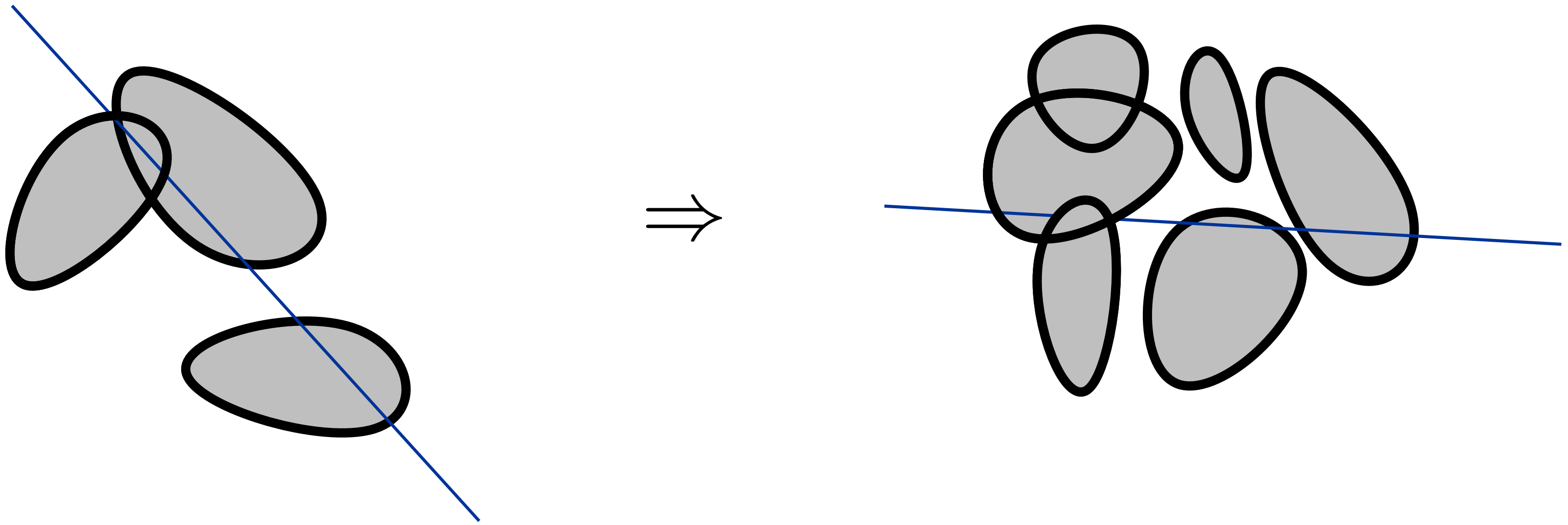
Katchalski's fractional transversal problem:

$T(k)$-families and α-transversals in the plane

Andreas F. Holmsen
KAIST

Katchalski's fractional transversal problem

Given a finite family F of convex sets in the plane such that any *three* can be intersected by a line. Does there always exist a line that intersects at least $\frac{2}{3}|F|$ members of F ?



Definitions

$F = \{A_1, \ldots, A_n\}$: Family of convex sets in the plane.

Common transversal : A straight line that intersects every member of F.

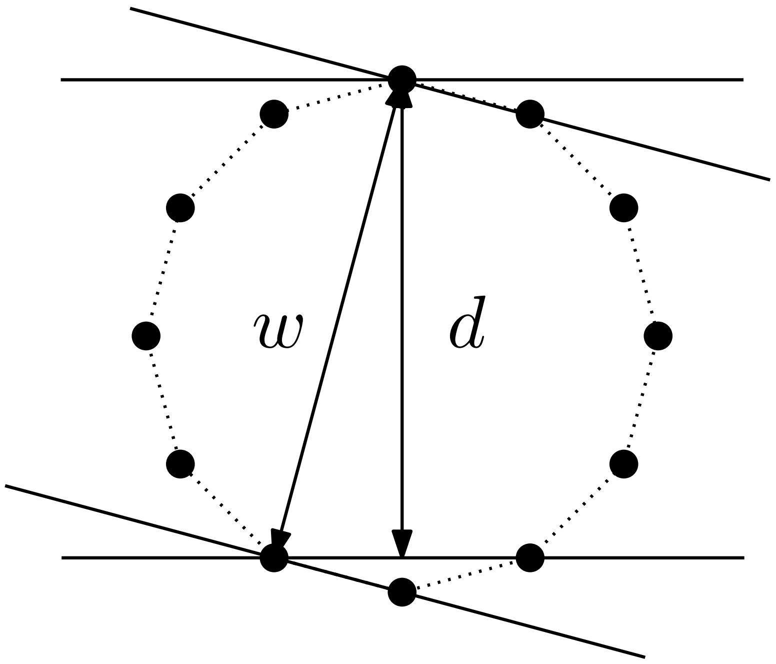
$T(k)$ - *family* : Every subfamily of size at most k has a common transversal.

α - *transversal* : A straight line that intersects at least αn members of F $(0 \leq \alpha \leq 1)$.

No Helly type theorem for line transversals

For every positive integer k there exists a $T(k)$-family that does not have a transversal.

Regular $(k+1)$ - gon.

