Sharing Joints, in Moderation
A Grounshaking Clash between
Algebraic and Combinatorial Geometry

Micha Sharir

Tel Aviv University

Joint work with
György Elekes (in memoriam), Haim Kaplan, and Eugenii Shustin

- A nice, but not so major problem in combinatorial geometry

- Open for nearly 20 years

- Completely solved [Guth-Katz 2008]
By a new, algebraic geometry machinery

- Which we “trivialize”, and extend in many ways

- The beginning of a long and beautiful friendship

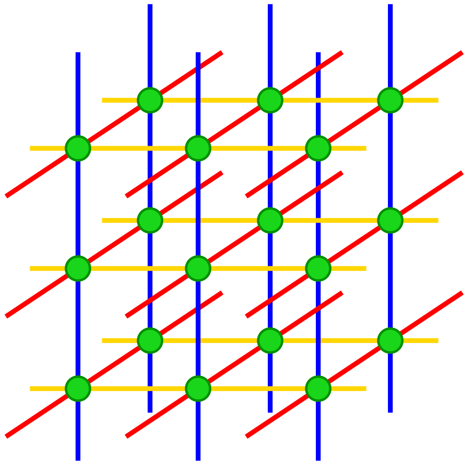
Joints (in 3-space)

L – Set of n lines in $\mathbb{R}^3$

Joint: Point incident to three non-coplanar lines of L

The Joints Problem. Show:
The number of joints in L is $O(n^{3/2})$

Worst-case tight:



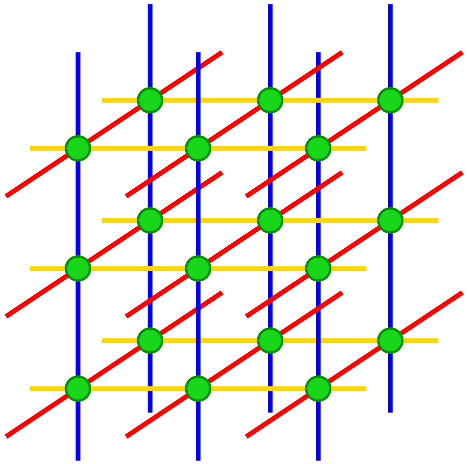
Joints in d dimensions

L – Set of n lines in $\mathbb{R}^d$

Joint: Point incident to d lines of L,
not all in a common hyperplane

Show: The number of joints in L is $O(n^{d/(d-1)})$

Again, worst-case tight (same grid construction)



The Joints Problem

Long (but sparse) history (exclusively in 3D):

$O(n^{7/4}) = O(n^{1.75})$ joints
[Chazelle et al. 1992]

$O(n^{23/14}) = O(n^{1.643})$ joints
[Sharir 1994]

$O^*(n^{112/69}) = O(n^{1.6232})$ joints
[Feldman, Sharir 2005]

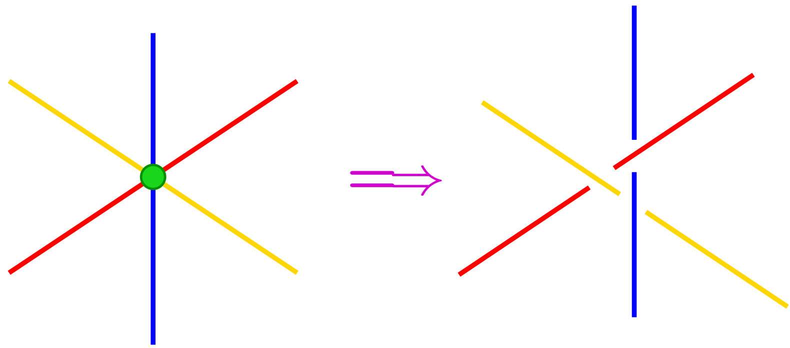
$O^*(n^{3/2+\varepsilon}/\theta^{1/2+\varepsilon})$ "θ-fat" joints
[Bennett, Carbery, Tao 2005]

$O(n^{5/3})$ line-joint incidences
[Sharir, Welzl 2004]

The Joints Problem

Mildly related to cycles in the “depth order” of the lines:

Small perturbation of the lines turns a joint into a cycle



(Much harder problem; limited progress;
Arises in spatial visibility problems)

The Joints Problem

Older bounds obtained by “traditional” methods:

Forbidden subgraphs
Space decomposition
Plücker coordinates
Duality of sorts,

but no real progress ...

