Tamás Fleiner

Stable and Crossing Structures

Stellingen
behorende bij het proefschrift van
Tamás Fleiner
Stable and Crossing Structures

i. If the set of entries of an $n \times n$ matrix is the set of integers from 1 to n^2 then there is a row or a column of the matrix that has two neighbouring entries with difference at least n.

ii. If a rectangle is tiled with rectangles each having an edge of integer length then the original rectangle has an edge of integer length.

iii. Let A, B, C and D be different points of the plane such that AB and CD are perpendicular. Let point P be the intersecition of lines AC and BD, Q be the intersection of AD and BC and R be the intersection of AB and CD. Then P', Q and R are collinear, where P' is the mirror image of P on CD.

iv. Assume we have a token on position $(0, 0)$ of an infinite tableaux, and all other positions are empty. In a move we are allowed to remove a token from position (i, j) provided that positions $(i + 1, j)$ and $(i, j + 1)$ are empty and to put two new tokens on these positions. Then it is impossible that after some moves all positions with coordinate sum at most two become empty.

v. Let us assume that between any two cities of a country there is a train or a coach connection in at least one direction. Then there is a city C of the country such that the inhabitants of C can travel from C to any other city of the country by using only one of a train pass and a coach pass.

vi. If $H \subset \mathbb{Z}^n$ is a Hilbert basis and vector x can uniquely be expressed as a nonnegative integer combination of H then at most n elements of H have positive coefficient in the expression.

vii. A d-dimensional 0/1-polytope has $O((d-2)!)$ vertices. (Tamás Fleiner, Volker Kaibel and Günter Rote, Upper bounds on the maximal number of facets of 0/1-polytopes, *European J. Combin.*, 21(1): 121–130, 2000.)

viii. If $\mathcal{M} \setminus e$ is a binary matroid with no dual Fano minor for some matroid $\mathcal{M}$ and element e, then $\mathcal{M}$ has a basis of its cycle lattice that consists of circles of $\mathcal{M}$. (Tamás Fleiner, Winfried Hochstättler, Monique Laurent and Martin Loebl, Cycle bases for lattices of binary matroids with no Fano dual minor and their one-element extensions, *J. Combin. Theory Ser. B* 77(1): 25–38, 1999.)

ix. The proof of Pevzner for the linear size of 3-cross-free families is not easy to read. (P. A. Pevzner, Non-3-crossing families and multicommodity flows, in *Selected topics in discrete mathematics (Moscow, 1972–1990)*, pages 201–206. Amer. Math. Soc., Providence, RI, 1994.)

x. Playing soccer on a conference is not the wisest thing one can do.

xi. If a bicycle was stored for n days on the same place, and one wants to send it home on the $(n+1)$st day, it does not mean that it will not get stolen by then.

xii. There is an essential difference between főtétel-feltétel and főttétel feltéttel.

xiii. Although from an old babybed it is quite easy to install a 'fregoli', this practical Hungarian invention is not known in The Netherlands.

xiv. De Belastingdienst kent de euro niet.

Stable and crossing structures

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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de
Technische Universiteit Eindhoven, op gezag van de
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door

Tamás Fleiner

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Dit proefschrift is goedgekeurd door de promotoren:

prof.dr.ir. A.M.H. Gerards
en
prof.dr. A. Schrijver

Contents

Introduction

This thesis consists of two parts. The first one (Chapter II) concerns crossing and uncrossing related problems that emerge in Combinatorial Optimization. In the second part (Chapter III), we describe a link between stable matching-related results in Combinatorial Optimization, Mathematical Economics, Set-, Lattice-, Graph- and Game Theory. Beyond establishing a theoretical background of several known theorems, we include new observations as well. Chapter I contains the preliminaries.

In Chapter II, we describe three crossing-related results: an algorithmic, a structural and an extremal one. In Section 6, Theorem 6.4, we generalize the well-known uncrossing algorithm of Hurkens *et al.* [54] to a more general model originated from a theorem of Frank and Jordán (Theorem 4.2) [37]. Our result will be published as:

> Tamás Fleiner. Uncrossing the system of pairs of sets. (Accepted for publication in *Combinatorica*).

Motivated by the same theorem of Frank and Jordán, we prove in Section 7 a conjecture of Frank, concerning symmetric posets, that model the structure of crossing sets (see Theorem 7.2). The minmax formula that we prove there will be used in 7.3, where we explain some consequences for the problem of l_1-embedding of metric spaces. The main result of this section also appears in

> Tamás Fleiner. Covering a symmetric poset by symmetric chains. *Combinatorica*:17(3), 339–344, 1997.

We close Chapter II with a result on extremal set-systems. We consider a conjecture of Karzanov about the maximum size of a set-system where the number of pairwise crossing sets in the system is restricted. In Theorem 8.3 we confirm the conjecture when that number is at most two. This provides a simple and straightforward proof for a fairly complicated theorem of Pevzner [76]. This result will be published as

> Tamás Fleiner. The size of 3-cross-free families. (Accepted for publication in *Combinatorica*).

Chapter III is probably the most interesting part of this thesis. It is based on a simple observation, namely that the fixed-point theorem of Knaster and Tarski (the special case of Theorem 10.1 for subsetlattices: see also [62]) naturally explains several well-known theorems from different areas of Mathematics. The most surprising fact about these interconnections is that in spite of their straightforward nature so far they remained unobserved.

In Chapter III, we provide new proofs for several generalizations of the stable marriage theorem of Gale and Shapley (Theorem 9.7, see also [42]), well-known in Game Theory and Mathematical Economics. Already the origin of this theorem is unusual. It was published in the *American Mathematical Monthly*, a journal meant for students and for the general public rather than for specialists. The paper describes a model about n men and n women, each ranking the members of the opposite sex according to their preferences as a marriage partner. A natural question is whether there exists a so-called stable marriage scheme in which no potential partners would quit their marriages to marry each other. To justify the existence of such a matching, the authors introduced the so-called deferred acceptance algorithm and proved that in finite time, it terminates with a stable scheme. Gale and Shapley end their paper as follows.