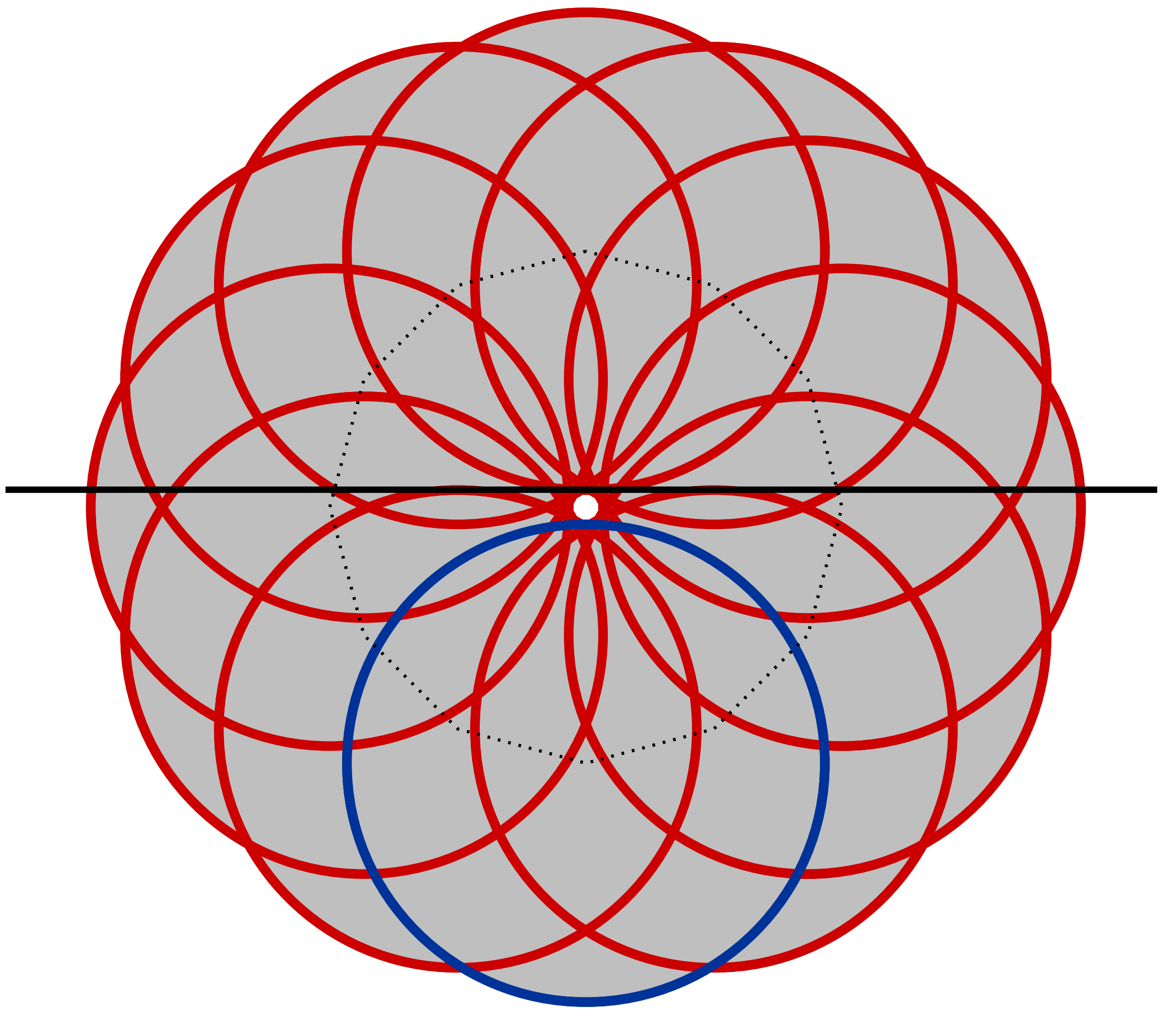


$d < w$

No Helly type theorem for line transversals

For every positive integer k there exists a $T(k)$-family that does not have a transversal.

F has a $\frac{k}{k+1}$ - transversal.



A basic result

Theorem. (Katchalski-Liu, 1980)

For every $k \geq 3$ there exists a maximal number $\alpha(k) \in (0,1)$ such that every $T(k)$-family has an $\alpha(k)$-transversal. Moreover,

$$\lim_{k \to \infty} \alpha(k) = 1$$

Problem. Determine the function $\alpha(k)$.

Some conjectures concerning $\alpha(k)$

Conjecture. (Katchalski, 1978)

$$\alpha(3) = \frac{2}{3}$$

Conjecture. (Eckhoff, 2008)

$$\alpha(k) = \frac{k-1}{k}$$

Previous results

Hadwiger-Debrunner (1963) : $\alpha(4) \geq \frac{1}{4}$

Eckhoff (1973) : $\alpha(4) \geq \frac{1}{2}$

Kramer (1974) : $\alpha(3) \geq \frac{1}{5}$

Eckhoff (1993) : $\alpha(3) \geq \frac{1}{4}$

Gallai - type theorems

Katchalski- Liu (1980) : $\lim_{k \to \infty} \alpha(k) = 1$

Eckhoff (2008) : $1 - \frac{\sqrt{2}}{\sqrt{k-1}} \leq \alpha(k)$, $k \geq 4$

Gallai type theorems for transversals

Theorem. (Eckhoff, 1973)
If F is a $T(4)$-family, then there is a partition $F = F_1 \cup F_2$ such that each F_i has a transversal.

Theorem. (Eckhoff, 1993)
If F is a $T(3)$-family, then there is a partition $F = F_1 \cup F_2 \cup F_3 \cup F_4$ such that each F_i has a transversal.

Remark. Eckhoff has conjectured that *three* parts suffice, which is best possible.

(p, q) - theorem for transversals

Theorem. (Alon-Kalai, 1995)

For every $p \geq q \geq 3$ there exists a minimal positive integer $h(p, q)$ such that the following holds:

If every subfamily of F of size p contains a subfamily of size at least q which has a transversal, then there exists a partition $F = F_1 \cup \cdots \cup F_m$, with $m \leq h(p, q)$, such that each F_i has a transversal.

A direct argument by Alon-Kalai (1995) method gives $\alpha(3) \geq \frac{1}{20}$ (reproduced in Matoušek: Lectures on Discrete Geometry).

Recent results

A generalization of the $T(3)$-property and a Tverberg type lemma.

Hadwiger's transversal theorem: Now in color !

