

**ADDENDUM TO “A BURGESS-LIKE SUBCONVEX BOUND FOR TWISTED
L-FUNCTIONS”**

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We present a detailed proof of (3.12) in [BHM], starting from the formula

$$R_{\psi^*}(m; w'') := \prod_{\substack{p^\alpha \| w'' \\ p^\alpha \| D}} \left(\sum_{\beta \geq 0} \frac{\psi^{*2}(p^\beta)}{p^{\beta(1+2it)}} r(p^{v_p(m)}; p^{\alpha+\beta}) \right).$$

We will calculate explicitly the factors on the right hand side:

$$R_{\psi^*, p^\alpha}(m; w'') := \sum_{\beta \geq 0} \frac{\psi^{*2}(p^\beta)}{p^{\beta(1+2it)}} r(p^{v_p(m)}; p^{\alpha+\beta}).$$

The Ramanujan sums satisfy

$$r(m; q) = \sum_{d|(m, q)} d\mu(q/d),$$

therefore we have

$$r(p^{v_p(m)}; p^\gamma) = \begin{cases} 0, & \gamma \geq v_p(m) + 2; \\ -p^{v_p(m)}, & \gamma = v_p(m) + 1; \\ p^\gamma - p^{\gamma-1}, & 1 \leq \gamma \leq v_p(m). \end{cases}$$

Using the notation

$$B_\kappa := \sum_{0 \leq \beta \leq \kappa} \frac{\psi^{*2}(p^\beta)}{p^{2it\beta}}$$

we obtain

$$(1) \quad R_{\psi^*, p^\alpha}(m; w'') = \begin{cases} 0, & \alpha \geq v_p(m) + 2; \\ -p^{v_p(m)}, & \alpha = v_p(m) + 1; \\ p^\alpha B_{v_p(m)-\alpha} - p^{\alpha-1} B_{v_p(m)-\alpha+1}, & 1 \leq \alpha \leq v_p(m). \end{cases}$$

In particular,

$$|R_{\psi^*, p^\alpha}(m; w'')| \leq \begin{cases} 0, & \alpha \geq v_p(m) + 2; \\ p^{v_p(m)}, & \alpha = v_p(m) + 1; \\ (v_p(m) + 1)(p^\alpha + p^{\alpha-1}), & 1 \leq \alpha \leq v_p(m). \end{cases}$$

It is now clear that

$$|R_{\psi^*}(q; w''_q)| \leq \prod_{\substack{p^\alpha \| w''_q \\ p^\alpha \| D \\ \alpha \leq v_p(q)+1}} (v_p(q) + 1) \left(1 + \frac{1}{p}\right) p^{\min(v_p(q), \alpha)}.$$

We specify \tilde{v}_q in such a way that

$$p^\alpha \| w''_q \quad \text{and} \quad \alpha \leq v_p(q) + 1 \quad \iff \quad p^\alpha \| \tilde{v}_q.$$

That is,

$$\tilde{v}_q := \prod_{\substack{p^\alpha \| w_q'' \\ \alpha \leq v_p(q)+1}} p^\alpha.$$

In addition, we define v_q as

$$v_q := \left(\frac{w''}{(w'', 2)}, q \right).$$

Let us now consider an arbitrary $p^\alpha \| \tilde{v}_q$.

Case 1. If $\alpha = v_p(q) + 1$, then $v_p(v_q) = v_p(q)$, $\alpha = v_p(v_q) + 1$, therefore by (1)

$$|R_{\psi^*, p^\alpha}(v_q; \tilde{v}_q)| = p^{v_p(v_q)} = p^{\min(v_p(q), \alpha)}.$$

Case 2. If $\alpha \leq v_p(q)$ and $p = 2$, then $v_p(v_q) = \alpha - 1$, therefore by (1)

$$|R_{\psi^*, p^\alpha}(v_q; \tilde{v}_q)| = p^{v_p(v_q)} = \frac{1}{2} p^{\min(v_p(q), \alpha)}.$$

Case 3. If $\alpha \leq v_p(q)$ and $p > 2$, then $v_p(v_q) = \alpha$, therefore by (1)

$$|R_{\psi^*, p^\alpha}(v_q; \tilde{v}_q)| = \left| p^\alpha - p^{\alpha-1} \left(1 + \frac{\psi^{*2}(p)}{p^{2it}} \right) \right| \geq \left(1 - \frac{2}{p} \right) p^{\min(v_p(q), \alpha)}.$$

It is now clear that

$$|R_{\psi^*}(v_q; \tilde{v}_q)| \geq \prod_{\substack{p^\alpha \| w_q'' \\ p^\alpha \| D \\ \alpha \leq v_p(q)+1}} \kappa(p) p^{\min(v_p(q), \alpha)},$$

where

$$\kappa(p) := \begin{cases} \frac{1}{2}, & p = 2; \\ 1 - \frac{2}{p}, & p > 2. \end{cases}$$

It follows that

$$\left| \frac{R_{\psi^*}(q; w_q'')}{R_{\psi^*}(v_q; \tilde{v}_q)} \right| \leq \prod_{p|(q, D)} (v_p(q) + 1) \frac{1 + \frac{1}{p}}{\kappa(p)} \leq 3\tau(D)\tau(q).$$

REFERENCES

- [BHM] V. Blomer, G. Harcos, P. Michel, *A Burgess-like subconvex bound for twisted L-functions (with Appendix 2 by Z. Mao)*, Forum Math. **19** (2007), 61–105.