> Finally, we call attention to one additional aspect of the preceding analysis which may be of interest to teachers of mathematics. This is the fact that our result provides a handy counterexample to some of the stereotypes which non-mathematicians believe mathematics to be concerned with.
>
> Most mathematicians at one time or another have probably found themselves in the position of trying to refute the notion that they are people with "a head for figures" or that they "know a lot of formulas." At such times it may be convenient to have an illustration at hand to show that mathematics need not be concerned with figures, either numerical or geometrical. For this purpose we recommend the statement and proof of our Theorem 1[1]. The argument is carried out not in mathematical symbols but in ordinary English; there are no obscure or technical terms. Knowledge of calculus is not presupposed. In fact, one hardly needs to know how to count. Yet any mathematician will immediately recognize the argument as mathematical, while people without mathematical training will probably find difficulty in following the argument, though not because of unfamiliarity with the subject matter.

We shall point out that the stable marriage theorem is a straightforward consequence of the fixed point theorem of Knaster and Tarski. The deferred acceptance algorithm turns out to be a set-function iteration, that finds a fixed point of an appropriate monotone mapping. I believe that our observation about the interconnection between the theory of stable matchings and of lattices improves the understanding of the former structures. I also believe that this does not contradict at all the above citation of Gale and Shapley; it only proves that by introducing certain "obscure and technical terms" some already simple and robust argument in the English language might be simplified even further.

To point out some other consequences of the stable matching theorem to Graph Theory, we deduce a result of Sands *et al.* (Theorem 13.1, see also [90]) on monochromatic paths and the theorem of Pym (Theorem 13.2, see also [77, 78]) on path linking also in Chapter III. Another interesting observation is that the Mendelsohn-Dulmage theorem (Theorem 10.3, see also [71]) (that can be seen as a generalization of the

[1] The theorem about the existence of a stable marriage scheme.

Cantor-Bernstein theorem (Theorem 10.2)) also follows easily from the stable marriage theorem. However, we cannot prove the Kundu-Lawler theorem (Theorem 17.1, see also [65]), a matroid-generalization of the Mendelsohn-Dulmage theorem, directly from the stable marriage theorem of Gale and Shapley. Instead, we show a matroid generalization of this latter theorem about the existence of matroid-kernels, and that implies the Kundu-Lawler theorem similarly as Mendelsohn-Dulmage follows from Gale-Shapley.

This matroid theorem yields an appropriate model in which other facts about stable matchings can be generalized. In this course, we point out an abstract property of set-functions: the increasing property. It is a sufficient condition for the so-called lattice property of comonotone kernels (these are generalized stable matchings). This increasing property holds for all cases in which the lattice property has been proved. Under different names, this very same condition was used by Feder [30] and Subramanian [96] when they formulated the stable roommates problem (the nonbipartite generalization of the stable marriage problem) as a specific stable network problem. (See sections 15 and 19.)

Finally, based on the lattice property, we prove an extension of earlier results of Vande Vate [99] and Rothblum [89] by characterizing the kernel-polytope for increasing comonotone set-functions in Corollary 20.4. Our linear description is especially interesting because it also characterizes the matroid kernel polytope for the case where injective functions define the kernel. The matroid-kernel polytope for constant functions is closely related to the matroid intersection polytope, described by Edmonds [25]. A possible generalization of our result, the linear description of the kernel-polytope for the non-injective case is left to the reader as an open problem. We close the discussion by showing in Theorem 20.7 that without requiring the increasing property, the optimization over the kernel-polytope is NP-complete.

The most interesting results in Chapter III is contained in

> Tamás Fleiner. A fixed point approach to stable matchings and some applications. (Submitted to the *Mathematics of Operations Research.*)

Game Theory seems to me an area of mathematics wrongly neglected by the Combinatorial Optimization and Graph Theory community. There are examples of particularly important results found by Game Theorists now extensively used by the latter groups. Such results are linear programming duality (see Theorem 5.6), the partial solution of the conjecture of Berge and Duchet on perfect graphs (see Conjecture 9.5) by Boros and Gurvich, or the use of stable matchings in the result of Galvin on list colourings of the edges of bipartite graphs (see Theorem 9.9), to mention just three of them. Or an other example, closer to our topic is the theorem of Kelso and Crawford (Theorem 16.2 in 16.1), which is though somewhat weaker than our main tool in Chapter III (Theorem 11.3), several results we present there already follow from it. I hope that the interconnections explained in Chapter III are going to help the communication between these disciplines by bringing them a tiny bit closer to each other.

Acknowledgement

Back in 1995, as a freshly graduated master's student I received an e-mail from Aart Blokhuis saying that the group of his friend, Lex Schrijver, would like to find a PhD student on Combinatorial Optimization. I think that by applying for this position, I succeeded to make one of the wisest decisions of my life. From the following four years stay in The Netherlands I have learnt a lot, met so many new people (some of whom I even call a friend), and experienced so much that this period seems to be a collection of nice memories that I always happy to remember. Now, that Aart is in the core committee of my promotion I thank him for his e-mail and for all the positive experience that followed from it during my work.

I also acknowledge my supervisor and promoter, Bert Gerards, who helped not only by carefully reading my write-ups and suggesting shortcuts and other ideas, but by patiently listening to my new 'proofs' and helping to find the mistake that always lied somewhere in the argument. He could always reassure me when after a considerable time I did not see at all where my work would lead. Although, Bert did not think it was appropriate to publish the new results of this dissertation together, I am convinced that he also has a share in them. Beyond his availability, I thank my second promoter, Lex Schrijver, especially for his questions he asked from me that turned into Section 8 and Chapter III of this dissertation. I have learnt a lot from both Bert and Lex. I am also grateful to my master's supervisor, András Frank, for his questions that motivate sections 6 and 7, and for the invitation to participate on his workshop in Bonn.

To recollect my memories about staying at CWI, I certainly have to mention the audience of the GRATCOS. (The silly acronym stands for the graph theory and combinatorial optimization seminar). Hence I thank the kindness of my coauthor, Monique Laurent, who often practiced her fantastic ability of explaining something and writing on the blackboard in the same time much faster than one could make notes simply by copying the board. On the seminar, Rudi Pendavingh was one of the first people who understood the speaker, and with Judith Keijsper we were always happy when after some discussion, we started to grasp it, as well. But the reason I acknowledge them is the time we spent and the fun we had together apart from the seminar. It was indeed my fault that I broke my elbow in soccer-play, and it was meant to be a joke when I claimed that Judith pushed me. I thank Leen Stougie for being helpful. I appreciate it very much. I also have nice memories about my office-mates, Hein van der Holst, Jack Koolen, Romeo Rizzi and Tamara Dakic. Beyond acknowledging Jack for pointing out the interconnection described in 7.3, I thank him the for chats we had. I appreciate his personality a lot. With Romeo, it was always possible to discuss some nice mathematics when I was overwhelmed by my own silly problems. With Tamara, we also did a couple of things together and beyond words in our languages, we found other common roots, as well. Last but not least, I should mention Andrei Kotlov, another regular member of the GRATCOS. It was always instructive to listen to his talks on the seminar or elsewhere. I am also grateful to his wife, Joanna.