The Joints Problem

And then a miracle happened...

[L. Guth and N. H. Katz, Algebraic methods in discrete analogs of the Kakeya problem, arXiv:0812.1043v1, 4 Dec 2008]

A new algebraic proof technique, solving the 3D problem:

The number of joints in a set of n lines in 3-space is $O(n^{3/2})$

- Proof uses basic tools from algebraic geometry

(E.g., Bézout's Theorem)

- Somewhat involved

- We "trivialize" it, and extend it to any $d \geq 3$:

Theorem: The number of joints in a set of n lines in $\mathbb{R}^d$ is $O(n^{d/(d-1)})$

Note added in press:

In an unbelievable development:

Theorem: The number of joints in a set of n lines in $\mathbb{R}^d$ is $O(n^{d/(d-1)})$

[Kaplan, Sharir, Shustin]
On lines and joints
arXiv:0906.0558, posted June 2, 2009

[Quilodrán]
The joints problem in $\mathbb{R}^n$
arXiv:0906.0555, posted June 2, 2009

One algebraic tool

S set of m points in $\mathbb{R}^d$

Claim: There exists a d-variate polynomial $p(x_1, \ldots, x_d)$ of degree b, vanishing at all the points of S, for

$$\binom{b+d}{d} \geq m+1 \qquad \text{or} \qquad b \approx (d!m)^{1/d}$$

Proof: A d-variate polynomial p of degree b has $M = \binom{b+d}{d}$ monomials

Requiring p to vanish at m points $\Longrightarrow$
$m < M$ linear homogeneous equations in the coefficients of the monomials

Always has a nontrivial solution □

Proof of the bound on joints:

L – Set of n lines in $\mathbb{R}^d$

J – Set of their joints; put $m = |J|$

Assume to the contrary that $m > An^{d/(d-1)}$
($A \approx d$ a constant; to be fixed)

Step 1: Pruning

As long as L has a line ℓ incident to $< m/(2n)$ joints,

Remove ℓ from L and its incident joints from J

Left with subsets $L_0 \subseteq L$, $J_0 \subseteq J$, with

- $|J_0| > m/2$
- Each $\ell \in L_0$ is incident to $\geq m/(2n)$ points of J_0
- Each $a \in J_0$ is a joint of L_0

Step 2: Vanishing

Construct a polynomial p vanishing at all the points of J_0

Of degree $b \leq (d!m)^{1/d}$

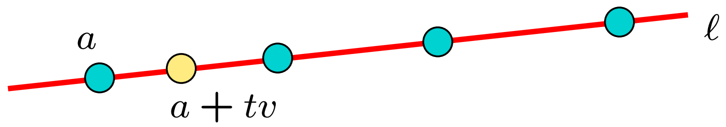
Crucial: Every line of L_0 contains more than b points of J_0:

$$\frac{m}{2n} > (d!m)^{1/d}, \qquad \text{or} \qquad m > \underbrace{(2^d d!)^{1/(d-1)}}_{A} n^{d/(d-1)}$$

$p = 0$ on more than b points on a line $\ell \implies p \equiv 0$ on ℓ

So $p \equiv 0$ on every line of L_0

Step 3: Differentiating



Fix $a \in J_0$ and an incident line of L_0 $\quad \ell = \{a + tv \mid t \in \mathbb{R}\}$

$$p(a+tv) = p(a) + (\nabla p(a) \cdot v)t + O(t^2)$$

for t small

$p(a+tv) \equiv 0$ for all t and $p(a) = 0 \implies \nabla p(a) \cdot v = 0$

Step 3: Differentiating

$\nabla p(a) \cdot v = 0$
for all directions v of lines of L_0 incident to a

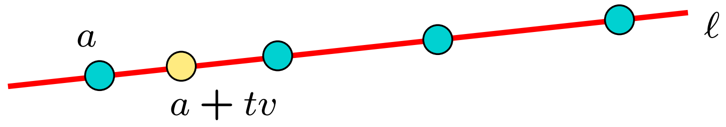
a is a joint $\Longrightarrow$ $\nabla p(a) = 0$

