda \oplus \lambda\chi$. Then $\beta \cong \beta \otimes \chi$, so $\beta \cong \mathrm{Ind}_K^{\mathbb{Q}}(\kappa)$ for some one-dimensional κ and $\tau^\mu = \mathrm{Ind}_K^{\mathbb{Q}}(\kappa \oplus \lambda|_K)$.
4. $\tau^\mu \cong \lambda \oplus \lambda\chi \oplus \nu \oplus \nu\chi$. Then $\tau^\mu \cong \mathrm{Ind}_K^{\mathbb{Q}}(\lambda|_K \oplus \nu|_K)$. ■

5 Compatibility of Twists

So far we have constructed, for each quadratic character μ of K in some dense set $\mathcal{M}$, a continuous semisimple representation

$$\rho^{\mu} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_{\ell})$$

such that

$$L^{S(\mu)}(s, \mathrm{Ind}_K^{\mathbb{Q}}(\rho^{\mu})) = L^{S(\mu)}(s, I_K^{\mathbb{Q}}(\pi \otimes \mu)),$$

where $S(\mu)$ abbreviates $S_{\mathbb{Q}}(\pi \otimes \mu)$ and both sides involve Euler factors of degree 4 over $\mathbb{Q}$. As we want to compare Euler factors over K it is useful to rewrite the previous equation (using restriction and base change) as

$$L^{S(\mu)}(s, \rho^{\mu}) L^{S(\mu)}(s, (\rho^{\mu})^c) = L^{S(\mu)}(s, \pi \otimes \mu) L^{S(\mu)}(s, (\pi \otimes \mu)^c), \tag{5.1}$$

where now $S(\mu)$ abbreviates $S_K(\pi \otimes \mu)$ and all L-functions involve Euler factors of degree 2 over K. Note that $S(\mu)$ includes all the rational primes where ρ^{μ} or $(\rho^{\mu})^c$ is ramified.

Our aim is to show that the Galois representations ρ^{μ} are globally compatible in the sense that they can be replaced by twists $\rho \otimes \mu$ of some fixed ρ. This will be achieved in three lemmata. Recall our assumption that we are not in a case covered by Remark 1.5.

LEMMA 5.1. *$\rho^{\mu}|_L$ is irreducible for all $\mu \in \mathcal{M}$ and all quadratic extensions L/K.* □

PROOF. Assume that $\rho^{\mu}|_L$ is reducible for some $\mu \in \mathcal{M}$ and some quadratic extension L/K. Let Ψ be an irreducible summand of $\rho^{\mu}|_L$ and for a prime v of K outside $S(\mu)$ let $\{\alpha'_v, \beta'_v\}$ denote the Langlands parameters of $\pi \otimes \mu$. By continuity Ψ takes values in a finite extension E of $\mathbb{Q}_{\ell}$. Applying restriction and base change in (5.1) we see that if w is a place of L lying above a place v of K outside $S(\mu) \cup \mathrm{disc}(L/K)$ then $\Psi(\mathrm{Frob}_w) \in \{(\alpha'_v)^f, (\beta'_v)^f, (\alpha'_{v^c})^f, (\beta'_{v^c})^f\}$, where $f = (L_w : K_v)$. Hence in fact $\Psi(\mathrm{Frob}_w)$ is either one of the Langlands parameters of the base changes $(\pi \otimes \mu)_L$ or $(\pi \otimes \mu)_L^c$ at w. Applying the results of [18, Section 3] we conclude that $(\pi \otimes \mu)_L$ is not cuspidal which by [18, Lemma 2] means that we are in Case 1 of Remark 1.5. This contradiction proves the lemma. ■

LEMMA 5.2. *Let $\mu \in \mathcal{M}$ and let δ be a quadratic character of K.*

1. *$\rho^{\mu} \otimes \delta \not\cong \rho^{\mu}$ for δ nontrivial.*
2. *$\rho^{\mu} \otimes \delta \not\cong (\rho^{\mu})^c$ in all cases.* □

PROOF. Assume first that δ is nontrivial, and denote by L/K the corresponding quadratic extension.

