LECTURE NOTES: BOUNDED GAPS BETWEEN PRIMES

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We give an informal discussion of a recent breakthrough result concerning the gaps between primes. We begin with the statement of the famous Twin Prime Conjecture.

Conjecture 1 (Euclid)**.** *The equation $p-p'=2$ has infinitely many solutions in prime numbers p, p'.*

Theorem 1 (Zhang)**.** *There is a constant $H>0$ such that the inequality $2\leqslant p-p'\leqslant H$ has infinitely many solutions in prime numbers p, p'. The value $H=70\,000\,000$ is admissible.*

Theorem 2 (PolyMath8)**.** *In Zhang's theorem, the value $H=4680$ is admissible.*

A useful way to think about Conjecture 1 is as follows: for $\mathscr{H}=\{0,2\}$ there are infinitely many $n\in\mathbb{N}$ such that the translate $n+\mathscr{H}$ consists of primes. The first idea behind Theorem 1 is the expectation that by enlarging $\mathscr{H}$ the translates $n+\mathscr{H}$ will contain more primes, and the task of detecting at least two primes in them becomes more tractable. Enlarging, however, should be done with care. For example, it would not be wise to take $\mathscr{H}=\{0,2,4\}$, because it represents all the residues mod 3, prohibiting three simultaneous primes in any translate $n+\mathscr{H}$ with $n>3$. A better choice is $\mathscr{H}=\{0,2,6\}$ where there is no such congruence obstruction. We call a finite set $\mathscr{H}\subset\mathbb{N}$ admissible if it does not represent all the residues modulo any prime.

Conjecture 2 (Hardy–Littlewood)**.** *Let $\mathscr{H}\subset\mathbb{N}$ be admissible. There are infinitely many $n\in\mathbb{N}$ such that the translate $n+\mathscr{H}$ consists of primes.*

In fact there is a precise quantitative form of this conjecture, supported by heuristic arguments and computer experiments, but we shall not need it here. This brings us to the second idea behind Theorem 1: some translates $n+\mathscr{H}$ probably contain much more primes than the others, so it seems worthwhile to introduce some nonnegative weights $\nu(n)\geqslant 0$ in such a way that $\nu(n)$ is more likely to be large when $n+\mathscr{H}$ is rich in primes. In fact $n+\mathscr{H}$ contains at least two primes infinitely often once we can prove, for any sufficiently large x,

$$\sum_{x\leqslant n\leqslant 2x}\nu(n)\sum_{h\in\mathscr{H}}\theta(n+h)>\log(3x)\sum_{x\leqslant n\leqslant 2x}\nu(n).$$

Here $\theta(p)$ is defined as $\log p$ when p is prime, and 0 otherwise. If $\mathscr{H}$ consists of $k=|\mathscr{H}|$ elements, then it suffices to show

$$\forall h\in\mathscr{H}:\ \sum_{x\leqslant n\leqslant 2x}\nu(n)\theta(n+h)>\frac{1}{k}\log(3x)\sum_{x\leqslant n\leqslant 2x}\nu(n). \tag{1}$$

Theorem 1 (resp. Theorem 2) relies on the validity of (1) for $k=3\,500\,000$ (resp. $k=632$) and some $\nu:\mathbb{N}\to\mathbb{R}_{\geqslant 0}$.

The construction of the original weights $\nu(n)\geqslant 0$ is due to Goldston–Pintz–Yıldırım and is inspired by the work of Selberg. Further refinements were given by Soundararajan, Motohashi–Pintz, Farkas–Pintz–Révész, and the PolyMath8 group. Ideally we would choose $\nu(n)$ to be supported on $n\in\mathbb{N}$ such that $n+\mathscr{H}$ consists of primes. A convenient analytic alternative would be

$$\nu(n):=\sum_{d\mid P(n)}\mu(d)\log^k\left(\frac{P(n)}{d}\right),\qquad P(n):=\prod_{h\in\mathscr{H}}(n+h),$$

which is (nontrivially) nonnegative and supported on n such that $P(n)$ has at most k distinct prime factors. These weights lead to serious complications (e.g. large divisors $d \mid P(n)$ make the error term unmanageable), which is not surprising as the positivity of the left hand side of (1) for these weights would already imply a variant of Conjecture 2. To remedy this, we can introduce a smooth cutoff at $d \approx R$ (to make the error term more manageable), increase k slightly to $k+l$ (to allow more non-primes in $n+\mathscr{H}$), and square the resulting expression (to guarantee nonnegativity):

$$\nu(n) := \left(\sum_{d \mid P(n)} \mu(d) \left(1 - \frac{\log d}{\log R} \right)_+^{k+l} \right)^2, \qquad P(n) := \prod_{h \in \mathscr{H}} (n+h).$$

