ADDENDUM TO "A BURGESS-LIKE SUBCONVEX BOUND FOR TWISTED L-FUNCTIONS"

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1. We present a detailed proof of (3.12) in [BHM07], starting from the formula

$$R_{\psi^*}(m;w''):=\prod_{\substack{p^\alpha\| w''\\ p^\alpha\| D}}\left(\sum_{\beta\geqslant 0}\frac{\psi^{*2}(p^\beta)}{p^{\beta(1+2it)}}r\big(p^{v_p(m)};p^{\alpha+\beta}\big)\right).$$

We will calculate explicitly the factors on the right hand side:

$$R_{\psi^*,p^\alpha}(m;w''):=\sum_{\beta\geqslant 0}\frac{\psi^{*2}(p^\beta)}{p^{\beta(1+2it)}}r\big(p^{v_p(m)};p^{\alpha+\beta}\big).$$

The Ramanujan sums satisfy

$$r(m;q)=\sum_{d\mid(m,q)}d\mu(q/d),$$

therefore we have

$$r\big(p^{v_p(m)};p^\gamma\big)=\begin{cases}0, & \gamma\geqslant v_p(m)+2;\\ -p^{v_p(m)}, & \gamma=v_p(m)+1;\\ p^\gamma-p^{\gamma-1}, & 1\leqslant\gamma\leqslant v_p(m).\end{cases}$$

Using the notation

$$B_\kappa:=\sum_{0\leqslant\beta\leqslant\kappa}\frac{\psi^{*2}(p^\beta)}{p^{2it\beta}}$$

we obtain

$$R_{\psi^*,p^\alpha}(m;w'')=\begin{cases}0, & \alpha\geqslant v_p(m)+2;\\ -p^{v_p(m)}, & \alpha=v_p(m)+1;\\ p^\alpha B_{v_p(m)-\alpha}-p^{\alpha-1}B_{v_p(m)-\alpha+1}, & 1\leqslant\alpha\leqslant v_p(m).\end{cases}\tag{1}$$

In particular,

$$\big|R_{\psi^*,p^\alpha}(m;w'')\big|\leqslant\begin{cases}0, & \alpha\geqslant v_p(m)+2;\\ p^{v_p(m)}, & \alpha=v_p(m)+1;\\ \big(v_p(m)+1\big)(p^\alpha+p^{\alpha-1}), & 1\leqslant\alpha\leqslant v_p(m).\end{cases}$$

It is now clear that

$$\big|R_{\psi^*}\big(q;w''_q\big)\big|\leqslant\prod_{\substack{p^\alpha\| w''_q\\ p^\alpha\| D\\ \alpha\leqslant v_p(q)+1}}\big(v_p(q)+1\big)\left(1+\frac{1}{p}\right)p^{\min(v_p(q),\alpha)}.$$

We specify $\tilde{v}_q$ in such a way that

$$p^\alpha\| w''_q\quad\text{and}\quad\alpha\leqslant v_p(q)+1\qquad\Longleftrightarrow\qquad p^\alpha\|\tilde{v}_q.$$

That is,

$$\tilde{v}_q := \prod_{\substack{p^\alpha \| w''_q \\ \alpha \leqslant v_p(q)+1}} p^\alpha.$$

In addition, we define v_q as

$$v_q := \left(\frac{w''}{(w'',2)}, q \right).$$

Let us now consider an arbitrary $p^\alpha \| \tilde{v}_q$.

Case 1. If $\alpha = v_p(q)+1$, then $v_p(v_q) = v_p(q)$, $\alpha = v_p(v_q)+1$, therefore by (1)

$$\left| R_{\psi^*,p^\alpha}(v_q;\tilde{v}_q) \right| = p^{v_p(v_q)} = p^{\min(v_p(q),\alpha)}.$$

Case 2. If $\alpha \leqslant v_p(q)$ and $p=2$, then $v_p(v_q) = \alpha - 1$, therefore by (1)

$$\left| R_{\psi^*,p^\alpha}(v_q;\tilde{v}_q) \right| = p^{v_p(v_q)} = \frac{1}{2}\, p^{\min(v_p(q),\alpha)}.$$

Case 3. If $\alpha \leqslant v_p(q)$ and $p>2$, then $v_p(v_q) = \alpha$, therefore by (1)

$$\left| R_{\psi^*,p^\alpha}(v_q;\tilde{v}_q) \right| = \left| p^\alpha - p^{\alpha-1}\left(1 + \frac{\psi^{*2}(p)}{p^{2it}}\right) \right| \geqslant \left(1 - \frac{2}{p}\right) p^{\min(v_p(q),\alpha)}.$$

It is now clear that

$$\left| R_{\psi^*}(v_q;\tilde{v}_q) \right| \geqslant \prod_{\substack{p^\alpha \| w''_q \\ p^\alpha \| D \\ \alpha \leqslant v_p(q)+1}} \kappa(p) p^{\min(v_p(q),\alpha)},$$

where

$$\kappa(p) := \begin{cases} \frac{1}{2}, & p=2; \\ 1 - \frac{2}{p}, & p>2. \end{cases}$$

It follows that

$$\left| \frac{R_{\psi^*}(q;w''_q)}{R_{\psi^*}(v_q;\tilde{v}_q)} \right| \leqslant \prod_{p \mid (q,D)} \left(v_p(q)+1\right) \frac{1+\frac{1}{p}}{\kappa(p)} \leqslant 3\tau(D)\tau(q).$$

2. We update Lemma 7.1 in [BHM07] and its proof. The lemma holds in the stronger form that $N \mid (4M)^2$. In the proof, the sentence "We see from Section 1.3.2 of [Wa81] etc." should be replaced by: "We know from the work of Niwa [Ni75] and Cipra [Ci83] that the newform corresponding to the representation $\pi \otimes \chi\chi_{-4}^k$ has level dividing $2M$, that is, $c(\pi_p \otimes \chi\chi_{-4}^k) \leqslant v_p(2M)$." At the end of the proof, $N \leqslant (4M)^2$ should be replaced by $N \mid (4M)^2$.

References

[BHM07] V. Blomer, G. Harcos, P. Michel, *A Burgess-like subconvex bound for twisted L-functions (with Appendix 2 by Z. Mao)*, Forum Math. **19** (2007), 61–105.

[Ci83] B. Cipra, *On the Niwa-Shintani theta-kernel lifting of modular forms*, Nagoya Math. J. **91** (1983), 49–117.

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