Assume that $\rho^\mu \otimes \delta \cong \rho^\mu$. Then $\mathrm{Hom}(\delta, \rho^\mu \otimes (\rho^\mu)^\vee) \neq 0$. Since $\rho^\mu \otimes (\rho^\mu)^\vee = \mathrm{End}(\rho^\mu)$, we see by Schur's Lemma that $\rho^\mu|_L$ is reducible as $\delta|_L$ is trivial. This is a contradiction to Lemma 5.1 and establishes the first part of the lemma.

Assume that $\rho^\mu \otimes \delta \cong (\rho^\mu)^c$. Then $(\rho^\mu)^c \otimes \delta^c \cong \rho^\mu$, hence in fact $\rho^\mu \otimes (\delta\delta^c) \cong \rho^\mu$. By the first part of the lemma this forces $\delta\delta^c = 1$, hence $\delta = \delta^c$ since δ is quadratic. This implies, using (5.1), that

$$L^T(s, \rho^\mu \otimes \delta) L^T(s, (\rho^\mu \otimes \delta)^c) = L^T(s, \pi \otimes \mu\delta) L^T(s, (\pi \otimes \mu\delta)^c),$$

where T is a finite set of primes in K containing $S(\mu)$. Using again the assumption $\rho^\mu \otimes \delta \cong (\rho^\mu)^c$ we obtain

$$L^T(s, (\rho^\mu)^c) L^T(s, \rho^\mu) = L^T(s, \pi \otimes \mu\delta) L^T(s, (\pi \otimes \mu\delta)^c),$$

hence by (5.1), multiplicity one, and base change, we have in fact

$$I_K^{\mathbb{Q}}(\pi \otimes \mu\delta) \cong I_K^{\mathbb{Q}}(\pi \otimes \mu).$$

Base changing this to K and comparing the cuspidal representations (in the isobaric sums), we are forced to have

$$\pi \otimes \mu\delta \cong (\pi \otimes \mu)^c, \tag{5.2}$$

since $\pi \otimes \mu\delta \cong \pi \otimes \mu$ falls under Case 1 of Remark 1.5. Here $\delta = \delta^c$, so there is a quadratic character ϵ of $\mathbb{Q}$ such that $\delta = \epsilon|_K$. Regarding δ and ϵ as idèle class characters, δ is the pull-back of ϵ by the norm map $N_{K/\mathbb{Q}}$ and so its restriction to the idèle classes of $\mathbb{Q}$ is the trivial character. Regarding δ as Galois character this means that its transfer to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is trivial. It follows from Hilbert's theorem 90 (as in the proof of [18, Lemma 1]) that we may write $\delta = \nu/\nu^c$ for some character ν of K. Plugging this in (5.2) we obtain

$$\pi \otimes \mu\nu \cong (\pi \otimes \mu\nu)^c,$$

hence we are in Case 2 of Remark 1.5. This contradiction establishes the second part of the lemma for nontrivial δ.

It remains to prove the lemma for trivial δ, i.e. $\rho^\mu \not\cong (\rho^\mu)^c$. However, this is immediate from (5.1) and the multiplicity one theorem since we are assuming that $\pi\otimes\mu \not\cong (\pi\otimes\mu)^c$. ■

Lemma 5.3. *There is a continuous semisimple representation*

$$\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

which is unramified outside S and for all $\mu \in \mathcal{M}$ satisfies

$$\rho^\mu \oplus (\rho^\mu)^c \cong (\rho\otimes\mu) \oplus (\rho\otimes\mu)^c.$$ □

