

# Uniform Approximate Functional Equation for Principal L-Functions

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## 1 Introduction

In their discussion of families of L-functions and the corresponding (sub)convex estimates, Iwaniec and Sarnak [2] introduced the analytic conductor  $C = C(\pi)$  of a cusp form  $\pi$  on  $GL_m$  over a number field  $F$ . As they pointed out, the Phragmén-Lindelöf principle implies that the central value  $L(1/2, \pi)$  is at most  $C^{1/4+\epsilon}$ , an estimate referred to as the convexity or trivial bound. The central value contains important arithmetic information, so it is often very useful (in fact crucial) to replace the exponent  $1/4$  by any smaller value (note that the generalized Lindelöf hypothesis, which in turn is implied by the generalized Riemann hypothesis, asserts that any positive exponent is permissible). It is sometimes possible to realize this improvement by placing the L-function into a family or a fake family and averaging some moment of the corresponding central values with well-chosen weights (called amplifiers). A crucial ingredient in such an argument is to express the central values  $L(1/2, \pi)$  in the family by an “approximate functional equation,” that is, a sum of two Dirichlet series which essentially have  $\sqrt{C}$  terms. For details we refer the reader to [2].

In the present paper, we try to perform the calculation for the entire family of  $\pi$ 's on  $GL_m$  over  $F$  ( $m$  and  $F$  are fixed). First, we obtain an exact representation of the central value with uniform decay properties (Theorem 2.1). This formula is most useful for families whose Archimedean parameters remain bounded. The second representation (Theorem 2.5), inspired by a recent result of Ivić [1], has a more explicit main term

at the cost of an error term. This formula works best in families where the Archimedean parameters simultaneously grow large.

The proofs are based on standard Mellin transform techniques combined with recent progress on the Ramanujan-Selberg conjectures achieved by Luo, Rudnick, and Sarnak [3]. More precisely, we use the bounds of [3] at the Archimedean places to see that the Mellin integrals can be evaluated in a sufficiently large half plane, while the bounds of [3] at the non-Archimedean places enter through the work of Molteni [4] by providing the necessary estimates for the Dirichlet coefficients of the L-functions. A variant of the method yields similar formulae for products of central values (e.g., for higher moments).

## 2 Statement of results

Let  $F$  be a number field of degree  $d$  and  $\pi = \otimes_v \pi_v$  be an irreducible cuspidal automorphic representation of  $GL_m$  over  $F$  with unitary central character and contragradient representation  $\tilde{\pi}$ . The corresponding L-functions are defined for  $\Re s > 1$  by absolutely convergent Dirichlet series as

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad L(s, \tilde{\pi}) = \sum_{n=1}^{\infty} \frac{\bar{a}_n}{n^s}, \tag{2.1}$$

and these are connected by a functional equation of the form

$$N^{s/2} L(s, \pi_{\infty}) L(s, \pi) = \kappa N^{(1-s)/2} L(1-s, \tilde{\pi}_{\infty}) L(1-s, \tilde{\pi}). \tag{2.2}$$

Here  $N$  is the conductor (a positive integer),  $\kappa$  is the root number (of modulus 1) and

$$L(s, \pi_{\infty}) = \prod_{j=1}^{md} \pi^{-s/2} \Gamma\left(\frac{s + \mu_j}{2}\right), \quad L(s, \tilde{\pi}_{\infty}) = \prod_{j=1}^{md} \pi^{-s/2} \Gamma\left(\frac{s + \bar{\mu}_j}{2}\right) \tag{2.3}$$

are the products of the L-functions of  $\pi_v$  and  $\tilde{\pi}_v$ , respectively, at the Archimedean places  $v$ . We define the analytic conductor of  $\pi$  (at  $s = 1/2$ ) as

$$C = \frac{N}{\pi^{md}} \prod_{j=1}^{md} \left| \frac{1}{4} + \frac{\mu_j}{2} \right|. \tag{2.4}$$

(Hopefully no confusion arises from the fact that  $\pi$  denotes both a representation and a real constant. The meaning should be clear from the context.) We have the following uniform approximate functional equation expressing  $L(1/2, \pi)$  as a sum of two Dirichlet series.