Further, I acknowledge Ron Aharoni for inviting me to the Technion and working together. I learnt really a lot from him; a part of it is described in Section 18. I thank Krzysztof Apt for helping me to find the reference for Tarski's theorem. I am grateful to András Biró for recalling Tarski's theorem when I told him my not so appealing proof for Theorem 11.3. His contribution gave a new impulse to my work on stable matchings. I also thank Michele Conforti for the discussions we had. After the fatal soccer-play, he forced the thorough tests in the hospital when my arm was declared to be unharmed. Without him, I might had severe problems with my elbow by now. I thank Bill Cunningham for inviting me to the Fields Institute and Sasha Karzanov for the discussions we had there. I also acknowledge András Lukács who pointed out what is described in the beginning of Section 16 and thank András Sebő for the invitation to Grenoble that turned out to be especially fruitful, in spite of that probably no publication will follow from solving the problem of the broken cooling pipe with a single bike repair kit. Further, I thank András for the discussions in Toronto and for the suggestion to simplify section 19.

There were other people, not so close to Combinatorial Optimization, that are still responsible that my stay in The Netherlands was so pleasant. My officemate, Ard Overkamp helped really a lot in furnishing the apartment after I arrived to Amsterdam. He also brought me to the juggling club of the CWI, where unfortunately I could not show too much talent. But as he prophesied back then, now I can claim that I have learnt at least something after these four years. (Namely, certain easy patterns with three balls). It is not at all his fault that I could not reach my secret aim, to learn to ride the monocycle. By revealing his supernatural ability, my other officemate, Kanat Çamlibel, has learned to cycle after three sessions, but lost interest to continue it, making him a painful loss for the cycling sport but a superb win for mathematics if he is at least as good at that. (I am sure, he is.) To Sindo Núñez Queija it was always nice to talk to. I appreciate his personality a lot. We also had funny discussions with Bert Zwart, and I feel a bit sorry because I could not score the ultimate goal for him, i.e. to produce a thesis with the same number of pages and references. With Ignacio Pagonabarraga and Anand Yethiray we spent quite some time together and had fun. It is even more true to Stefi Cavallar and Chris Mair, who also helped in all possible respect, like in moving or by dragging me away when I was without family and had long hours in CWI to finish the write-up of this dissertation.

Adri Steenbeek always helped when I had trouble with computers. Before the reader would think that all Hungarians whom I thank are called András, I acknowledge another former classmate, László Nádai for his help in the printing of this thesis.

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At last, I thank my wife, Rita Fleiner for supporting me in my work in spite of all the difficulties that she had to face. Still, she helped me more than I could help her during these times. I also appreciate the effort of our families when they regularly came to visit and help us. I should also mention Zsófia Fleiner, who did not help directly in my job, but being an exceptionally easy child, made parental obligations as simple as it gets.

I am not sure whether the above considerable list is complete. I hope that all contributors received some reward for their share and that I could cheer them up a bit so that they enjoyed the time we spent together at least as much as I did. I tried my best to prepare this dissertation, and of course, any shortage or mistake in it is entirely my responsibility.

August, 2000, Budapest

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Chapter I

Preliminaries

This chapter contains an overview of notions and results used in this thesis. It is somewhat more extensive than strictly necessary so as to provide some background of the areas we are going to touch.

The notation we use is fairly standard. Symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote the set of natural, integer, rational and real numbers, respectively. The powerset of a set X is denoted by $2^X := \{Y : Y \subseteq X\}$. In particular, $\emptyset, X \in 2^X$. If k is a natural number then $[k]$ denotes the set of the first k positive integers: $[k] = \{1, 2, \ldots, k\}$ and for a set X, $\binom{X}{k}$ denotes the family of all k-element subsets of X.

1 Partial orders and lattices

1.1 Partial orders

A *partial order* is a binary relation $\leq$ on a groundset (say X) with the following three properties:

$$\text{for all } x \in X, \quad x \leq x \tag{1.1}$$

$$\text{for all } x, y \in X, \quad \text{if } x \leq y \text{ and } y \leq x \text{ then } x = y \tag{1.2}$$

$$\text{for all } x, y, z \in X, \quad \text{if } x \leq y \text{ and } y \leq z \text{ then } x \leq z. \tag{1.3}$$

Property (1.1) is called *reflexivity*, (1.2) is *antisymmetry*, and (1.3) is *transitivity*. If $x \leq y$ or $y \leq x$, we say that x and y are *comparable*, otherwise they are *uncomparable*. If x and y are uncomparable and there are elements l and u of X such that $l \leq x$, $l \leq y$, $x \leq u$ and $y \leq u$ then we say that x and y are *crossing elements*. If $\leq$ is a partial order, then we introduce the notation $\geq$ for its converse

$$x \geq y \quad \text{if} \quad y \leq x,$$

moreover we define

$$x < y \text{ if } x \leq y \text{ and } x \neq y.$$

For the above partial order, an element x of X is a *minimal element* if there is no $y \in X$ such that $y < x$. It is a *maximal element* if there is no $y \in X$ such that $x < y$. If X has an element that is less than any other element of the groundset then it is called the *zero-element* of X and denoted by $\mathbf{0}$. It there is an element that is greater than any other element of the groundset then it is called the *unit-element* and denoted by $\mathbf{1}$.

A partial order is called a *linear order* (or sometimes a *total order*) if any two elements x and y of X are comparable.

A *partially ordered set* or shortly a *poset* is a pair $P = (X, \leq)$, where $\leq$ is a partial order on set X. P is *finite* if X is a finite set (and then the *size* of P is $|X|$), otherwise P is *infinite*. For poset $P = (X, \leq)$ and subset Y of X the restriction $\leq|_{Y^2}$ is a partial order on Y. By an abuse of notation we denote this poset by $P|_Y = (Y, \leq) \subseteq P$. Because of this declaration, we have to be careful with the meaning of certain notions, like with the one of "crossing elements". That is, to decide the crossing property of two elements x and y, besides the partial order, we should also know the other elements of the groundset of the poset.

