ERRATUM TO “TWISTED L-FUNCTIONS OVER NUMBER FIELDS AND HILBERT'S ELEVENTH PROBLEM”

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1. The sentence on lines -3 to -2 of [BH, p. 7] should read as follows: The bundle H is trivial, because any $\varphi(s_0) \in H(s_0)$ extends uniquely to a section $\varphi \in H$ such that the restriction of $\varphi(s)$ to $\mathcal{K}$ is independent of $s \in \mathbb{C}$.

2. Lines -11 to -9 of [BH, p. 11] should read as follows: By [BrMo, §4], the functions $\tilde{W}_{q/2,\nu}$ $(q \in \mathbb{Z})$ for fixed ν and fixed parity $\kappa \in \{0,1\}$ form an orthonormal basis of the Hilbert space $L^2(\mathbb{R}^\times, d^\times y)$ which justifies our normalization:

$$L^2(\mathbb{R}^\times, d^\times y) = \bigoplus_{q \equiv \kappa \pmod 2} \mathbb{C}\tilde{W}_{\frac{q}{2},\nu}, \qquad \langle \tilde{W}_{\frac{q}{2},\nu}, \tilde{W}_{\frac{q'}{2},\nu} \rangle = \delta_{q,q'}. \tag{25}$$

3. On line -7 of [BH, p. 30], we stated incorrectly that any element $g \in \mathrm{GL}_2(\mathbb{A})$ can be written as

$$g = z\tilde{\gamma}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}(\tilde{k}_\infty \times \tilde{k}_{\mathrm{fin}})$$

for some $z \in Z(\mathbb{A})$, $\tilde{\gamma} \in \mathrm{GL}_2(K)$, $\tilde{k}_\infty \times \tilde{k}_{\mathrm{fin}} \in \mathrm{SO}_2(K_\infty) \times \mathcal{K}(\mathfrak{c}_\pi)$, and $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in P(\mathbb{A})$, where $y = y_\infty \times y_{\mathrm{fin}}$ is such that all coordinates of y_∞ exceed δ and y_{fin} takes values from a finite set depending only on K and $\mathfrak{c}_\pi$. Instead, we can only deduce that

$$g = z\tilde{\gamma}\left(\begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} \times h\right)(\tilde{k}_\infty \times \tilde{k}_{\mathrm{fin}}),$$

where $\left(\begin{smallmatrix} y' & x' \\ 0 & 1 \end{smallmatrix}\right) \in P(K_\infty)$ with $y'_1, \dots, y'_d > \delta$, and $h \in \mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$ takes values from a finite set depending only on K and $\mathfrak{c}_\pi$. That is, our mistake was to assume that the matrices h are upper triangular.

As we shall explain below, the weaker statement suffices for the proof of Lemma 5 in [BH]. More precisely, we shall use the theory of newforms to reduce the general case to the upper triangular case for which the proof of Lemma 5 works. For simplicity we shall assume that the central character of π is trivial, this is the only case of Lemma 5 used in [BH].

Let $\phi \in V_\pi(\mathfrak{c}_\pi)$ be arbitrary, not necessarily of pure weight. If g is decomposed as above, then

$$\phi(g) = \psi\left(\begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix}\right),$$

where ψ denotes the right h-translate of ϕ. In particular, ψ is invariant under some fixed congruence subgroup $\left(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}\right)\mathcal{K}(\mathfrak{c})\left(\begin{smallmatrix} 0 & 1 \\ z^{-1} & 0 \end{smallmatrix}\right)$, where $z \in \mathbb{A}_{\mathrm{fin}}^\times$ and $\mathfrak{c} \subseteq \mathfrak{c}_\pi$, cf. [Mi, pp. 177–178]. Now $\psi(g'\left(\begin{smallmatrix} 0 & 1 \\ z^{-1} & 0 \end{smallmatrix}\right))$ as a function of g' is invariant under $\mathcal{K}(\mathfrak{c})$, hence by (14) and (17) we have a unique decomposition

$$\psi\left(g'\begin{pmatrix} 0 & 1 \\ z^{-1} & 0 \end{pmatrix}\right) = \sum_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\pi^{-1}} \psi_{\mathfrak{t}}\left(g'\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right),$$

where $\psi_{\mathfrak{t}} \in V_\pi(\mathfrak{c}_\pi)$ are newforms. In other words,

$$\psi(g') = \sum_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\pi^{-1}} \psi_{\mathfrak{t}}\left(g'\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}\right).$$