**Theorem 2.1.** There is a smooth function  $f : (0, \infty) \rightarrow \mathbb{C}$  and a complex number  $\lambda$  of modulus 1 depending only on the Archimedean parameters  $\mu_j$  ( $j = 1, \dots, md$ ) such that

$$L\left(\frac{1}{2}, \pi\right) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} f\left(\frac{n}{\sqrt{C}}\right) + \kappa\lambda \sum_{n=1}^{\infty} \frac{\bar{a}_n}{\sqrt{n}} \bar{f}\left(\frac{n}{\sqrt{C}}\right). \tag{2.5}$$

The function  $f$  and its partial derivatives  $f^{(k)}$  ( $k = 1, 2, \dots$ ) satisfy the following uniform growth estimates at 0 and infinity:

$$f(x) = \begin{cases} 1 + O_{\sigma}(x^{\sigma}), & 0 < \sigma < \frac{1}{m^2 + 1}; \\ O_{\sigma}(x^{-\sigma}), & \sigma > 0; \end{cases} \tag{2.6}$$

$$f^{(k)}(x) = O_{\sigma,k}(x^{-\sigma}), \quad \sigma > k - \frac{1}{m^2 + 1}. \tag{2.7}$$

The implied constants depend only on  $\sigma, k, m,$  and  $d$ . □

**Remark 2.2.** The range  $0 < \sigma < 1/(m^2 + 1)$  in (2.6) can be widened to  $0 < \sigma < 1/2$  for all representations  $\pi$  which are tempered at  $\infty$ , that is, conjecturally for all  $\pi$ . Similarly, upon the Ramanujan-Selberg conjecture the range of  $\sigma$  in (2.7) can be extended to  $\sigma > k - 1/2$ .

The following corollaries are simple consequences of [Theorem 2.1](#) combined with an average form of the (finite) Ramanujan conjecture recently obtained by Molteni [[4](#), Theorem 4],

$$\sum_{n \leq x} |a_n| = O_{\epsilon, m, d}(x^{1+\epsilon} C^{\epsilon}). \tag{2.8}$$

**Corollary 2.3.** For any positive numbers  $\epsilon$  and  $A$ ,

$$L\left(\frac{1}{2}, \pi\right) = \sum_{n \leq C^{1/2+\epsilon}} \frac{a_n}{\sqrt{n}} f\left(\frac{n}{\sqrt{C}}\right) + \kappa\lambda \sum_{n \leq C^{1/2+\epsilon}} \frac{\bar{a}_n}{\sqrt{n}} \bar{f}\left(\frac{n}{\sqrt{C}}\right) + O_{\epsilon, A}(C^{-A}). \tag{2.9}$$

The implied constant depends only on  $\epsilon, A, m,$  and  $d$ . □

**Corollary 2.4.** For any  $\epsilon > 0$ , there is a uniform convexity bound

$$L(1/2, \pi) \ll_{\epsilon} C^{1/4+\epsilon}. \tag{2.10}$$

The implied constant depends only on  $\epsilon, m,$  and  $d$ . □

In a family of representations  $\pi$ , it is often desirable to see that the weight functions  $f$  do not vary too much. In fact, assuming that the Archimedean parameters are not too small, we can replace  $f$  by an explicit function  $g$  (independent of  $\pi$ ) and derive an approximate functional equation with a nontrivial error term, that is, an error substantially smaller than the convexity bound furnished by [Corollary 2.4](#). To state the result we introduce

$$\eta = \min_{j=1, \dots, md} \left| \frac{1}{4} + \frac{\mu_j}{2} \right|. \tag{2.11}$$

**Theorem 2.5.** Let  $g : (0, \infty) \rightarrow \mathbb{R}$  be a smooth function with the functional equation  $g(x) + g(1/x) = 1$  and derivatives decaying faster than any negative power of  $x$  as  $x \rightarrow \infty$ . Then, for any  $\epsilon > 0$ ,

$$L\left(\frac{1}{2}, \pi\right) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} g\left(\frac{n}{\sqrt{C}}\right) + \kappa\lambda \sum_{n=1}^{\infty} \frac{\bar{a}_n}{\sqrt{n}} g\left(\frac{n}{\sqrt{C}}\right) + O_{\epsilon, g}(\eta^{-1} C^{1/4+\epsilon}), \tag{2.12}$$

where  $\lambda$  (of modulus 1) is given by [\(3.4\)](#) and the implied constant depends only on  $\epsilon$ ,  $g$ ,  $m$ , and  $d$ . □