Poset $P = (X, \leq)$ is a *chain* if $\leq$ is a linear order. It is an *antichain* if $x \leq y$ implies $x = y$, that is, different elements are uncomparable. The *length* of a finite chain $C = (X, \leq)$ is $l(C) := |X|$. By a *well-ordered set* we mean a chain $C = (X, \leq)$, with the property that every nonempty subset Y of X has a $\leq$-minimal element. A *well-order* is a linear order that defines a well-ordered set. A poset in which each chain is well-ordered is called a partial well-ordered set. The corresponding partial order is called a *partial well-ordering*, or *pwo*.

An example of a chain is $\mathbb{R}|_X = (X, \leq)$, where X is a set of real numbers. (Here, $\leq$ is the usual order on $\mathbb{R}$: $x \leq y$ if $y - x$ is positive.) For $X = \mathbb{R}$ or $X = \mathbb{Q}$, $\mathbb{R}|_X$ is not well-ordered, but $\mathbb{R}|_{\mathbb{N}}$ is a well-ordered set.

For poset $P = (X, \leq)$ and elements $a, b \in X$, the *interval* between a and b is denoted by $X_a^b := \{x \in X : a \leq x \leq b\}$. A subset Y of X is called *convex* if $X_a^b \subseteq Y$ for any $a, b \in Y$.

If $P = (X, \leq)$ is a poset then the *height* of element x of X is

$$h(x) := h_P(x) := \max\{l(P|_C) : P|_C \text{ is a chain of } P \text{ with maximum } x\} .$$

The *height* (or *length*) *of poset* P is $h(P) := \max\{h(x) : x \in X\}$. If $h(x)$ or $h(P)$ is not defined, it is said to be infinite.

Posets P_i (for $i \in I$) *cover* poset $P = (X, \leq)$ if $P_i = P|_{Y_i}$ for $i \in I$ and $\bigcup_{i \in I} Y_i = X$. The *size* of this cover is $|I|$ if I is a finite set, otherwise it is infinite. The height and covers of a poset are related via a well-known minmax relation.

Theorem 1.1. *For poset $P = (X, \leq)$ the minimum size of an antichain cover of P is equal to $h(P)$.*

Proof. As a chain and an antichain of P have at most one element in common, it is clear that $h(P)$ is not less than the minimum in the theorem. Hence, if $h(P) = \infty$ then there is no finite antichain cover of P, so the minimum size of an antichain cover is also infinite. Otherwise, define $X_i := \{x \in X : h(x) = i\}$. Then $A_i := (X_i, \leq)$ is an antichain cover of P for $i \in [h(P)]$. $\square$

If in Theorem 1.1, we interchange the role of chains and antichains we get Dilworth's theorem. Its proof is not so trivial as the one above. (We will give it later on.) The *width* of a poset is the maximum size of its antichains if the maximum is finite, otherwise it is infinite.

Theorem 1.2 (Dilworth [18]). *The width of poset $P = (X, \leq)$ is equal to the minimum size of a chain cover of P.* $\square$

For poset $(X, \leq)$ and elements a, b of X we say that a *covers* b if $a \neq b$ and $X_b^a = \{a, b\}$. Note the unfortunate coincidence that we use the same term "cover" for different notions: here we used the word *cover* in algebraic sense, unlike for *covering posets*, which rather has a set theoretical meaning. If $\mathbf{0} \in X$ then elements that cover $\mathbf{0}$ are called *atoms*, and if $\mathbf{1} \in X$ then elements covered by $\mathbf{1}$ are *co-atoms*. Sometimes we represent a finite poset $P = (X, \leq)$ by its *Hasse diagram*, that is we represent the elements of X by points and we add arrows from element x to element y of X if y covers x. Often, instead of arrows, we use only line segments with the understanding that the corresponding arrow should point to the higher end of the segment. With this convention, the diagram of an antichain is a set of points representing the elements of X, along a horizontal line, without line segments in between. The diagram of a finite chain is a set of points along a vertical line with line segments connecting consecutive points. Observe that for a finite poset P, the diagram of P determines the partial order, as $a \leq b$ if and only if there is a sequence $a = x_1, x_2, \ldots, x_k = b$ such that x_{i+1} covers x_i. Figure 1.1 shows the diagram of two special posets, N_5 and M_3.

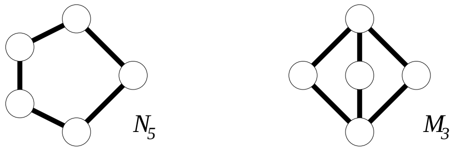

Figure 1.1: Diagrams of poset N_5 and M_3.

If $P = (X, \leq)$ and $Q = (Y, \preceq)$ are posets then $f : X \to Y$ is called *monotone* from P to Q, if $x \leq x'$ implies $f(x) \preceq f(x')$ for all $x, x' \in X$. P and Q are said to be *isomorphic* posets if there is a bijection f between X and Y such that both f and its inverse map f^{-1} are monotone.

Our next example of a poset is $P(\mathcal{F}) = (\mathcal{F}, \subseteq)$, where $\mathcal{F} \subseteq 2^X$ is a family of subsets of groundset X and $\subseteq$ is the usual set-inclusion relation. If $\emptyset \in \mathcal{F}$ then $\emptyset$ is the zero-element of $P(\mathcal{F})$, if $X \in \mathcal{F}$, then it is its unit-element. (It is not true though that the zero-element of such a poset must be $\emptyset$.) If $A, A \cup \{x\} \in \mathcal{F}$ for some $x \in X \setminus A$ then $A \cup \{x\}$ covers A. Atoms of $P(2^X)$ are sets of the form $\{x\}$ and co-atoms are sets of type $X \setminus \{x\}$, for $x \in X$. For a finite groundset X, $h(P(2^X)) = |X|$ and a well-known theorem of Sperner says that the width of $P(2^X)$ is $\binom{|X|}{\lceil \frac{|X|}{2} \rceil}$. For $A \subseteq B$, the interval $P(2^X)_A^B$ is isomorphic to poset $P(2^{B \setminus A})$ and the ismorphism is given by $f(Y) := Y \setminus A$.

1.2 Lattices

For a poset $P = (X, \leq)$, element a of X is the *greatest lower bound* or *meet* of subset Y of X if $a \leq y$ for all elements y of Y and if $a' \leq y$ for all y of Y then $a' \leq a$.