All first-order derivatives of p vanish at all the points of J_0

For each line $\ell \in L_0$, p_{x_i}, which has degree $b-1$, vanishes at more than b points of $\ell \Longrightarrow$ $p_{x_i} \equiv 0$ on ℓ

All first-order derivatives of p vanish on all the lines of L_0

Step 3: Iterating



Fix $a \in J_0$ and an incident line of L_0 $\quad \ell = \{a + tv \mid t \in \mathbb{R}\}$

$$p_{x_i}(a+tv) = p_{x_i}(a) + (\nabla p_{x_i}(a) \cdot v)t + O(t^2)$$

for t small

$p_{x_i}(a+tv) \equiv 0$ for all t and $p_{x_i}(a) = 0 \Longrightarrow \nabla p_{x_i}(a) \cdot v = 0$

Step 3: Iterating

Arguing exactly as above:
All second-order derivatives of p vanish on all the lines of L_0

Keep on going:
All partial derivatives of p, of any order, vanish on all the lines of L_0

Contradiction: Eventually reach derivatives with constant nonzero values!

The end

Some personal notes

György Elekes passed away in September 2008

About 6–7 years earlier, communicated ideas about joints in arrangements of lines, showing:

Number of incidences between n equally inclined lines (lines forming a fixed angle with the z-axis) in space and their joints is $O(n^{3/2}\log^{1/2} n)$

A very special case of a much harder problem he worked on, related to distinct distances in the plane
coming up soon!

Some personal notes

The paper he sent me in 2002 contained a scientific will:

... If I knew for sure that during the next thirty years — which is a loose upper bound for my life span — no new method would be developed to completely solve the $n^{4/3}$ *problem, then I would immediately suggest that we publish all we have in a joint paper.*

However, at the moment, I think we had better wait for the big fish (à la Wiles :))

By the way, in case of something unexpected happens to me (car accident, plane crash, a brick on the top of my skull) I definitely ask you to publish anything we have, at your will.

Gyuri

Some personal notes

Elekes's son contacted me, having found a copy of this note, and asked me to fulfill the will

I even managed to slightly improve Elekes's bound to $O(n^{3/2})$ (but only for equally inclined lines!)

I sent the revised note to János Pach asking him whether it would fit in DCG

Some personal notes

János's answer was merciless but very valuable:

Dear Micha:
Have you seen arXiv:0812.1043
Title: Algebraic Methods in Discrete Analogs of the Kakeya Problem
Authors: Larry Guth, Nets Hawk Katz

If the proof is correct, DCG is not a possibility for the Elekes-Sharir note.

Cheers, János

Some personal notes

And since then I haven't been sleeping much...

New Results:

- Simpler and extended proof of the Guth-Katz bound

  Already done

- Extensions of the algebraic technique (only in 3D):

•• The number of incidences between n lines and their joints is $O(n^{3/2})$

•• The max number of incidences between n lines and m of their joints is $\Theta(m^{1/3}n)$ (for $m \geq n$)

New Results:

•• The max number of incidences between n lines and m arbitrary points is $\Theta(m^{1/3}n)$ (for $m \geq n$), if
(i) No plane contains more than n points, and
(ii) Each point is incident to at least three lines

(In particular, $3m = O(m^{1/3}n) \Longrightarrow m = O(n^{3/2})$)

New Results

•• Proof of Bourgain's conjecture
(simpler variant of the proof of [Guth-Katz])

Bourgain's Conjecture (now Theorem)

L a set of n lines in 3-space
P a set of points in 3-space
No plane contains more than $n^{1/2}$ lines of L
Each line of ℓ contains at least $n^{1/2}$ points of P

Then $|P| = \Omega(n^{3/2})$

Best previous bound: $\Omega(n^{11/8})$
[Solymosi, Tóth, 2008]

New Results (Distinct Distances)

- Incidences between points and helices in $\mathbb{R}^3$

Related to distinct distances in the plane
(an [Erdős 46] classic)