**Remark 2.6.** This should be compared with the main result of Ivić [\[1\]](#) (in the light of the next remark). It is apparent that the formula is really of value when the family under consideration satisfies  $\eta \gg C^\delta$  with some fixed  $\delta > 0$ .

**Remark 2.7.** We obtain similar expressions for any value  $L(1/2 + it, \pi)$  on the critical line by twisting  $\pi$  with the one-dimensional representation  $|\det|^{it}$ . Under the twist the conductor remains  $N$ , the root number becomes  $\kappa N^{-it}$ , and the Archimedean parameters change to  $\mu_j + it$  ( $j = 1, \dots, md$ ). Accordingly, the analytic conductor  $C$  needs to be adjusted as well. Also, similar formulae hold when  $\pi$  is not irreducible but a tensor product of finitely many irreducible representations. This observation might be useful for studying higher moments of the central values in families.

### 3 Proof of Theorem 2.1

By a result of Luo, Rudnick, and Sarnak [\[3, Theorem 1\]](#),

$$\Re \mu_j \geq \frac{1}{m^2 + 1} - \frac{1}{2}, \quad j = 1, \dots, md. \tag{3.1}$$

(The Ramanujan-Selberg conjecture asserts that  $\pi_\infty$  is tempered, upon which we could replace the right-hand side by 0.) Therefore the function

$$F(s, \pi_\infty) = \left\{ \mathbf{N}^s \frac{L\left(\frac{1}{2} + s, \pi_\infty\right)L\left(\frac{1}{2}, \tilde{\pi}_\infty\right)}{L\left(\frac{1}{2} - s, \tilde{\pi}_\infty\right)L\left(\frac{1}{2}, \pi_\infty\right)} \right\}^{1/2} \tag{3.2}$$

is holomorphic in the half plane  $\Re s > -1/(m^2 + 1)$ . With this notation we can rewrite the functional equation (2.2) as

$$F(s, \pi_\infty)L\left(\frac{1}{2} + s, \pi\right) = \kappa\lambda F(-s, \tilde{\pi}_\infty)L\left(\frac{1}{2} - s, \tilde{\pi}\right), \tag{3.3}$$

where

$$\lambda = \frac{L\left(\frac{1}{2}, \tilde{\pi}_\infty\right)}{L\left(\frac{1}{2}, \pi_\infty\right)}. \tag{3.4}$$

Note that  $F(0, \pi_\infty) = 1$  and

$$\bar{F}(s, \pi_\infty) = F(\bar{s}, \tilde{\pi}_\infty) \tag{3.5}$$

follow from definitions (3.2) and (2.3). Similarly,  $\lambda$  is of modulus 1.

We also fix an entire function  $H(s)$  which satisfies the growth estimate

$$H(s) \ll_{\sigma, A} (1 + |s|)^{-A}, \quad \Re s = \sigma; \tag{3.6}$$

on vertical lines. In addition, we assume that  $H(0) = 1$  and  $H(s)$  is symmetric with respect to both axes,

$$H(s) = H(-s) = \bar{H}(\bar{s}). \tag{3.7}$$

Such a function can be obtained as the Mellin transform of a smooth function

$$h : (0, \infty) \longrightarrow \mathbb{R} \tag{3.8}$$

with a total mass 1 with respect to the measure  $dx/x$ , functional equation  $h(1/x) = h(x)$  and derivatives decaying faster than any negative power of  $x$  as  $x \rightarrow \infty$ :

$$H(s) = \int_0^\infty h(x)x^s \frac{dx}{x}. \tag{3.9}$$

Using those two auxiliary functions and taking an arbitrary  $0 < \sigma < 1/(m^2 + 1)$ , we can express the central value  $L(1/2, \pi)$  via the residue theorem as

$$L\left(\frac{1}{2}, \pi\right) = \frac{1}{2\pi i} \int_{(\sigma)} L\left(\frac{1}{2} + s, \pi\right) F(s, \pi_\infty) H(s) \frac{ds}{s} - \frac{1}{2\pi i} \int_{(-\sigma)} L\left(\frac{1}{2} + s, \pi\right) F(s, \pi_\infty) H(s) \frac{ds}{s}. \tag{3.10}$$