Element b of X is said to be the *lowest upper bound* or *join* of Y if $b \geq y$ for all elements y of Y and if $b' \geq y$ for all $\in Y$ implies $b \leq b'$.

Poset $L = (X, \leq)$ is a *lattice* if any two elements x, y of X have both a meet (denoted by $x \wedge y$) and a join (denoted by $x \vee y$). If L is a lattice, its partial order $\leq$ is called a *lattice-order.*

Lattices can also be defined through the above lattice operations: $L = (X, \wedge, \vee)$ is a lattice if for all $x, y, z \in X$

$$x \wedge x = x \text{ and } \qquad x \vee x = x \tag{1.4}$$

$$x \wedge y = y \wedge x \text{ and } \qquad x \vee y = y \vee x \tag{1.5}$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \text{ and } \qquad x \vee (y \vee z) = (x \vee y) \vee z \tag{1.6}$$

$$x \wedge (x \vee y) = x \text{ and } \qquad x \vee (x \wedge y) = x \tag{1.7}$$

If $L = (X, \wedge, \vee)$ satisfies properties (1.4-1.7) then the binary relation $\leq$ on X defined by $x \leq y$ if $x = x \wedge y$ is a lattice-oder. This way, $(X, \leq)$ becomes a lattice and $\wedge$ and $\vee$ will be its meet and join operations, respectively. In what follows we will call the above triples $(X, \wedge, \vee)$ also lattices.

1.3 Homomorphisms and sublattices

Observe that the above definitions define different structures: lattice as a special binary relation on a groundset, and lattice as an abstract algebra with two special operations ($\wedge$ and $\vee$) on a groundset. Although, as we have seen, these definitions are essentially the same, yet there is a point that one should be careful about. Namely, the definitions of substructures depend on the way we think about the lattice.

Poset $L' = (X', \leq)$ is a *lattice subset* of poset $L = (X, \leq)$, if $X' \subseteq X$ and $\leq$ induces a lattice-order on X'. Structure $L' = (X', \wedge', \vee')$ is a *sublattice* of lattice $L = (X, \wedge, \vee)$, if $X' \subseteq X$ and $\wedge'$ and $\vee'$ are the restrictions of $\wedge$ and $\vee$ on X', i.e. $x \wedge' y = x \wedge y$ and $x \vee' y = x \vee y$ for all $x, y \in X'$. It is easy to see that every sublattice L' of L is also a lattice subset of L: the partial order on L' defined from the lattice operations is the restricted partial order to L'. But if the restriction of $\leq$ is a lattice order on some subset X' of X, it does not mean that the corresponding lattice operations $\wedge'$ and $\vee'$ are restrictions of $\wedge$ and $\vee$ to X'.

Another difference between lattices as abstract algebras and lattices as special posets is the meaning of homomorphism (structure-preserving mapping) in the two cases. If $L = (X, \wedge, \vee)$ and $L' = (X', \wedge', \vee')$ are posets then $\phi : X \rightarrow X'$ is a *lattice-homomorphism* from L to L' if $\phi(x \wedge y) = \phi(x) \wedge' \phi(y)$ and $\phi(x \vee y) = \phi(x) \vee' \phi(y)$ for all $x, y \in X$. Clearly, if ϕ is a lattice-homomorphism then ϕ is necessarily monotone between posets L and L', but the converse is not true.

However, if both f and f^{-1} are monotone for some bijective function f between lattices, then f and f^{-1} are both lattice-homomorphisms. This observation shows that the notion of lattice-isomorphism does not depend on the way we look at lattices.

If we view lattices as special posets, then the analogous notion of a lattice-homomorphism is strict monotonicity. Function $f : X \to Y$ is *strictly monotone* for posets $P = (X, \leq)$ and $Q = (Y, \preceq)$ if

$$x \leq x' \text{ if and only if } f(x) \preceq f(x')$$

for any pair of elements x and x' of X. It is clear that if $f : X \to Y$ is strictly monotone from lattice $P = (X, \leq)$ to poset $Q = (Y, \preceq)$, then $f(X)$ is a lattice subset of Q.

Let $\mathcal{L}$ be some class of lattices. We say that lattice $L = (X, \wedge, \vee)$ is a *free lattice in class* $\mathcal{L}$ generated by poset $P = (Y, \leq)$ if $Y \subseteq X$, the lattice order of L on Y is identical with $\leq$ and for any lattice L' in $\mathcal{L}$ and for any monotone function f from P to L' there exists a unique lattice-homomorphism extension ϕ of f from L to L'. In other words, L is a smallest lattice in which every element can be obtained from Y by lattice operations and if a lattice-identity is true on L then this identity is true on the whole class $\mathcal{L}$.

1.4 Special lattices

Our example for a poset in 1.2, $P(2^X) = (2^X, \subseteq)$ is also an example of a lattice with meet and join operations $\cap$ and $\cup$, respectively. This example shows that by "tapering", we can generalize set theoretical relation $\subseteq$ to partial order relation $\leq$, and set theoretical operations $\cap$ and $\cup$ to lattice operations $\wedge$ and $\vee$.

If poset $P = (X, \leq)$ has both a zero- and a unit-element, and for some $x \in X$ there is an element $\overline{x}$ such that $x \wedge \overline{x} = \mathbf{0}$ and $x \vee \overline{x} = \mathbf{1}$, then we say that $\overline{x}$ is a *complement* of x in P. If all elements of X have a complement then P is called a *complemented poset*. In accordance with this, we say that lattice $L = (X, \wedge, \vee)$ is a *complemented lattice* if for all $x \in X$ there is a *complement* $\overline{x}$ of x with $x \wedge \overline{x} = \mathbf{0}$ and $x \vee \overline{x} = \mathbf{1}$.