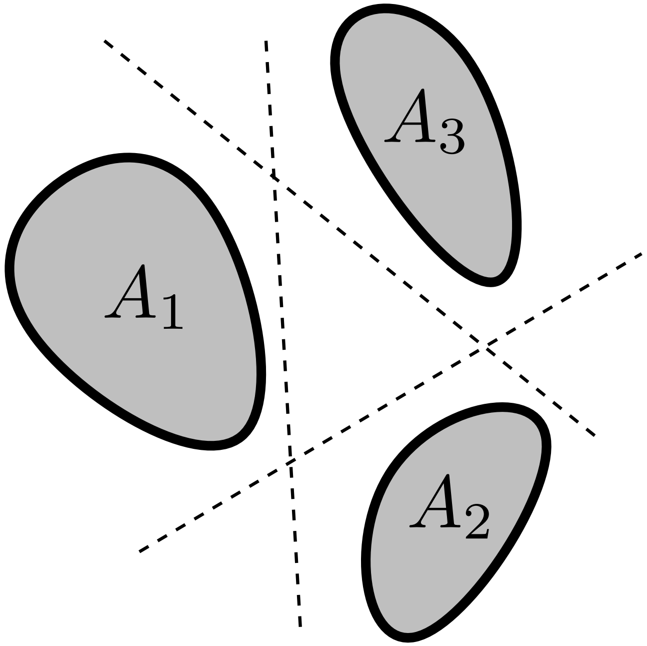
The space of transversals and $T(k)$-families.

A construction: $\alpha(3) \leq \frac{1}{2}$

The $T(3)$ - property

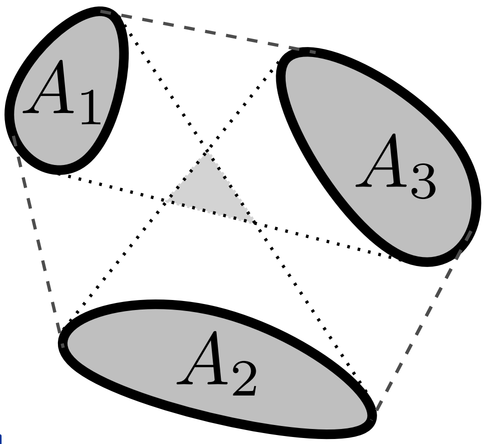
Three convex sets $\{A_1, A_2, A_3\}$ have a transversal if and only if there is a partition $\{A_j\} \cup \{A_i, A_k\}$ such that

$$A_j \cap \text{conv}(A_i \cup A_k) \neq \emptyset$$



Tight triples

A triple of convex sets $\{A_1, A_2, A_3\}$ is called *tight* if the following holds:



$$\bigcup_{1 \le i < j \le 3} \operatorname{conv}(A_i \cup A_j) = \operatorname{conv}\left(\bigcup_{i=1}^{3} A_i\right)$$

Tight triples

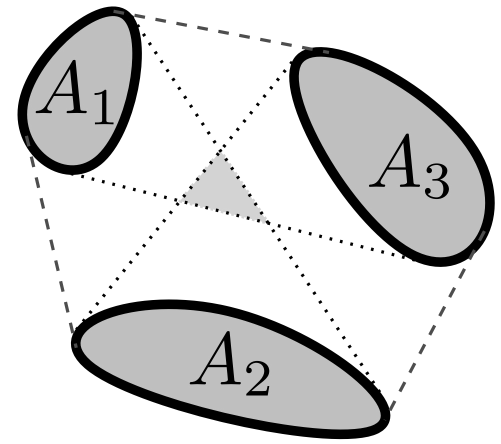
$\{A_1, A_2, A_3\}$ is tight

$\Updownarrow$

$$\bigcap_{1 \le i < j \le 3} \text{conv}(A_i \cup A_j) \neq \emptyset$$

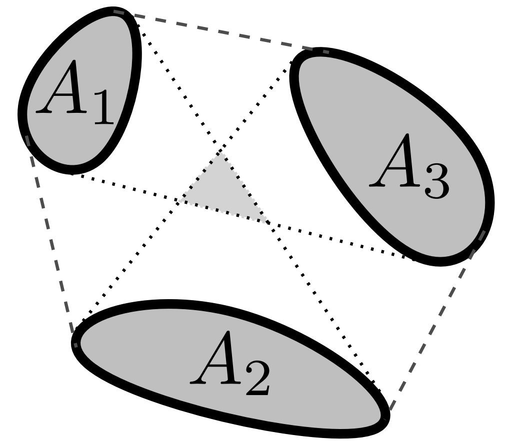
$\Updownarrow$

$$\bigcup_{1 \le i < j \le 3} \text{conv}(A_i \cup A_j) \quad \text{is topologically trivial.}$$



Tight triples

Observation. If $\{A_1, A_2, A_3\}$ has a transversal, then $\{A_1, A_2, A_3\}$ is tight.



Lemma. Let $F = \{A_1, \ldots, A_{2k}\}$. If every triple of F is tight, then there is a partition of F into disjoint pairs $F = P_1 \cup \cdots \cup P_k$ ($|P_i| = 2$) such that

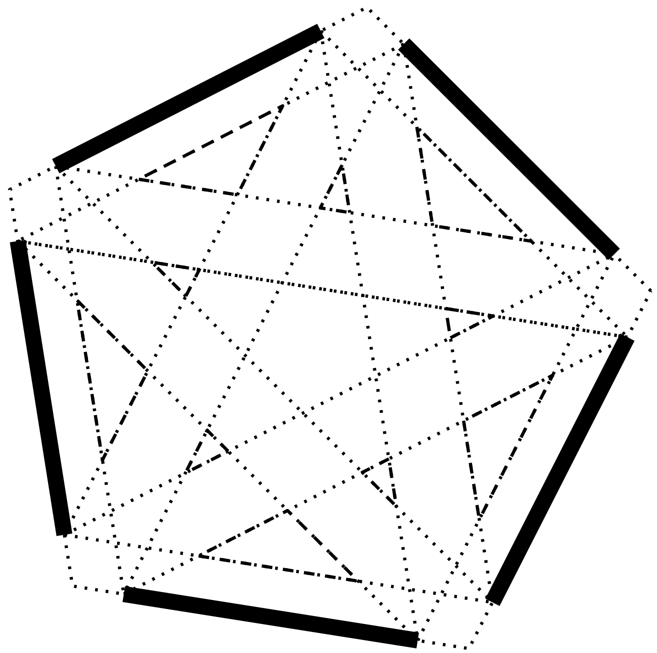
$$\text{conv}(P_1) \cap \cdots \cap \text{conv}(P_k) \neq \emptyset.$$

Tight triples

Theorem. If every triple of F is tight, then F has a $\frac{1}{8}$ - transversal.