Proof. For any quadratic character λ write $\delta_\lambda := \lambda\lambda^c$. Let $\tilde{\rho}^\mu := \rho^\mu \otimes \mu^{-1}$. Our goal is to show that either $\tilde{\rho}^\mu$ or $(\tilde{\rho}^\mu)^c\delta_\mu$ is independent of $\mu \in \mathcal{M}$. For any prime $v \notin S$ denote by $\{\alpha_v, \beta_v\}$ the set of inverse roots of the Hecke polynomial of π at v; then for any prime $v \notin S(\mu)$ the set of eigenvalues of $(\rho^\mu \oplus (\rho^\mu)^c)(\mathrm{Frob}_v)\cdot(\mu^{-1})(v) = (\tilde{\rho}^\mu \oplus (\tilde{\rho}^\mu)^c\delta_\mu)(\mathrm{Frob}_v)$ is

$$\{\alpha_v, \beta_v, \alpha_{v^c}\delta_\mu(v), \beta_{v^c}\delta_\mu(v)\}$$

by (5.2).

Given $\mu, \mu' \in \mathcal{M}$ the corresponding set of eigenvalues coincide over the splitting field $F := K(\delta_\mu\delta_{\mu'})$ of degree at most 2. By the Chebotarev density theorem and continuity, so do their characteristic polynomials. Viewing the representations as $\overline{\mathbb{Q}}_l[\mathrm{Gal}(\overline{K}/K)]$-representations, a general theorem on semisimple modules of algebras over fields of characteristic 0 [2, Ch. 8, Sec. 12.2, Prop. 3] then tells us that their semisimplifications are isomorphic. But since the ρ^μ are semisimple, so is the restriction to the normal subgroup $\mathrm{Gal}(\overline{K}/F)$ and we get

$$(\tilde{\rho}^\mu \oplus (\tilde{\rho}^\mu)^c\delta_\mu)|_F \cong (\tilde{\rho}^{\mu'} \oplus (\tilde{\rho}^{\mu'})^c\delta_{\mu'})|_F. \tag{5.3}$$

By Lemma 5.1, $\tilde{\rho}^\mu|_F$ is irreducible so either we have $\tilde{\rho}^\mu|_F \cong \tilde{\rho}^{\mu'}|_F$ or we have $(\tilde{\rho}^\mu)^c\delta_\mu|_F \cong \tilde{\rho}^{\mu'}|_F$. Let us fix some element $\mu_0 \in \mathcal{M}$, and for each $\mu \in \mathcal{M}$ such that the second case holds for $\mu' = \mu_0$ replace $\tilde{\rho}^\mu$ by $(\tilde{\rho}^\mu)^c\delta_\mu$ (this corresponds to replacing the original ρ^μ by $(\rho^\mu)^c$ which is legitimate). By this change we have achieved that $\tilde{\rho}^\mu|_F \cong \tilde{\rho}^{\mu_0}|_F$ for all $\mu \in \mathcal{M}$, that is, $\tilde{\rho}^\mu \cong \tilde{\rho}^{\mu_0} \otimes \psi_{\mu,\mu_0}$ for some character ψ_{μ,μ_0} of $\mathrm{Gal}(F/K)$. We shall regard ψ_{μ,μ_0} as a quadratic character of $\mathrm{Gal}(\overline{K}/K)$ trivial on $\mathrm{Gal}(\overline{K}/F)$.

Note that $\psi_{\mu_0,\mu_0} = 1$, so that for general $\mu, \mu' \in \mathcal{M}$ the definition

$$\psi_{\mu,\mu'} := \psi_{\mu,\mu_0}\psi_{\mu',\mu_0} \tag{5.4}$$

is unambiguous. This character satisfies

$$\tilde{\rho}^{\mu} \cong \tilde{\rho}^{\mu'} \otimes \psi_{\mu,\mu'}, \tag{5.5}$$