Lattice L is *modular* if modular property (1.8) holds for its elements.

$$\text{If } x \leq z \text{ then } x \vee (y \wedge z) = (x \vee y) \wedge z. \tag{1.8}$$

If L is a modular lattice with no infinite chain then *modular equation* (1.9) holds for its height function:

$$h(x) + h(y) = h(x \wedge y) + h(x \vee y) \qquad \text{for all } x, y \in X. \tag{1.9}$$

Function h is called *submodular* if (1.9) holds with "$\geq$" instead of "$=$", and it is called *supermodular* if "$\leq$" stands for "$=$" in (1.9). There is a well-known characterization of modular lattices in terms of forbidden substructures:

Theorem 1.3. *Lattice L is modular if and only if it does not contain a sublattice isomorphic to lattice N_5 on Figure 1.1.*

Proof. It is easy to see that N_5 is not modular. On the other hand, let $x \leq z$ and $y \in X$. Clearly, $x \leq (x \vee y)$, $x \leq z$ and $y \wedge z \leq (x \vee y)$, $y \wedge z \leq z$, hence $x \vee (y \wedge z) \leq (x \vee y) \wedge z$. Thus if modular law (1.8) is not true for x, y, z then $x \vee (y \wedge z) < (x \vee y) \wedge z$. It is easy to check that elements $x \vee (y \wedge z), (x \vee y) \wedge z, x \vee z, y, y \wedge z$ form a sublattice isomorphic to N_5. $\square$

Theorem 1.4. *The free modular lattice generated by 3 elements x, y, z is isomorphic to lattice L_{28}, and the free modular lattice generated by a chain $x_1 \leq x_{\leq} \ldots \leq x_n$ and another element y is lattice L^n, represented by the diagrams on Figure 1.2.*

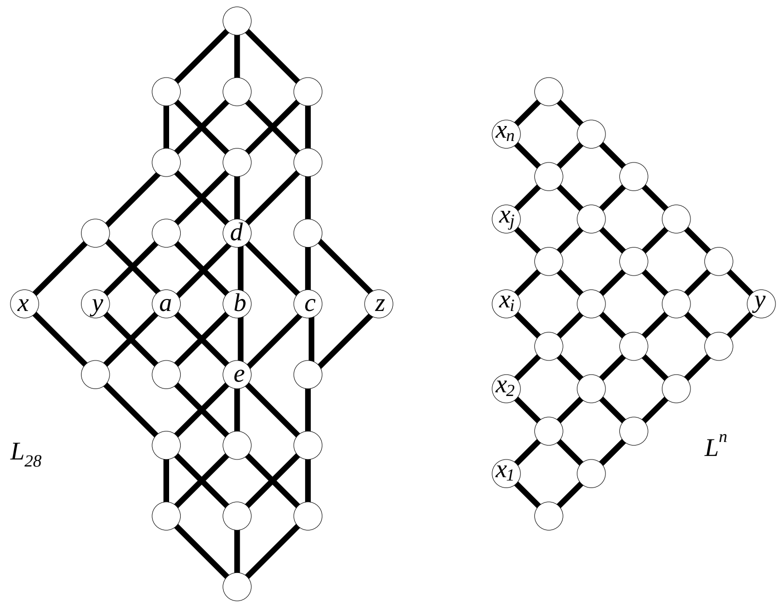

Figure 1.2: Diagrams of lattice L_{28} and L^n.

Proof. It is finite work to check that L_{28} and L^n are both modular lattices containing only the generators and elements that are generated by the generators. Also, if an identity holds on L_{28} or on L^n then it follows from the modular law (1.8). □

We remark that the free modular lattice generated by four elements is not a finite one. One can strengthen the modular property even further to the following distributive law:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{1.10}$$

Lattice L is called *distributive* if distributive property (1.10) holds for its elements. Note that for distributive lattices the "symmetric pair" of (1.10) holds:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Distributive lattices are modular, and lattices that isomorphic to some sublattice of $(2^X, \cap, \cup)$ are distributive by de Morgan's law. Again, there is a well-known characterization of distributive lattices:

Theorem 1.5. *Lattice L is distributive if and only if it does not contain a sublattice isomorphic to lattice N_5 or M_3 on Figure 1.1.*

Proof. (Sketch) Neither N_5 nor M_3 is modular. On the other hand, if L does not contain N_5 or M_3 then L is modular by Theorem 1.3. It is easy to see that if distributive law (1.10) does not hold for some x, y, z, then there is a homomorphism from L_{28} to L which maps elements a, b, c, d, e of L_{28} into different elements of L, exhibiting a sublattice of L isomorphic to M_3. $\square$

Another well-known theorem says that distributive lattices are essentially the same as subsetlattices:

Theorem 1.6. *If lattice L is distributive then L is isomorphic to some sublattice of $(2^X, \cap, \cup)$ for some set X.* $\square$

The interested reader can find further material on partially ordered sets and lattices in the book of Birkhoff [5].

2 Graphs

A *simple graph* is a pair $G = (V, E)$ consisting of a *vertex-set* V and of an *edge-set* E of unordered pairs of vertices. Elements of V are called *vertices*, elements of E are *edges*. For different vertices u, v of V we denote edge $\{u, v\}$ by uv. We say that edge uv is *incident with* u and v. A *multigraph* is a similar structure with the difference that E is a multiset[1] of unordered pairs and singletons of vertices. Pairs in E are called edges, just like for simple graphs, and singleton elements of E are *loops*.

A *simple directed graph* (or shortly *simple digraph* is a pair $D = (V, A)$ consisting of a *vertex-set* V and of an *arc-set* A of ordered pairs of vertices. A *directed (multi)graph* (or shortly *digraph*) is a similar structure where A is a multiset containing arcs and singleton vertices. Similarly, the abbreviated notion $uv \in A$ stands for arc (u, v). We shall use this abbreviation only if it is either clear or not important whether the particular edge is directed or not. We say that arc $a = uv$ is *incident with tail* u and *head* v.

For (di)graphs G, we introduce operators V, E and A such that $V(G), E(G)$ and $A(G)$ denotes the set of vertices, edges and arcs of G, respectively.

We call a directed graph $\vec{G}$ an *orientation* of an undirected graph G if $V(\vec{G}) = V(G)$ and $uv \in E(G)$ if and only if $v \in A(\vec{G})$ or $vu \in A(\vec{G})$ (or both). If $\vec{G}$ is an orientation of simple graph G then we call G the *underlying (undirected) graph* of $\vec{G}$.

The *complement* of simple graph $G = (V, E)$ is $\overline{G} := (V, \binom{V}{2} \setminus E)$.

Undirected graph $K_n := ([n], \binom{[n]}{2})$ is the *clique* or *complete graph* on n vertices, its complement $\overline{K}_n = ([n], \emptyset)$ is the *coclique* or *empty graph* on n vertices. $P_n := ([n+1], E(P_n))$ is the *path of length* n, where $E(P_n) := \{\{i, i+1\} : i \in [n]\}$, $C_n := ([n], E(P_{n-1}) \cup \{\{1, n\}\})$ is the *cycle of length* n. A cycle is *odd* if its length is odd.