Here we combined inequality (3.6), Lemma 3.1 below, and the fact that  $L(1/2 + s, \pi)$  grows moderately on the lines  $\Re s = \pm\sigma$ . The last property follows from the Phragmén-Lindelöf principle. Applying a change of variable  $s \mapsto -s$  in the second integral, we get, by the functional equations (3.3) and (3.7),

$$L\left(\frac{1}{2}, \pi\right) = \frac{1}{2\pi i} \int_{(\sigma)} L\left(\frac{1}{2} + s, \pi\right) F(s, \pi_\infty) H(s) \frac{ds}{s} + \frac{\kappa\lambda}{2\pi i} \int_{(\sigma)} L\left(\frac{1}{2} + s, \tilde{\pi}\right) F(s, \tilde{\pi}_\infty) H(s) \frac{ds}{s}. \tag{3.11}$$

By another change of variable  $s \mapsto \bar{s}$  in the second integral we observe, using (2.1), (3.5), and (3.7), that this integral is the complex conjugate of the first one. Therefore, we obtain the representation (2.5) of Theorem 2.1 by defining

$$f\left(\frac{x}{\sqrt{C}}\right) = \frac{1}{2\pi i} \int_{(\sigma)} x^{-s} F(s, \pi_\infty) H(s) \frac{ds}{s}. \tag{3.12}$$

Then, for any nonnegative integer  $k$  we also have

$$f^{(k)}(x) = \frac{(-1)^k}{2\pi i} \int_{(\sigma)} x^{-s-k} C^{-s/2} F(s, \pi_\infty) H(s) s(s+1) \cdots (s+k-1) \frac{ds}{s}. \tag{3.13}$$

When  $k = 0$ , the integrand in this expression is holomorphic for  $\Re s > -1/(m^2 + 1)$  with the exception of a simple pole at  $s = 0$  with residue 1. So in this case we are free to move the line of integration to any nonzero  $\sigma > -1/(m^2 + 1)$ , but negative  $\sigma$ 's will pick up an additional value 1 from the pole at  $s = 0$ . When  $k > 0$ , the integrand is holomorphic in the entire half plane  $\Re s > -1/(m^2 + 1)$ , so the line of integration can be shifted to any  $\sigma > -1/(m^2 + 1)$  without changing the value of the integral. Henceforth, by (3.6) and (3.13), the truth of inequalities (2.6) and (2.7) are reduced to the following.

**Lemma 3.1.** For any  $\sigma > -1/(m^2 + 1)$ , there is a uniform bound

$$C^{-s/2} F(s, \pi_\infty) \ll_\sigma (1 + |s|)^{m d \sigma / 2}, \quad \Re s = \sigma. \tag{3.14}$$

The implied constant depends only on  $\sigma, m$ , and  $d$ . □

We start with the following simple estimate.

**Lemma 3.2.** For any  $\alpha > -\sigma$ , there is a uniform bound

$$\frac{\Gamma(z + \sigma)}{\Gamma(z)} \ll_{\alpha, \sigma} |z + \sigma|^\sigma, \quad \Re z \geq \alpha. \tag{3.15}$$

□

Proof of **Lemma 3.2.** The function  $\Gamma(z + \sigma)/\Gamma(z)$  is holomorphic in a neighborhood of  $\Re z \geq \alpha$ . For  $|z| > 2|\sigma|$  we get, using Stirling’s formula,

$$\frac{\Gamma(z + \sigma)}{\Gamma(z)} \ll_\sigma \left| \frac{(z + \sigma)^{z + \sigma - 1/2}}{z^{z - 1/2}} \right| \ll_\sigma |z + \sigma|^\sigma. \tag{3.16}$$