More generally, we can consider, for any sufficiently smooth $g : \mathbb{R} \to \mathbb{R}$ supported on $[0,1]$,

$$\nu(n) := \left(\sum_{d \mid P(n)} \mu(d) g\left(\frac{\log d}{\log R} \right) \right)^2, \qquad P(n) := \prod_{h \in \mathscr{H}} (n+h). \tag{2}$$

For technical reasons we restrict the weights $\nu(n)$ to integers $n \in \mathbb{N}$ such that $P(n)$ has no prime factor less than $\log\log\log x$. Then the two sides of (1) can be evaluated asymptotically under an assumption that the primes are very evenly distributed with respect to all the square-free moduli $[d_1, d_2] \leqslant R^2$ that come from the d-sum in $\nu(n)$ after squaring out.

Assumption 1. *For any $A > 0$ we have*

$$\sum_{\substack{q \leqslant R^2 \\ q \text{ square-free}}} \max_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \left| \sum_{\substack{x \leqslant n \leqslant 2x \\ n \equiv a \pmod q}} \theta(n) - \frac{x}{\phi(q)} \right| \ll_A \frac{x}{\log^A x}. \tag{3}$$

If we specify R as a fixed power of x, the criterion (1) becomes

$$\frac{\log R}{\log x} > \frac{1}{k(k-1)} \cdot \frac{\int_0^1 g^{(k)}(t)^2 \, t^{k-1} \, dt}{\int_0^1 g^{(k-1)}(t)^2 \, t^{k-2} \, dt}.$$

The minimal value of the second fraction on the right hand side equals $j_{k-2}^2/4$, where $j_{k-2} \approx k + 1.856 k^{1/3}$ is the first zero of the Bessel function J_{k-2}. Hence the above ideas guarantee two simultaneous primes in infinitely many translates of an admissible k-set $\mathscr{H} \subset \mathbb{N}$, assuming (3) holds for some R (a fixed power of x) satisfying

$$\frac{\log R}{\log x} > \frac{j_{k-2}^2}{4k(k-1)}.$$

The right hand side tends to $1/4$ from above, so it seems that in (3) we need to take $R = x^{1/4+\varpi}$ for some fixed $\varpi > 0$, in which case the condition on k boils down to

$$1 + 4\varpi > \frac{j_{k-2}^2}{k(k-1)}. \tag{4}$$

Unfortunately, (3) is only known to hold for R slightly below $x^{1/4}$, this being the famous Bombieri–Vinogradov theorem. To overcome this difficulty, we can modify the definition of $\nu(n)$ in such a way that the arising square-free moduli $q = [d_1, d_2]$ have further special properties. Then (3) is needed only for these more special moduli, and in fact only for residues that come from a fixed integer $a \in \mathbb{Z}$ independent of q:

Assumption 2. *For any $A > 0$, and for any $a \in \mathbb{Z}$ coprime to all the special square-free moduli $q \leqslant x^{1/2+2\varpi}$, we have*

$$\sum_{\substack{q \leqslant x^{1/2+2\varpi} \\ q \text{ special square-free}}} \left| \sum_{\substack{x \leqslant n \leqslant 2x \\ n \equiv a \pmod q}} \theta(n) - \frac{x}{\phi(q)} \right| \ll_A \frac{x}{\log^A x}. \tag{5}$$

In Zhang's paper "special" means "x^δ-smooth", and his arguments allow to deduce (5) from $828\varpi + 172\delta < 1$. In the PolyMath8 paper "special" means "quadruply x^δ-densely divisible", and (5) is proved under $600\varpi + 180\delta < 7$. In both cases there is some price to pay in the criterion (4), namely the left hand side needs to be lowered by an error term depending on ϖ and δ. In the PolyMath8 paper the analysis of the error term is so efficient that the resulting increment in k is only 2 (namely $k = 632$ is admissible, whereas (4) for $\varpi = 7/600$ would only allow $k \geqslant 630$).

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