For a (di)graph G and subsets U, W of $V(G)$ we introduce

$$D(U, W) := D_G(U, W) := \{e = uv \in E(G) : u \in U, v \in V\}, \tag{2.1}$$

$$D^+(U, W) := D^+_G(U, W) := \{e = uv \in A(G) : u \in U, v \in V\}, \tag{2.2}$$

$$D^-(U, W) := D^+(W, U). \tag{2.3}$$

[1] In a multiset, an element can have an integral multiplicity more than 1.

Further, we denote $D(U, V(G)\backslash U)$ by $D(U)$, and the *star* $D(\{u\})$ of vertex u by $D(u)$. For digraphs, we use similar notations with $+$ and $-$ superscripts. In particular, $D^+(u)$ and $D^-(u)$ denotes the *outstar* and *instar* of vertex u, respectively. Notation d stands for the cardinality of D, hence $d(U) := |D(U)|$, $d^+(U) := |D^+(U)|$, etc. Clearly, $d(u)$ is the number of edges incident with vertex u, in other words the *degree* of u.

Similarly to the incident edges, we define notations for the adjacent vertices as follows.

$$\Gamma(U, W) := \Gamma_G(U, W) = \{v \in W : uv \in E(G) \text{ for some } u \in U\}, \tag{2.4}$$

$$\Gamma^+(U, W) := \Gamma_G^+(U, W) = \{v \in W : uv \in A(G) \text{ for some } u \in U\}, \tag{2.5}$$

$$\Gamma^-(U, W) := \Gamma_G^-(U, W) = \{v \in W : vu \in A(G) \text{ for some } u \in U\} \tag{2.6}$$

for subsets U and W of $V(G)$. We define $\Gamma(U) := \Gamma(U, V(G) \setminus U)$ for $U \subseteq V$, and $\Gamma(u) := \Gamma(\{u\})$ for $u \in V$. Similarly to the degree-function d, γ denotes the cardinality of Γ, the number of neighbours of the certain set. For digraphs we use similar notations with $+$ and $-$ superscripts.

2.1 Paths in graphs

For a (di)graph G, we introduce "graph distances" between vertices of G: we say that $\mathrm{dist}_G(u, v) = k$ if the shortest (directed) path of G from x to y has length k. That is, $y \in \Gamma^{(k)}(x) \setminus \bigcup_{i=0}^{k-1} \Gamma^{(i)}(x)$.

Sequence $w = (v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$ is a *walk* in graph $G = (V, E)$ if $k \in \mathbb{N}$ and $e_i = v_{i-1}v_i \in E$ for $1 \leq i \leq k$. If it does not cause ambiguity we might identify walks with their edge- or vertex-sequence. $V(w) = \{v_0, v_1, \dots, v_k\}$ denotes the set of vertices, $E(w) = \{e_1, e_2, \dots, e_k\}$ the set of edges, $In(w) = v_0$ the *initial node*, and $End(w) = v_k$ the *end node* of the above walk w. We say that walk w is an *ST-walk* if $In(w) \in S$ and $End(w) \in T$. (In this context we may also denote one-element sets by their unique element, like in the term 'st-walk'.)

If $\mathcal{W}$ is a family of walks, then operators $V(\mathcal{W}), E(\mathcal{W}), In(\mathcal{W})$ and $End(\mathcal{W})$ denote the set of vertices, edges, initial nodes and end nodes of walks of $\mathcal{W}$, respectively. Family $\mathcal{W}$ is an *edge-disjoint* (*vertex-disjoint*) family if whenever $E(w) \cap E(w') \neq \emptyset$ ($V(w) \cap V(w') \neq \emptyset$) for $w, w' \in \mathcal{W}$ then $w = w'$.

If all the vertices $v_1, v_2, \dots, v_k$ of $V(w)$ of walk w are different then w is a *simple path*. A *circular path* is a walk w for which $In(w) = End(w)$, but all other vertices of w are different. An *infinite path* is a (one or both way) infinite sequence of different incident edges and vertices. A walk is a *general path* if it is either a simple path or a circular path or an infinite path.

2.2 Subgraphs, minors and connectivity

If $G = (V, E)$ is a graph with $e \in E$ and $v \in V$ then $G \setminus e = (V, E \setminus \{e\})$ is the graph obtained by the deleting of edge e, $G - v := (V \backslash \{v\}, \{e \in E : e \text{ is not incident with } v\})$ is the graph after the deletion of vertex v. For an edge $e = uv$ (directed or not) $G + e := (V, E \cup \{e\})$ is the graph obtained by adding edge e to G. For a set X of edges and a set Y of edges and vertices of G, $G + X - Y$ denotes the graph obtained by adding the elements in X as new edges to G and by deleting the edges and vertices in Y from the resulted graph. A *subgraph* of G is a graph of the form $G - Y$. If subgraph G' of graph G can be obtained by only deleting vertices of G then we say that G' is an *induced subgraph* of G or that G *spans* G'.

For an edge $e = uv$ of undirected graph G, G/e denotes the graph after the *contraction* of e. The vertex-set of the contracted graph is $V(G/e) = (V \setminus \{u, v\}) \cup \{v_e\}$, where $v_e \notin V$, and the edges of G/e are the edges of G not incident with u or v together with the (possibly loop-)edges of $D(u) \cup D(v) \setminus \{e\}$, after substituting ends u and v by v_e. G' is a *minor* of graph G if there are disjoint subsets X, Y of the edge-set of G, such that $G' = G \setminus X/Y$ is the graph obtained from G by deleting the edges in X and contracting the edges in Y. Observe that no matter in which order we execute deletions and contractions, we get the same minor.

Graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there are bijections $\phi_V : V \to V'$ and $\phi_E : E \to E'$ such that if $uv = e \in E$ then $\phi_e(e) = \phi_V(u)\phi_V(v)$. Often we consider isomorphic graphs the same. So, "graph G is a subgraph (or minor) of graph H" often means that there exists some subgraph (or minor) of H that is isomorphic to G. Similarly, the statement that G is (say) a clique on n vertices we mean that G is isomorphic to K_n.