Remark. The best possible value in the theorem is $\frac{2}{5}$.

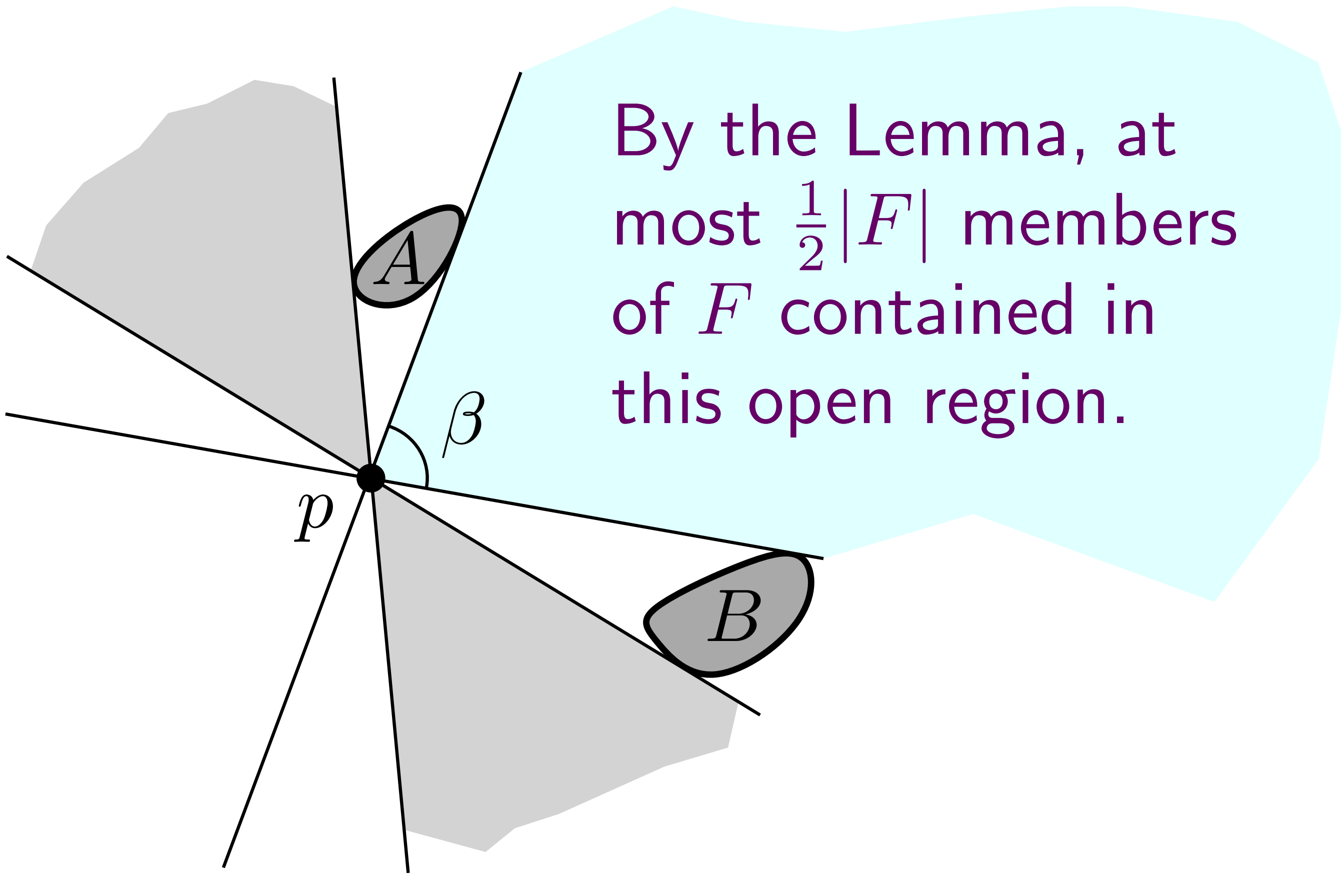
Remark. By the (p, q)-theorem for transversals F can be stabbed by a finite number of lines.



Tight triples

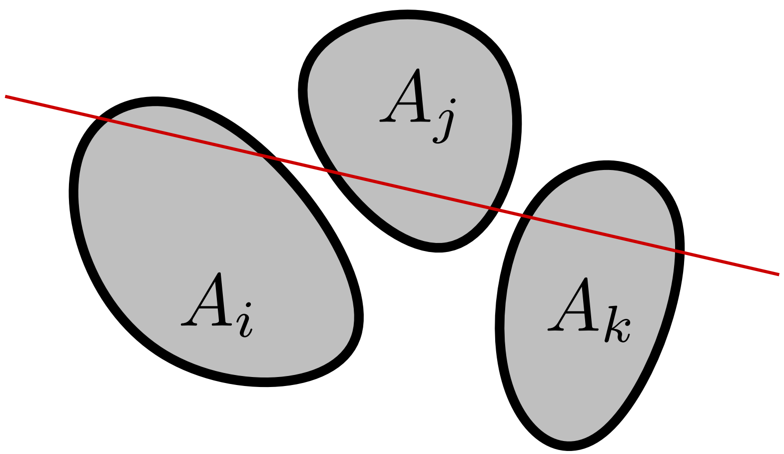
Theorem. If every triple of F is tight, then F has a $\frac{1}{8}$ - transversal.

Proof.