whence Lemma 5.2 tells us that in (5.3) we must have $\tilde{\rho}^{\mu}|_F \cong \tilde{\rho}^{\mu'}|_F$ for $F = K(\delta_\mu \delta_{\mu'})$ and $\psi_{\mu,\mu'}$ must be trivial on $\mathrm{Gal}(\overline{K}/F)$. It follows that $\psi_{\mu,\mu'} = 1$ or $\psi_{\mu,\mu'} = \delta_\mu\delta_{\mu'}$, since these are the only characters of $\mathrm{Gal}(\overline{K}/K)$ trivial on $\mathrm{Gal}(\overline{K}/F)$. We claim that either $\psi_{\mu,\mu'} = 1$ for all $\mu, \mu' \in \mathcal{M}$ or $\psi_{\mu,\mu'} = \delta_\mu\delta_{\mu'}$ for all $\mu, \mu' \in \mathcal{M}$. Assume the first alternative fails then there are $\mu_1, \mu_2 \in \mathcal{M}$ with $\psi_{\mu_1,\mu_2} = \delta_{\mu_1}\delta_{\mu_2} \neq 1$. For arbitrary $\mu \in \mathcal{M}$ we have $\psi_{\mu,\mu_1}\psi_{\mu,\mu_2} = \psi_{\mu_1,\mu_0}\psi_{\mu_2,\mu_0} = \psi_{\mu_1,\mu_2}$ by (5.4), hence $\psi_{\mu,\mu_1} \neq \psi_{\mu,\mu_2}$. Therefore $\psi_{\mu,\mu_1} = \delta_\mu\delta_{\mu_1}$ or $\psi_{\mu,\mu_2} = \delta_\mu\delta_{\mu_2}$. In fact these equations are equivalent since the two sides have equal products by (5.4), namely

$$\psi_{\mu,\mu_1}\psi_{\mu,\mu_2} = \psi_{\mu_1,\mu_0}\psi_{\mu_2,\mu_0} = \psi_{\mu_1,\mu_2} = \delta_{\mu_1}\delta_{\mu_2}.$$

We infer that $\psi_{\mu,\mu_1} = \delta_\mu\delta_{\mu_1}$ is valid for all $\mu \in \mathcal{M}$. This implies, for all $\mu, \mu' \in \mathcal{M}$,

$$\psi_{\mu,\mu'} = \psi_{\mu,\mu_0}\psi_{\mu',\mu_0} = \psi_{\mu,\mu_1}\psi_{\mu',\mu_1} = \delta_\mu\delta_{\mu'}.$$

If $\psi_{\mu,\mu'} = 1$ for all μ, μ' then (5.5) implies that $\tilde{\rho}^{\mu}$ is independent of μ. If $\psi_{\mu,\mu'} = \delta_\mu\delta_{\mu'}$ for all μ, μ' then (5.5) implies that $(\tilde{\rho}^{\mu})^c\delta_\mu$ is independent of μ. In both cases we denote the common value of these representations by ρ and verify that it satisfies the required properties. ■

6 End of Proof

We have shown that there is a dense set $\mathcal{M}$ of quadratic characters of K (see Definition 2.1) and a continuous irreducible semisimple representation

$$\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

unramified outside S such that for any $\mu \in \mathcal{M}$ we have

$$L^{S(\mu)}(s, \rho \otimes \mu)L^{S(\mu)}(s, (\rho \otimes \mu)^c) = L^{S(\mu)}(s, \pi \otimes \mu)L^{S(\mu)}(s, (\pi \otimes \mu)^c), \tag{6.1}$$

where $S(\mu)$ abbreviates $S_K(\pi \otimes \mu)$ (see Definition 2.2). Note that $S(\mu)$ is contained in the union of S and the set of primes in K where μ or μ^c is ramified.

