ADDENDUM TO "HYBRID BOUNDS FOR TWISTED L-FUNCTIONS"

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The aim of this *addendum* to our paper [BH] is two-fold. Firstly we show that the same method yields a more general version of Theorems 1 and 2 in [BH]. Secondly we correct an error in the proof of Proposition 3 in [BH].

Starting with the former, we observe that Theorems 1 and 2 in [BH] hold almost verbatim without the assumption that f has trivial nebentypus.

Theorem 1'. *Let f be a primitive (holomorphic or Maaß) cusp form of archimedean parameter μ as in* [BH, (1.2)], *level N and arbitrary nebentypus, and let χ be a primitive character modulo q. Then for $\Re s = 1/2$ and for any $\varepsilon > 0$ the twisted L-function $L(f \otimes \chi, s)$ satisfies*

$$L(f \otimes \chi, s) \ll_{\mu,\varepsilon} (N|s|q)^{\varepsilon} N^{\frac{4}{5}} (|s|q)^{\frac{1}{2}-\frac{1}{40}},$$

where the implied constant depends only on ε and μ.

Theorem 2'. *Let f be a primitive (holomorphic or Maaß) cusp form of archimedean parameter μ, level N and arbitrary nebentypus, and let χ be a primitive character modulo q. Then for $\Re s = 1/2$ and for any $\varepsilon > 0$ the twisted L-function satisfies*

$$L(f \otimes \chi, s) \ll_{\varepsilon} \left(|s|^{\frac{1}{4}} |\mu|^{\frac{1}{2}} N^{\frac{1}{4}} (N,q)^{\frac{1}{8}} q^{\frac{3}{8}} + |s|^{\frac{1}{2}} |\mu| N^{\frac{1}{2}} (N,q)^{\frac{1}{4}} q^{\frac{1}{4}} \right) (|s||\mu|Nq)^{\varepsilon}$$

if f is holomorphic, and

$$L(f \otimes \chi, s) \ll_{\varepsilon} \left(|s|^{\frac{1}{4}} (1+|\mu|)^3 N^{\frac{1}{4}} (N,q)^{\frac{1}{8}} q^{\frac{3}{8}} + |s|^{\frac{1}{2}} (1+|\mu|)^{\frac{7}{2}} N^{\frac{1}{2}} (N,q)^{\frac{1}{4}} q^{\frac{1}{4}} \right) (|s|(1+|\mu|)Nq)^{\varepsilon}$$

otherwise.

Remark 1. The factor $(N,q)^{\frac{1}{8}}$ can be improved to $(N,q)^{\frac{1}{8}}/(\frac{N}{N_\psi}, q)^{\frac{1}{4}}$, where $N_\psi \mid N$ is the conductor of the nebentypus. In particular, this factor can be omitted as long as $N_\psi^2 \mid N$, and this recovers [BH, Theorem 2] when $N_\psi = 1$.

Remark 2. Together with the convexity bound, we obtain

$$L(f \otimes \chi, s) \ll_{\varepsilon} (|s|(1+|\mu|)Nq)^{\varepsilon} |s|^{\frac{1}{2}} (1+|\mu|)^3 N^{\frac{1}{2}} q^{\frac{3}{8}}.$$

Recently there has been a lot of interest in such bounds. In [FKM, Section 1.4], Theorem 2' (with an unspecified dependence on f and s and in the case of trivial nebentypus) has been derived from a general result on exponential sums in Hecke eigenvalues of modular forms. Hoffstein [Ho] has devised an alternative method based on double Dirichlet series to derive Theorem 2' for holomorphic f and with an unspecified dependence on f and s. Finally, Munshi [Mu] and Wu [Wu] have improved the exponent in Theorem 1'.

We indicate the minor modifications to prove Theorem 2'. Then Theorem 1' follows as in [BH, Section 8]; we only remark that there is a small typo in [BH, (8.8)]: N_0 should be N.

Let f be a newform of level N and nebentypus character ψ of conductor $N_\psi \mid N$. The case $N_\psi = 1$ corresponds to the situation in [BH]. In [BH, (3.1)] we define

$$D := 3[N, N_\psi q]. \tag{1}$$

Valentin Blomer was supported by the Volkswagen Foundation and ERC Starting Grant 258713. Gergely Harcos was supported by OTKA grants K 101855 and K 72731 and ERC Advanced Grant 228005.

We claim that in [BH, (3.9)] we can prove the slightly stronger bound

(2)
$$\mathcal{Q}_k^{\text{holo}}(\ell) \ll_\varepsilon \left(\frac{1}{\sqrt{\ell}} + \left(\frac{\ell^{\frac14}(\frac{N}{N_\psi},q)^{\frac12}(N,q)^{\frac12}}{q^{\frac12}N^{\frac12}} + \frac{\ell^{\frac12}(\frac{N}{N_\psi},q)(N,q)^{1/2}}{q^{\frac12}N}\right)\left(\frac{1+|\tau|}{k}+1\right)\right)\big((1+|\tau|)D\ell\big)^{\varepsilon}$$

and similarly for $\mathcal{Q}(\ell)$. Then we can proceed with the analysis on [BH, pp. 65–66], with the only change that we use the new value of D in (1) and in the definition of L we replace $(N,q)^{\frac34}$ by $(\frac{N}{N_\psi},q)^{\frac12}(N,q)^{\frac14}$. Hence it remains to show (2).