Hadwiger's transversal theorem

Theorem. (Hadwiger, 1957)
Let $F = \{A_1, A_2, \ldots, A_n\}$ be a family of pairwise disjoint convex sets. If for every $1 \leq i < j < k \leq n$ there is a line that intersects A_i, A_j, A_k in the given order, then F has a transversal.

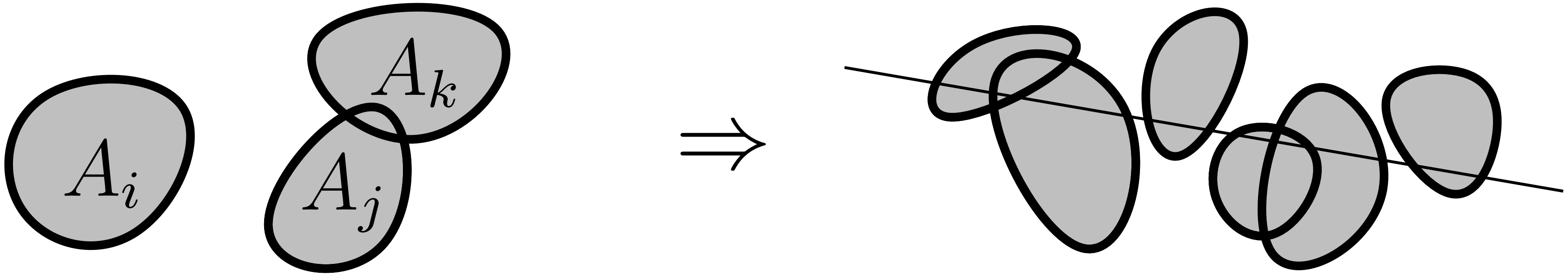


We can replace the ordering condition with the following:

$A_j \cap \mathrm{conv}(A_i \cup A_k) \neq \emptyset$

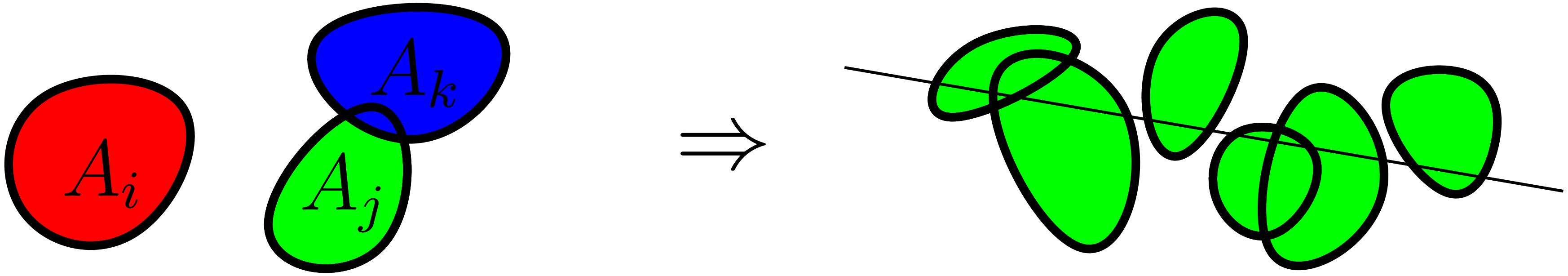
Hadwiger's transversal theorem

Theorem. (Wenger, 1990)
Let $F = \{A_1, A_2, \ldots, A_n\}$. If for every $1 \leq i < j < k \leq n$ we have $A_j \cap \text{conv}(A_i \cup A_k) \neq \emptyset$, then F has a transversal.