For any prime v of K outside S denote by $\{\alpha_v, \beta_v\}$ the set of inverse roots of the Hecke polynomial of π at v and by $\{\gamma_v, \delta_v\}$ the inverse roots of the Frobenius polynomial of ρ at v. We shall regard these as multisets (i.e. sets with multiplicities). By (6.1) for all $\mu \in \mathcal{M}$ unramified at v and v^c we have (as multisets)

$$\{\gamma_v\mu(v), \delta_v\mu(v), \gamma_{v^c}\mu(v^c), \delta_{v^c}\mu(v^c)\} = \{\alpha_v\mu(v), \beta_v\mu(v), \alpha_{v^c}\mu(v^c), \beta_{v^c}\mu(v^c)\}.$$

We need to show that (as multisets)

$$\{\gamma_v, \delta_v\} = \{\alpha_v, \beta_v\} \qquad \text{and} \qquad \{\gamma_{v^c}, \delta_{v^c}\} = \{\alpha_{v^c}, \beta_{v^c}\}. \tag{6.2}$$

If v is inert then the statement is trivial by the existence of some $\mu \in \mathcal{M}$ that is unramified at $v = v^c$.

If v is split then we can find $\mu_1, \mu_2 \in \mathcal{M}$ unramified at v and v^c such that $\mu_1(v) = \mu_1(v^c)$ but $\mu_2(v) \neq \mu_2(v^c)$. It follows that (as multisets)

$$\{\gamma_v, \delta_v, \gamma_{v^c}, \delta_{v^c}\} = \{\alpha_v, \beta_v, \alpha_{v^c}, \beta_{v^c}\} \tag{6.3}$$

and

$$\{\gamma_v, \delta_v, -\gamma_{v^c}, -\delta_{v^c}\} = \{\alpha_v, \beta_v, -\alpha_{v^c}, -\beta_{v^c}\}. \tag{6.4}$$

By forming the sums of both multisets and then adding and subtracting the two resulting equations we conclude that

$$\gamma_v + \delta_v = \alpha_v + \beta_v \qquad \text{and} \qquad \gamma_{v^c} + \delta_{v^c} = \alpha_{v^c} + \beta_{v^c}. \tag{6.5}$$

By forming the reciprocal sums of both multisets and then adding and subtracting the two resulting equations we conclude that

$$\gamma_v^{-1} + \delta_v^{-1} = \alpha_v^{-1} + \beta_v^{-1} \qquad \text{and} \qquad \gamma_{v^c}^{-1} + \delta_{v^c}^{-1} = \alpha_{v^c}^{-1} + \beta_{v^c}^{-1}. \tag{6.6}$$

Let us focus on the left hand sides of (6.5) and (6.6). If the left hand side of (6.6) designates a nonzero common value then we divide by it the left hand side of (6.5) and obtain $\gamma_v\delta_v = \alpha_v\beta_v$. Together with the left hand side of (6.5) this yields $\{\gamma_v, \delta_v\} = \{\alpha_v, \beta_v\}$, hence in fact the entire (6.2) upon using (6.3) again. In the same way (6.2) follows if the right hand side of (6.6) designates a nonzero common value.

We are left with the subtle case when both sides of (6.6) designate zero as common value. Then (6.3) and (6.4) simplify to the same multiset equation

$$\{\gamma_v, -\gamma_v, \gamma_{v^c}, -\gamma_{v^c}\} = \{\alpha_v, -\alpha_v, \alpha_{v^c}, -\alpha_{v^c}\} \tag{6.7}$$

and (6.2) simplifies to

$$\gamma_v^2 = \alpha_v^2 \qquad \text{and} \qquad \gamma_{v^c}^2 = \alpha_{v^c}^2. \tag{6.8}$$

We need to deduce (6.8) from (6.7). Taking squares in (6.7) and halving multiplicities we see that it really is equivalent to the multiset equation

$$\{\gamma_v^2, \gamma_{v^c}^2\} = \{\alpha_v^2, \alpha_{v^c}^2\}. \tag{6.9}$$

Now we observe that in the present situation

$$\alpha_v^2 = -\omega(v) = -\omega(v^c) = \alpha_{v^c}^2,$$

whence in fact (6.9) yields

$$\gamma_v^2 = \gamma_{v^c}^2 = \alpha_v^2 = \alpha_{v^c}^2,$$

so that (6.8) holds as needed.