The rest of the values of  $z$  (i.e., those with  $\Re z \geq \alpha$  and  $|z| \leq 2|\sigma|$ ) form a compact set, so for these we simply have

$$\frac{\Gamma(z + \sigma)}{\Gamma(z)} \ll_{\alpha, \sigma} 1 \ll_{\alpha, \sigma} |z + \sigma|^\sigma. \tag{3.17}$$

■

Proof of **Lemma 3.1.** For  $\sigma = 0$  the statement trivially follows from definitions (2.3) and (3.2). So let  $s = \sigma + it$  where  $\sigma \neq 0$ . Pick any  $j = 1, \dots, md$  and apply **Lemma 3.2** with

$$\alpha = \frac{1}{2(m^2 + 1)} - \frac{\sigma}{2}, \quad z = \frac{1}{4} + \frac{\mu_j}{2} - \frac{\sigma}{2} + \frac{it}{2}, \tag{3.18}$$

to yield

$$\frac{\Gamma\left(\frac{1}{4} + \frac{\mu_j}{2} + \frac{\sigma}{2} + \frac{it}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{\mu_j}{2} - \frac{\sigma}{2} + \frac{it}{2}\right)} \ll_{\sigma, m} \left| \frac{1}{4} + \frac{\mu_j}{2} + \frac{\sigma}{2} + \frac{it}{2} \right|^\sigma. \tag{3.19}$$

This is the same as

$$\frac{\Gamma\left(\frac{1}{4} + \frac{\mu_j}{2} + \frac{s}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{\mu_j}{2} - \frac{s}{2}\right)} \ll_{\sigma, m} \left| \frac{1}{4} + \frac{\mu_j}{2} + \frac{s}{2} \right|^\sigma. \tag{3.20}$$

Observe that by  $|s| \geq \sigma$  and  $|1/4 + \mu_j/2| \gg_m 1$  (see (3.1)) we have, on the right-hand side,

$$\left| \frac{1}{4} + \frac{\mu_j}{2} + \frac{s}{2} \right| \ll_{\sigma, m} \left| \frac{1}{4} + \frac{\mu_j}{2} \right| |s|. \tag{3.21}$$

Therefore by taking a product over all  $j = 1, \dots, md$  we get, using (2.3) and (2.4),

$$\left| \pi^{m ds} \frac{L\left(\frac{1}{2} + s, \pi_\infty\right)}{L\left(\frac{1}{2} - s, \tilde{\pi}_\infty\right)} \right|^{1/2} \ll_{\sigma, m, d} \left( \frac{\pi^{md} C}{N} \right)^{\sigma/2} |s|^{md\sigma/2}, \quad \Re s = \sigma. \tag{3.22}$$

By (3.2), this is equivalent to (3.14), completing the proof of Lemma 3.1 and Theorem 2.1. ■

#### 4 Proof of Theorem 2.5

We can assume that  $H(s)$  is the Mellin transform of  $h(x) = -xg'(x)$ . Indeed,  $h : (0, \infty) \rightarrow \mathbb{R}$  is a smooth function with functional equation  $h(1/x) = h(x)$  and derivatives decaying faster than any negative power of  $x$  as  $x \rightarrow \infty$ , therefore  $H(s)$  is entire and satisfies (3.6) and (3.7). Also,

$$H(0) = - \int_0^\infty g'(x) dx = g(0+) = 1. \tag{4.1}$$

Equivalently,  $H(s)/s$  is the Mellin transform of  $g(x)$ , because by partial integration it follows that

$$- \int_0^\infty g'(x) x^s dx = s \int_0^\infty g(x) x^s \frac{dx}{x}. \tag{4.2}$$

In any case,  $g(x)$  can be expressed as an inverse Mellin transform

$$g(x) = \frac{1}{2\pi i} \int_{(\sigma)} x^{-s} H(s) \frac{ds}{s}. \tag{4.3}$$

The idea is to compare  $g(x)$  with the function  $f(x)$  given by (3.12). We have, for any  $\sigma > 0$ ,

$$f(x) - g(x) = \frac{1}{2\pi i} \int_{(\sigma)} x^{-s} \{ C^{-s/2} F(s, \pi_\infty) - 1 \} H(s) \frac{ds}{s}. \tag{4.4}$$