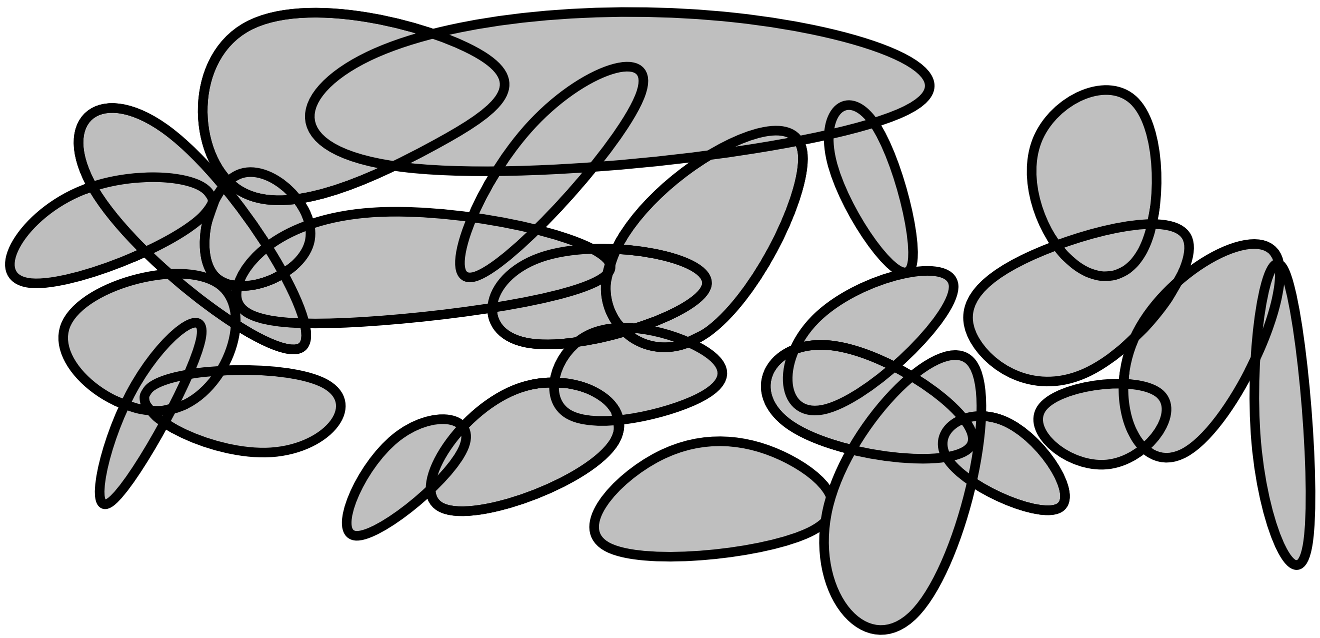


Hadwiger's transversal theorem

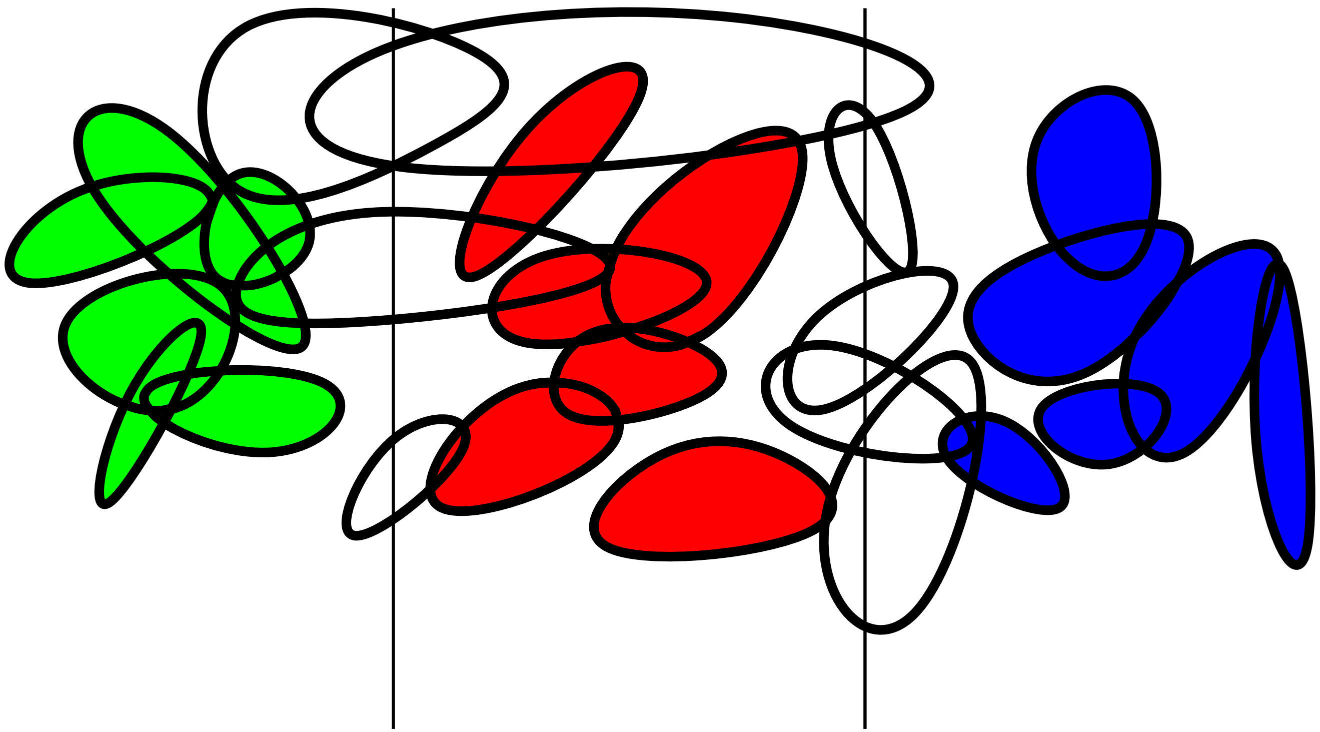
Theorem. (Arocha-Bracho-Montejano, 2008)
Let $F = F_1 \cup F_2 \cup F_3 = \{A_1, A_2, \ldots, A_n\}$. If for every $1 \leq i < j < k \leq n$ where A_i, A_j, A_k belong to distinct parts (F_p's) we have $A_j \cap \text{conv}(A_i \cup A_k) \neq \emptyset$, then one of the F_p has a transversal.



Application to general $T(3)$-families



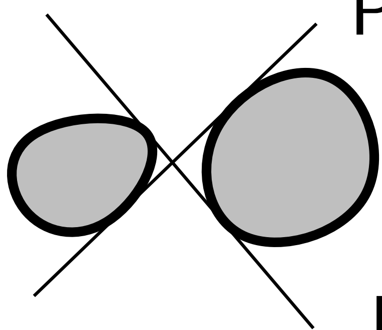
Application to general $T(3)$-families



$\Rightarrow \frac{1}{5}$ - transversal.

The space of transversals

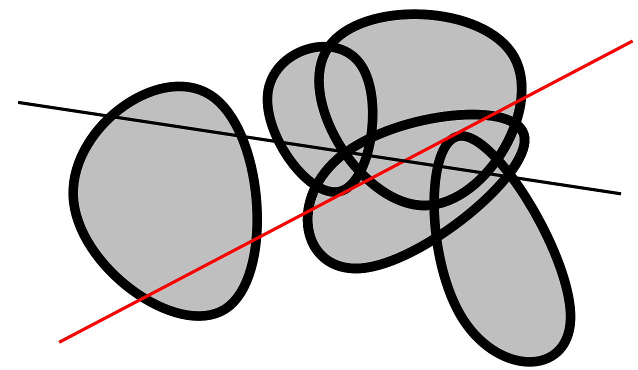
Disjoint pairs:



Positive separating tangent

Negative separating tangent

Observation. Suppose F contains at least one disjoint pair. Then F has a transversal if and only if a positive separating tangent of some disjoint pair of F is transversal to F.



Lower bound for $\alpha(k)$

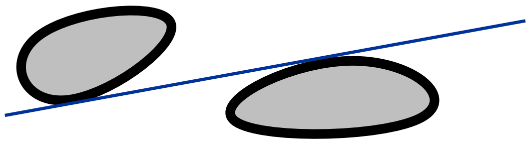
Suppose F contains $d\binom{n}{2}$ intersecting pairs, $0 \leq d < 1$.