- Some conjectures about number of such incidences

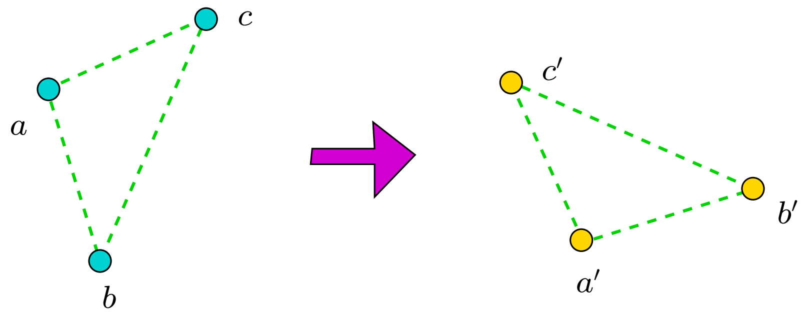
- If true, would imply:

Number of distinct distances in any set S of s points in the plane
is always $\Omega(s/\log s)$

- Almost tight—Erdős's upper bound is $O(s/\sqrt{\log s})$

New Results (Distinct Distances)

• Several sharp upper bounds on the number of incidences,
And some implications:
(Still not fully resolving the conjectures)

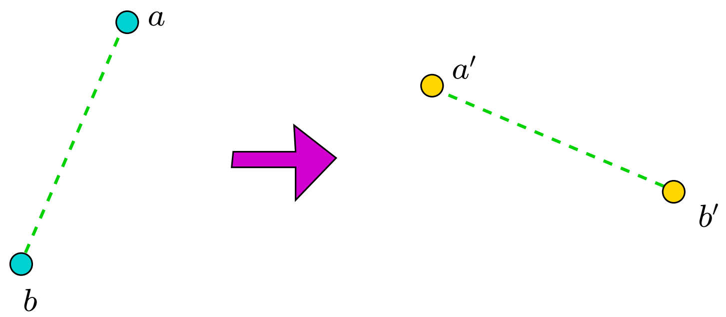


•• Number of rotations (rigid motions) which map (at least) three points of S to three other points of S is $O(s^3)$

Worst-case tight for collinear triples

New Results (Distinct Distances)

•• Major open question: How many rotations map a pair of points of S to another such pair? $O(s^3)$?



•• **Theorem:** Number of distinct (mutually non-congruent) triangles spanned by S is $\Omega(s^2/\log s)$

Almost worst-case tight:
The grid gives $O(s^2)$ distinct triangles (Or fewer?)

Philosophical Interlude / Oxymoron:

- Such a "clash of disciplines" does not happen often
- When it does, the landscape changes and
- The gold rush should begin
- I should not be stressing this too much

(Keep all the gold to myself...)

Extensions: Incidences and flat points

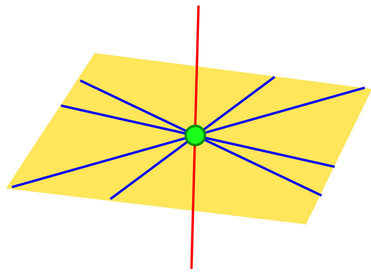
Only in 3D

- To please the connoisseurs
- And to awe the multitudes...

Incidences with joints

Same general approach (make a polynomial vanish "everywhere"), but:

When pruning "light" lines, cannot remove the incident joints



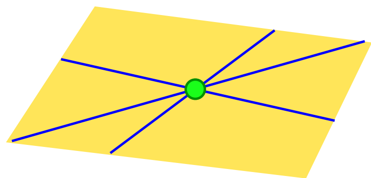
So some joints become "flat"
(but still incident to many (coplanar) lines)

Handling flat points: More algebraic tools

(Developed by [Guth-Katz] for solving Bourgain's problem)

In a nutshell:

Point a is linearly flat for a polynomial p if
(i) a is a regular point, and
(ii) p vanishes on three distinct coplanar lines through a



Claim: If a is linearly flat then the second fundamental form of p vanishes at a

Handling flat points: An additional (gory) algebraic twist

Claim: If a is linearly flat then the second fundamental form of p vanishes at a

Translation: Take the second-order Taylor expansion

$$p(a+u) \approx p(a) + \nabla p(a) \cdot u + \frac{1}{2} u^T H_p(a) u$$

where H_p is the Hessian matrix

$$\begin{pmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{xy} & p_{yy} & p_{yz} \\ p_{xz} & p_{yz} & p_{zz} \end{pmatrix}$$