In fact, the integrand is holomorphic in the entire half plane  $\Re s > -1/(m^2 + 1)$ , so the line of integration can be shifted to any  $\sigma > -1/(m^2 + 1)$  without changing the value of the integral. In particular, the choice  $\sigma = 0$  is permissible, that is,

$$f(x) - g(x) = \frac{1}{2\pi i} \int_{-\infty}^\infty x^{-it} \{ C^{-it/2} F(it, \pi_\infty) - 1 \} H(it) \frac{dt}{t}. \tag{4.5}$$

Note that  $\chi^{-it}$  and  $C^{-it/2}F(it, \pi_\infty)$  are of modulus 1. For any  $\epsilon > 0$ , the values of  $t$  with  $|t| \geq \min(\eta, C^\epsilon)$  contribute  $O_{\epsilon, g, m, d}(\eta^{-1})$  to the integral. This follows from (3.6) and  $\eta \ll C^{1/md}$ . We estimate the remaining contribution via the following lemma.

**Lemma 4.1.** For any  $\epsilon > 0$ , there is a uniform bound

$$C^{-it/2}F(it, \pi_\infty) - 1 \ll_\epsilon |t|\eta^{-1}C^\epsilon, \quad |t| < \min(\eta, C^\epsilon). \tag{4.6}$$

The implied constant depends only on  $\epsilon, m$ , and  $d$ . □

*Proof.* As  $C^{-it/2}F(it, \pi_\infty)$  lies on the unit circle, it suffices to show that

$$\log \{C^{-it/2}F(it, \pi_\infty)\} \ll_{\epsilon, m, d} |t|\eta^{-1}C^\epsilon, \quad |t| < \min(\eta, C^\epsilon). \tag{4.7}$$

Here, the left-hand side is understood as a continuous function defined via the principal branch of the logarithm near  $t = 0$ . Using (2.3), (2.4), and (3.2) we can see that the derivative (with respect to  $t$ ) of the left-hand side is given by

$$\frac{i}{2} \Re \sum_{j=1}^{m,d} \left\{ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{\mu_j}{2} + \frac{it}{2} \right) - \log \left( \frac{1}{4} + \frac{\mu_j}{2} \right) \right\}, \tag{4.8}$$

so we can further reduce the lemma to

$$\frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{\mu_j}{2} + \frac{it}{2} \right) - \log \left( \frac{1}{4} + \frac{\mu_j}{2} \right) = O_{\epsilon, m, d}(\eta^{-1}C^\epsilon), \quad |t| < \min(\eta, C^\epsilon). \tag{4.9}$$

Here  $1/4 + \mu_j/2 + it/2$  has real part  $\gg_m 1$  by (3.1) and absolute value at least  $\eta/2$  by (2.11). Therefore, a standard bound yields

$$\frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{\mu_j}{2} + \frac{it}{2} \right) = \log \left( \frac{1}{4} + \frac{\mu_j}{2} + \frac{it}{2} \right) + O_m(\eta^{-1}). \tag{4.10}$$

For  $|t| < \min(\eta, C^\epsilon)$  we can also see that

$$\log \left( \frac{1}{4} + \frac{\mu_j}{2} + \frac{it}{2} \right) = \log \left( \frac{1}{4} + \frac{\mu_j}{2} \right) + O(\eta^{-1}C^\epsilon). \tag{4.11}$$

It follows from (3.1) that  $C \gg_{m, d} 1$ , therefore the last two estimates add up to (4.9) as required. ■

Returning to the integral (4.5), it follows from Lemma 4.1 that the values of  $t$  with  $|t| < \min(\eta, C^\epsilon)$  contribute at most  $O_{\epsilon, g, m, d}(\eta^{-1}C^{2\epsilon})$ . Altogether we have, by  $C \gg_{m, d} 1$ ,

$$f(x) - g(x) = O_{\epsilon, g, m, d}(\eta^{-1}C^{2\epsilon}). \tag{4.12}$$

We conclude [Theorem 2.5](#) by combining this estimate with [Corollary 2.3](#) and Molteni's bound [\(2.8\)](#).

## References

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