For $(\ell,D)=1$ we define the twisted divisor sum

$$\alpha_{1/2+i\tau}^{(\chi,\psi)}(\ell) = \sum_{\ell_1\ell_2=\ell}\psi(\ell_1)\chi(\ell_1)\overline{\chi(\ell_2)}\left(\frac{\ell_2}{\ell_1}\right)^{i\tau}$$

and the twisted Kloosterman-type sum

$$S_\psi(m_1,m_2,-\ell;c) := \sum_{\substack{a_1,a_2\,(c)\\ a_1a_2\equiv\ell\,(c)}}\psi(a_2)e\left(\frac{m_1a_1+m_2a_2}{c}\right).$$

If ψ is trivial, this coincides with the definition on [BH, p. 68]. Now [BH, (4.8)] holds in our more general setting if we replace $\alpha_{1/2+i\tau}^{(\chi)}(\ell)$ by $\alpha_{1/2+i\tau}^{(\chi,\psi)}(\ell)$ in the diagonal term and $S(m_1,m_2,-\ell;c)$ by $S_\psi(m_1,m_2,-\ell;c)$ in the off-diagonal term. This follows exactly as in [By, (5.3)], but for the convenience of the reader we provide the computation for the off-diagonal term. The off-diagonal term of the trace formula is

$$\sum_{m_1,m_2}\frac{(m_2/m_1)^{i\tau}}{(m_1m_2)^u}\chi(m_1)\overline{\chi(m_2)}\sum_{D|c}\frac{1}{c}\sum_{d|(\ell,m_1)}\psi(d)S_\psi(m_2,m_1\ell/d^2;c)\phi\left(\frac{4\pi\sqrt{m_1m_2\ell/d^2}}{c}\right).$$

We keep m_1,m_2 fixed, and manipulate the two inner sums over c and d. Since $(\ell,D)=1$, we can write these two sums as

$$\begin{aligned}&\sum_{d|(\ell,m_1)}\sum_{dD|c}\frac{d}{c}\psi(d)S_\psi(m_2,m_1\ell/d^2;c/d)\phi\left(\frac{4\pi\sqrt{m_1m_2\ell/d^2}}{c/d}\right)\\ =&\sum_{D|c}\frac{1}{c}\phi\left(\frac{4\pi\sqrt{m_1m_2\ell}}{c}\right)\sum_{d|(\ell,m_1,c)}d\psi(d)S_\psi(m_2,m_1\ell/d^2;c/d).\end{aligned}$$

We keep c fixed and consider the innermost sum over d. It equals

$$\begin{aligned}&\sum_{d|(\ell,m_1,c)}d\psi(d)\sideset{}{^*}\sum_{a\,(c/d)}\psi(a)e\left(\frac{am_2+\bar{a}m_1\ell/d^2}{c/d}\right)\\ =&\sum_{d|(\ell,m_1,c)}d\sideset{}{^*}\sum_{a\,(c/d)}\sum_{\substack{b\,(c/d)\\ ab\equiv\ell/d\,(c/d)}}\psi(ad)e\left(\frac{adm_2+bm_1}{c}\right)\\ =&\sum_{d|(\ell,m_1,c)}\sum_{\substack{a\,(c)\\(a,c)=d}}\sum_{\substack{b\,(c)\\ ab\equiv\ell\,(c)}}\psi(a)e\left(\frac{am_2+bm_1}{c}\right).\end{aligned}$$

If $d\mid(\ell,c)$ and $(a,c)=d$, then the b-sum vanishes unless $d\mid m_1$. Hence we can drop the condition $d\mid m_1$ in the first sum, because it is automatic. Next, if $(a,c)=d$, then the congruence $ab\equiv\ell\,(c)$ forces $d\mid\ell$, hence we can also drop the condition $d\mid\ell$, so that the previous display equals $S_\psi(m_1,m_2,-\ell;c)$ as desired.

In Step 4 on [BH, p. 69], we apply [By, Lemma 2] to our new situation. Here we see that the display after [By, (1.3)] has an additional factor $\psi(a_2)$, and hence the second display after [By, (1.4)] has an additional factor

$$\psi(-m_2 + c_2 q/N), \tag{3}$$

in the notation of [By]. In the notation of [BH], q/N in (3) equals c/q, which in turn is divisible by D/q. Our definition of D in (1) ensures that (3) is independent of c_2, hence the only change in the display [BH, (4.10)] is an extra factor $\psi(-m_2)$ inside the sum. Now (1) implies that the second line of [BH, (4.12)] becomes

$$(\ell q)^{\varepsilon}\left(\frac{\ell}{q}\right)^{\frac{1}{2}}\left(\frac{1+|\tau|}{a}+1\right)\left(\frac{(\frac{N}{N_\psi},q)(N,q)^{1/2}}{N}+\frac{(\frac{N}{N_\psi},q)^{1/2}(N,q)^{1/2}}{\ell^{\frac{1}{4}}N^{\frac{1}{2}}}\right),$$

and similarly for [BH, (4.13)]. The rest remains unchanged. This proves (2).

Finally, we would like to correct an error in the proof of [BH, Proposition 3]. On [BH, p. 75] it is assumed that V is independent of t. This is a priori not the case. Instead of the approximate functional equation [BH, (2.12)] one should use the uniform approximate functional equation of [BH1, Proposition 1]. This introduces an error of $D^{1/2}T^{-A}$ in (7.2) and the argument goes through as claimed.

We also remark that [BH, (7.3)] holds only on the support of the test function ψ (which is all that is needed), and for the display after [BH, (7.4)] one has to first write V as an inverse Mellin transform to separate variables.

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