Then $u^T H_p(a) u = 0$ for every u in the tangent plane at a

Vanishing second fundamental form

$$u^T \begin{pmatrix} p_{xx}(a) & p_{xy}(a) & p_{xz}(a) \\ p_{xy}(a) & p_{yy}(a) & p_{yz}(a) \\ p_{xz}(a) & p_{yz}(a) & p_{zz}(a) \end{pmatrix} u = 0$$

for every u in the tangent plane at a

$\Rightarrow$ Some (a-dependent) linear combinations of 2nd-order derivatives vanish at a

Express this as $q(a) = 0$ for some "global" polynomials q

Vanishing second fundamental form

Trick: e_1, e_2, e_3 – Coordinate unit vectors
$\nabla p(a) \times e_j$ lie in the tangent plane, so

$$(\nabla p(a) \times e_j)^T H_p(a)(\nabla p(a) \times e_j) = 0$$

for $j = 1, 2, 3$

$\equiv$ The three polynomials
$Q_p^{(j)} = (\nabla p \times e_j)^T H_p(\nabla p \times e_j)$ vanish at a
Conversely, if they vanish at a, the second fundamental form vanishes at a
Each $Q_p^{(j)}$ has degree $\leq (b-1) + (b-2) + (b-1) = 3b - 4$

Handling flat points — More tools from algebra

Call a regular point a flat for p if

$$p(a) = Q_p^{(1)}(a) = Q_p^{(2)}(a) = Q_p^{(3)}(a) = 0$$

Recall **Claim:** If the zero set $p = 0$ contains three coplanar lines meeting at a regular point a then a is a flat point

Claim: If a line ℓ contains $3b - 3$ flat points then ℓ is a flat line — all its points are flat

Proof: The four polynomials $p, Q_p^{(1)}, Q_p^{(2)}, Q_p^{(3)}$ vanish identically on ℓ

Handling flat points — More tools from algebra

How many flat lines can p have?

Claim: At most $b(3b-4)$ if it is square-free and has no linear factors

Proof: Uses Bézout's Theorem and other algebraic nuggets

Note: p has a linear factor $\Leftrightarrow$ $p \equiv 0$ on a plane
Every point / line on that plane is flat

Handling joints

Almost forgot: Handle points that are still joints after pruning

A joint a is a singular point of p:
We already showed that $\nabla p(a) = 0$ at a joint

A line ℓ with more than b joints is a singular line of P:
p_x, p_y, p_z, all of degree $b - 1$, vanish on $\geq b$ points on ℓ

Handling joints

How many singular lines can p have?

Claim: At most $b(b-1)$ if it is square-free

Proof: Again, follows from Bézout's Theorem

Incidences with joints – High-level overview

- Prune away light lines, but keep the joints (some may now be flat)

- Construct a square-free polynomial p to vanish at all the joints (Same ideas, but more complicated)

- Each heavy (surviving) line is either singular, or flat, or lies in a plane on which $p \equiv 0$ (a linear factor of p; at most b such planes)

•• At most $b(b-1) + b(3b-4) < 4b^2 \ll n$ singular / flat lines

•• Use the Szemerédi-Trotter bound $O(M^{2/3}N^{2/3} + M + N)$ for incidences in each vanishing plane

- Putting everything together, it somehow works...

Conclusions, Prospects

- Somehow, and strangely, p "senses" the geometry

- Find a more "geometric" proof? Geometric interpretation?

- Find other applications of the algebraic approach
(Joints / incidences with other curves? Higher dimensions?)

- Push further analysis of distinct distances via incidences in 3D

- Repeated distances in the plane?
Long overdue problem, suspected to require algebra

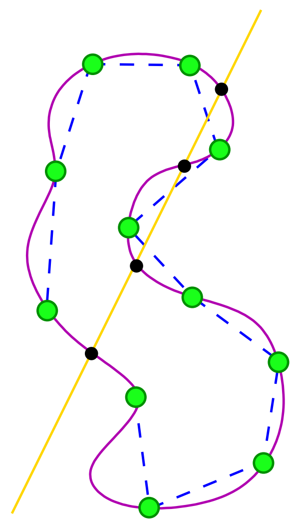
Whetting your appetite:
Spanning tree with small crossing number

- S – Set of n points in the plane

- Construct a bivariate polynomial p vanishing on S
Of degree $d \approx n^{1/2}$

- Use the zero set $Z: \;\; p = 0$ as a spanning tree
Shortcut its arcs into straight segments (if desired)

- Each line crosses Z in at most $d = O(n^{1/2})$ points!!



Thank You