We would like to apply the original proof of Lemma 5 to the terms on the right hand side, but the right shifts $\left(\begin{smallmatrix}0&z\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\0&t\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$ are not upper triangular. To remedy this, we use the fact if $c \in \mathbb{A}_{\mathrm{fin}}^\times$ represents $\mathfrak{c}_\pi$, then the right shift by $\left(\begin{smallmatrix}0&1\\c&0\end{smallmatrix}\right)$ is an involutive isometry $W : V_\pi(\mathfrak{c}_\pi) \to V_{\tilde{\pi}}(\mathfrak{c}_\pi)$, cf. [Mi, p. 180]. It follows that the newforms $\tilde{\psi}_{\mathfrak{t}} = W\psi_{\mathfrak{t}} \in V_{\tilde{\pi}}(\mathfrak{c}_\pi)$ satisfy

$$\psi(g') = \sum_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} \tilde{\psi}_{\mathfrak{t}}\left(g'\begin{pmatrix}0&z\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&t\end{pmatrix}\begin{pmatrix}0&1\\c&0\end{pmatrix}\right) = \sum_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} \tilde{\psi}_{\mathfrak{t}}\left(g'\begin{pmatrix}ctz&0\\0&1\end{pmatrix}\right).$$

Now for $g' = \left(\begin{smallmatrix}y'&x'\\0&1\end{smallmatrix}\right)$ as above, we can apply the original proof of Lemma 5 to each term on the right hand side, since $\left(\begin{smallmatrix}ctz&0\\0&1\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})$ is upper triangular and takes values from a finite set depending only on K and $\mathfrak{c}_\pi$. This way we obtain

$$\phi(g) = \psi\left(\begin{pmatrix}y'&x'\\0&1\end{pmatrix}\right) \ll_{\pi,K} \max_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} \|\tilde{\psi}_{\mathfrak{t}}\|_{S^{3d}} = \max_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} \|\psi_{\mathfrak{t}}\|_{S^{3d}}.$$

By the Gram-Schmidt process (38)–(39) we can orthogonalize the components $R_{\mathfrak{t}}\psi_{\mathfrak{t}}$. The transition matrix $(\alpha_{\mathfrak{t},\mathfrak{s}})$ is invertible and depends only on π, therefore the resulting vectors $\psi^{(\mathfrak{t})} \in R^{(\mathfrak{t})}V_\pi(\mathfrak{c}_\pi)$ satisfy

$$\phi(g) \ll_{\pi,K} \max_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} \|\psi_{\mathfrak{t}}\|_{S^{3d}} \ll_{\pi,K} \max_{\mathfrak{t}\mid\mathfrak{c}\mathfrak{c}_\pi^{-1}} \|\psi^{(\mathfrak{t})}\|_{S^{3d}} \leqslant \|\psi\|_{S^{3d}} = \|\phi\|_{S^{3d}}.$$

This is valid for any $g \in \mathrm{GL}_2(\mathbb{A})$, hence the proof of Lemma 5 is complete.

4. In lines -5 to -1 of [BH, p. 32], the ideal classes should be understood in the narrow sense, while the generator γ and the product r_1r_2 should be totally positive. Along with this change, the Kuznetsov formula [BH, (92)] should be corrected as follows: on the left hand side the restriction $\varepsilon_\pi = 1$ should be omitted, and on the right hand side the summation over U/U^2 should be restricted to U^+/U^2. A detailed proof of the corrected formula will appear in [Ma] for a wide class of test functions including the ones we need [BH, (95)]. The proof is similar to what we outlined on [BH, p. 33–35], but the analysis is carried out on the larger space[1]

$$\mathrm{FS} := L^2(\mathrm{GL}_2(K)Z(K_\infty)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K}(c)) = \bigoplus_{\omega\in\widehat{C(K)}} L^2(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K}(\mathfrak{c}),\omega).$$

In particular, whenever we refer to $L^2(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/T\mathcal{K}(\mathfrak{c}),\omega)$ in [BH], it should be understood as $L^2(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A})/\mathcal{K}(\mathfrak{c}),\omega)$. Accordingly, each restriction $\varepsilon_\pi = 1$ or $\varepsilon_\varpi = 1$ should be disregarded in the text, e.g. the notation preceding [BH, Theorem 2] should read

$$\int_{(\mathfrak{c})} f_\varpi\, d\varpi := \sum_{\pi\in\mathcal{C}(\mathfrak{c})} f_\pi + \int_{\varpi\in\mathcal{E}(\mathfrak{c})} f_\varpi\, d\varpi.$$

Then [BH, Lemma 6] and [BH, Theorems 2–3] remain valid, and for the latter we do not need to assume that π_1 and π_2 have the same signature character, cf. [BH, Remarks 11 & 13].

References

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[1] In retrospect, our mistake was to treat the group $\mathrm{O}_2(K_\infty)$ as if it were commutative, leading us to the false belief that the finite subgroup T acts by scalars on any $\pi \in \mathcal{C}(\mathfrak{c})$ and $\varpi \in \mathcal{E}(\mathfrak{c})$.

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