X : k-tuples containing at least one disjoint pair.

Y : k-tuples containing only intersecting pairs.

$$|X| \geq (1 - d^{k/2})\binom{n}{k}$$

$$\frac{|X|}{(1-d)\binom{n}{2}} \geq \binom{n}{k} / \binom{n}{2}$$



$$\Rightarrow \alpha(k) \geq \left(\frac{2}{k(k-1)}\right)^{\frac{1}{k-2}}$$

$\alpha(3) \geq \frac{1}{3}$, $\alpha(4) \geq 0.408\cdots$ ($\frac{1}{2}$), $\alpha(5) \geq 0.464\cdots$ ($\frac{1}{2}$), $\alpha(6) \geq 0.508\cdots$, ...

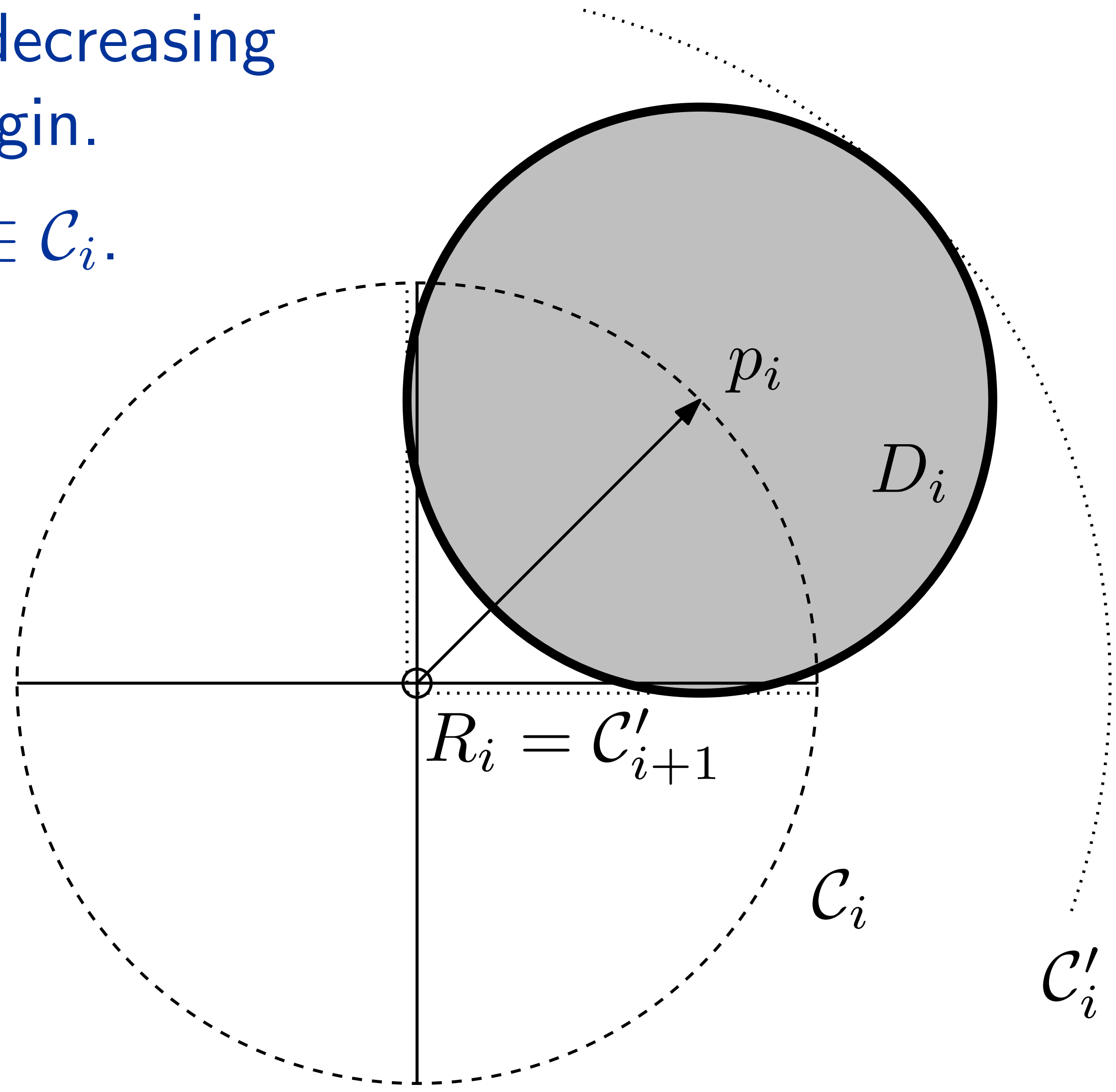
Construction

$\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ circles of decreasing radii centered at the origin.

D_i disk centered at $p_i \in \mathcal{C}_i$.

Directions p_i uniformly distributed

For all $1 \leq i < j \leq n$: Any pair of orthogonal lines that intersect in region R_j, at least one of them intersects D_i.