The proof of Theorem 1.1 is complete.

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References

[1] Arthur, J., and L. Clozel. *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula*. Annals of Mathematics Studies 120. Princeton, NJ: Princeton University Press, 1989.

[2] Bourbaki, N. *Éléments de Mathématique*. Vol. 27. Paris: Hermann, 1958.

[3] Blasius, D., and D. Ramakrishnan. "Galois groups over Q." In *Maass Forms and Galois Representations*. Mathematical Sciences Research Institute Publications 16, 33–77. New York: Springer, 1989.

[4] Clozel, L. *Motifs et Formes Automorphes: Applications du Principe de Fonctorialité, Automorphic Forms, Shimura Varieties, and L-Functions*. Perspectives on Mathematics 10, 77–159. Boston, MA: Academic Press, 1990.

[5] Deligne, P. *Formes Modulaires et Représentations ℓ-adiques*. Lecture Notes in Mathematics 179, 139–72. Berlin: Springer, 1971.

[6] Deligne, P., and J.-P. Serre. "Formes modulaires de poids 1." *Annales Scientifiques de l'Ecole Normale Supérieure* 7 (1974): 507–30.

[7] Friedberg, S., and G. Hoffstein. "Nonvanishing theorems for automorphic L-functions on GL_2." *Annals of Mathematics* 142 (1995): 385–423.

[8] Harris, M., D. Soudry, and R. Taylor. "l-adic representations associated to modular forms over imaginary quadratic fields I. Lifting to $GSp_4(\mathbb{Q})$." *Inventiones Mathematicae* 112 (1993): 377–411.

[9] Laumon, G. "Sur la cohomologie à supports compacts des variétés de Shimura pour $GSp(4)_{\mathbb{Q}}$." *Compositio Mathematica* 105 (1997): 267–359.

[10] Laumon, G. "Fonctions zêtas des variétés de Siegel de dimension trois." *Astérisque* 302 (2005): 1–66.

[11] Roberts, B. "Tempered representations and the theta correspondence." *Canadian Journal of Mathematics-Journal Canadien De Mathematiques* 50 (1998): 1105–18.

[12] Roberts, B. "Nonvanishing of global theta lifts from orthogonal groups." *Journal of the Ramanujan Mathematical Society* 14 (1999): 131–94.

[13] Roberts, B. "Global L-packets for GSp(2) and theta lifts." *Documenta Mathematica* 6 (2001): 247–314.

[14] Serre, J.-P. *Abelian l-adic Representations and Elliptic Curves*. New York: W. A. Benjamin, 1968.

[15] Takeda, S. "Some local-global non-vanishing results for theta lifts from orthogonal groups." (2006): preprint arXiv.org/abs/math/0603072.

[16] Taylor, R. "Galois representations associated to Siegel modular forms of low weight." *Duke Mathematical Journal* 63 (1991): 281–332.

[17] Taylor, R. "On the l-adic cohomology of Siegel threefolds." *Inventiones Mathematicae* 114 (1993): 289–310.

[18] Taylor, R. "l-adic representations associated to modular forms over imaginary quadratic fields. II." *Inventiones Mathematicae* 116 (1994): 619–43.

[19] Taylor, R. "Galois Representations." In *Proceedings of the International Congress of Mathematicians*, 449–74. Vol. 1. Beijing: Higher Education Press, 2002.

[20] Urban, E. "Sur les représentations p-adiques associées aux représentations cuspidales de $\mathrm{GSp}_{4/\mathbb{Q}}$." *Astérisque* 302 (2005): 151–76.

[21] Weissauer, R. "Four dimensional Galois representations." *Astérisque* 302 (2005): 67–150.

[22] Wiles, A. "On ordinary λ-adic representations associated to modular forms." *Inventiones Mathematicae* 94 (1988): 529–73.