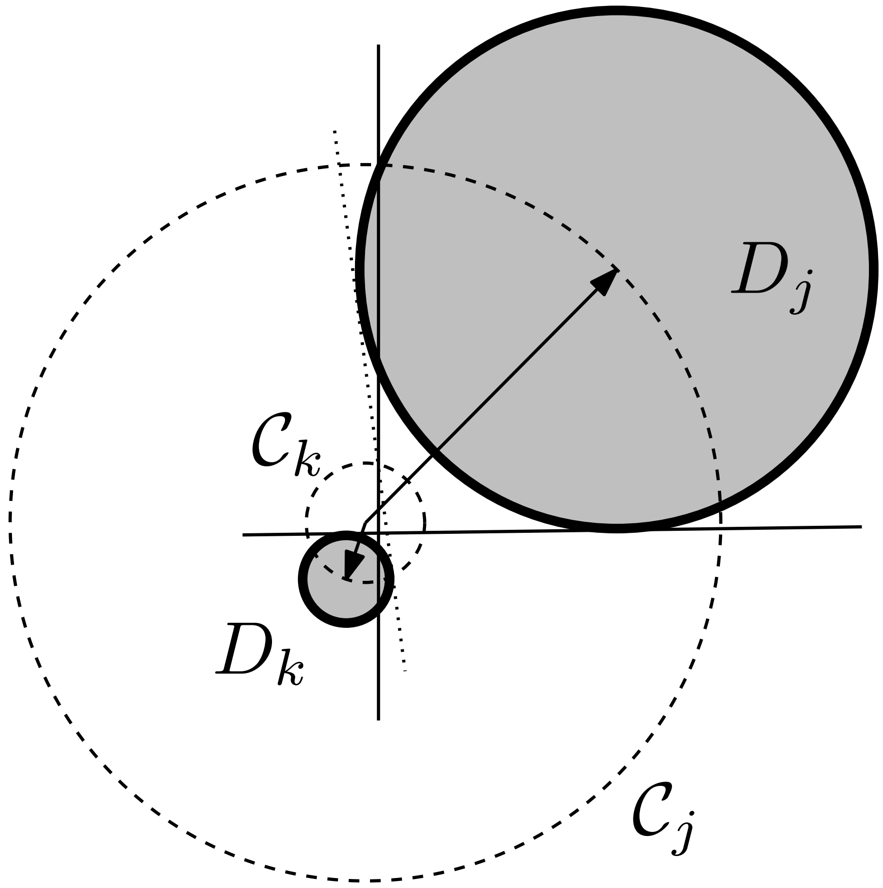


The $T(3)$ - property

Let $1 \leq i < j < k \leq n$. The separating tangents of D_j and D_k form an angle less than $\frac{\pi}{2}$,

A pair of orthogonal transversals to D_j and D_k intersect inside $\mathcal{C}_j \subset R_i$.

One of these lines intersects D_i.

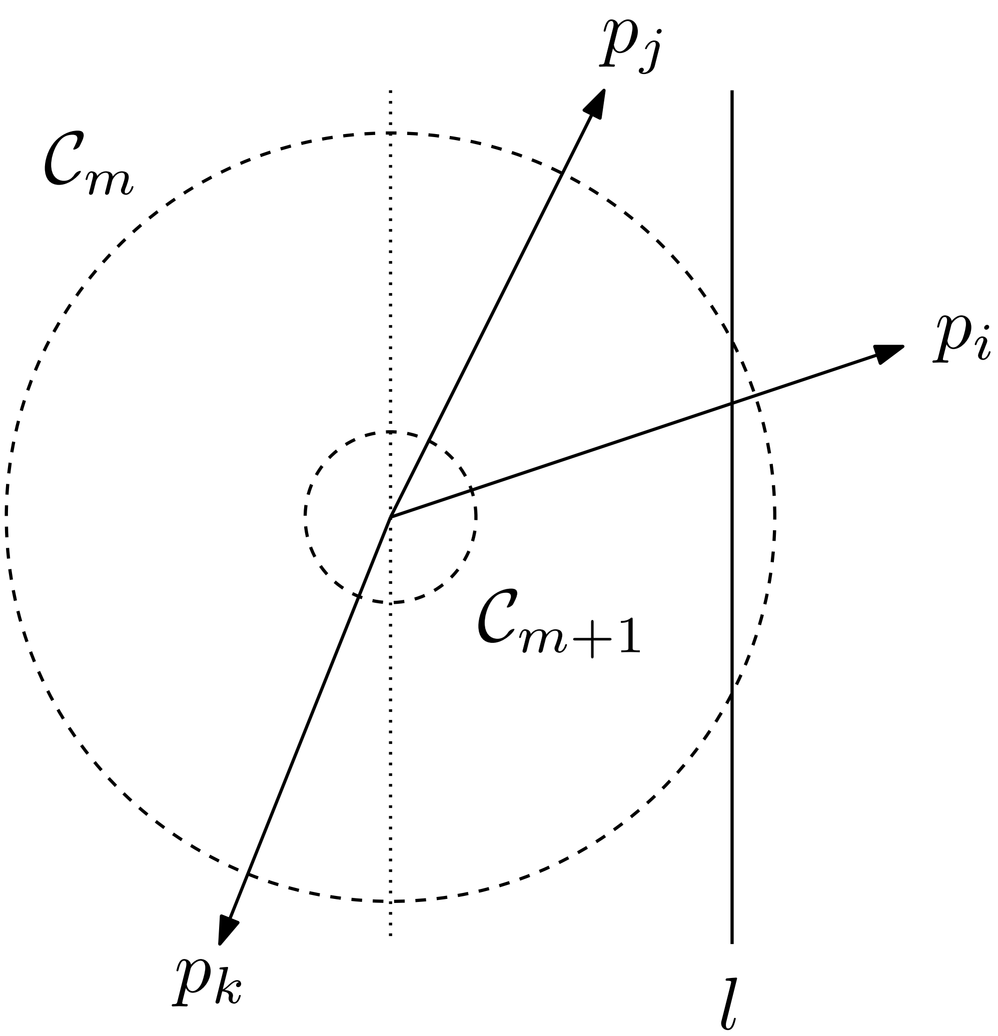


No α - transversal for $\alpha > \frac{1}{2}$

For a given line l let m be the greatest integer such that l intersects $\mathcal{C}_m$.

For $i < m$, l intersects D_i only if the angle with p_i is $\leq \frac{\pi}{4}$.

For $i > m$, l misses D_i.



Conclusion

Some progress...

$$\frac{1}{3} \leq \alpha(3) \leq \frac{1}{2}$$

$$\frac{1}{2} \leq \alpha(4) \leq 0.76\cdots$$

$$\frac{1}{2} \leq \alpha(5)$$

$$0.506 \leq \alpha(6)$$