For an undirected graph G, we define the equivalence relation $\sim$ on $V(G)$ by $v \sim u$ if there is a path between u and v. The equivalence classes of V under this relation are called the *components* of G. By $o(G)$ we denote the number of *odd components* of graph G, i.e. the number of components that have odd number of vertices. A vertex that is a component by itself is called an *isolated vertex*. We also use the name "component" for a subgraph of G induced by a component. In case of a directed graph, we distinguish between *weak components*, (these are the components of the underlying undirected graph), and *strong components*. The latter are the equivalence classes of the equivalence relation $\vec{\sim}$, defined by $u \vec{\sim} v$ if there is a directed path in G from u to v and a directed path from v to u, as well. Clearly, each weak component is the union of strong components. A graph with only one component is called *connected*. A digraph with only one weak component is *weakly connected*, and a digraph with only one strong component, is *strongly connected*.

(Di)graph $T = (V, E)$ is a *(directed) tree* if it is (weakly) connected and the underlying undirected graph does not have a cycle. A tree on n vertices has $n - 1$ edges. (Di)graph B is *bipartite* if the underlying undirected graph does not have a cycle subgraph of odd length. Or, equivalently, B is bipartite if $V(B)$ can be partitioned into two sets (colour-classes) such that all edges of $E(B)$ have ends in different parts. Note that trees are bipartite graphs. We denote by $K_{n,m} := ([n + m], \{ij : 1 \leq i \leq n < j \leq m\})$ the *complete bipartite graph* with n and m vertices in its colour-classes. For an undirected graph G, we define its *line-graph* $L(G)$ by $V(L(G)) := E(G)$ and $E(L(G)) := \{ef : e, f \in E(G) \text{ are adjacent edges}\}$.

2.3 Packing and covering in graphs

A *matching* is a subgraph M of graph G with $d_M(v) \leq 1$ for any $v \in V$, in other words, such that no two edges of M are adjacent. If $b : V \to \mathbb{N}$ then M is a *b-matching*, if

$d_M \leq b$. M is a *perfect b-matching* if $d_M = b$. A perfect 1-matching is also known as a *perfect matching* or a *1-factor* of G. By $\nu(G)$ we denote the *matching number* of G, that is the maximum number of edges of a matching of G, in other words the maximum number of independent edges of G. When it is clear from the context, we might use the term 'matching' for a set of edges of some matching. Another related quantity is the *independence* or *coclique number* $\alpha(G)$ of G, the maximum number of independent vertices of G, or in other words the maximum size of an induced coclique subgraph of G. Clearly, $\nu(G) = \alpha(L(G))$. Further interesting parameters of graph G are its *vertex cover number*

$$\tau(G) := \min\{|U| : U \subseteq V \text{ with } E(G - U) = \emptyset\} \tag{2.7}$$

and its *edge cover number*

$$\rho(G) := \min\{|F| : F \subseteq E \text{ and } d_{(V,F)} \geq 1\} = \tau(L(G)). \tag{2.8}$$

A set F of edges described in definition (2.8) is an *edge cover*, a set U of vertices in definition (2.7) is a *vertex cover*. It is easy to see that $\alpha(G) \leq \rho(G)$ and $\nu(G) \leq \tau(G)$ for a graph G. The following theorem of Gallai gives further connections between these parameters:

Theorem 2.1 (Gallai [44]). *$\alpha(G) + \tau(G) = |V|$, for any finite graph G. If G has no isolated vertex then $\nu(G) + \rho(G) = |V|$.* □

A well-known theorem of Kőnig gives another important relation for bipartite graphs:

Theorem 2.2 (Kőnig [61]). *For any finite bipartite graph G, $\nu(G) = \tau(G)$.* □

For general, not necessarily bipartite graphs, the Tutte-Berge formula [4] is a minmax relation for parameter ν.

Theorem 2.3 (Tutte-Berge formula [4]). *If $G = (V, E)$ is a finite undirected graph then*

$$\nu(G) = \min\left\{\frac{1}{2}\left[|V| + |X| - o(G - X)\right] : X \subseteq V\right\}. \tag{2.9}$$

□

The Edmonds-Gallai decomposition [24, 45, 46] of an undirected graph G is the partition of $V(G)$ into the following sets:

$$\begin{aligned} D(G) &:= \{v \in V : \exists \text{ maximum matching of } G \text{ not covering } v\}, \\ A(G) &:= \Gamma(D(G)), \\ C(G) &:= V \setminus (D(G) \cup A(G)) \end{aligned}$$

The main property of the decomposition is that $A(G)$ is an optimal choice for X in (2.9). This is stated in the Edmonds-Gallai decomposition theorem:

Theorem 2.4 (Edmonds-Gallai decomposition [22, 45, 46]). *If G is a finite graph, then*

$$\nu(G) = \frac{1}{2}\left[|V| + |A(G)| - o(G - A(G))\right] \ . \tag{2.10}$$

$\square$

Edmonds [24] also gave a polynomial-time method to construct the above decomposition.

3 Matroids

A *matroid* is a pair $\mathcal{M} = (E, \mathcal{I})$, of a finite *groundset* E and of a family of *independent sets* $\mathcal{I} \subseteq 2^E$ with the following properties:

$$\emptyset \in \mathcal{I}. \tag{3.1}$$

$$\text{If } I' \subseteq I \in \mathcal{I} \text{ then } I' \in \mathcal{I}. \tag{3.2}$$

$$\begin{gathered}\text{If } |I'| < |I| \text{ for } I, I' \in \mathcal{I} \text{ then there is an element} \\ e \in I \setminus I' \text{ such that } I' \cup \{e\} \in \mathcal{I}.\end{gathered} \tag{3.3}$$

The inclusionwise maximal members of $\mathcal{I}$ are the *bases* of $\mathcal{M}$. The set $\mathcal{B}$ of bases of a matroid has the following properties.

$$\mathcal{B} \neq \emptyset. \tag{3.4}$$

$$\text{If } B, B' \in \mathcal{B} \text{ then } |B| = |B'|. \tag{3.5}$$

$$\begin{gathered}\text{If } B, B' \in \mathcal{B} \text{ and } e \in B \setminus B' \text{ then there is an element} \\ f \text{ of } B' \setminus B \text{ such that } B \setminus \{e\} \cup \{f\} \in \mathcal{B}.\end{gathered} \tag{3.6}$$

Property (3.6) is often called the *weak basis-exchange axiom.* Subset C of E is a *circuit* of the above matroid $\mathcal{M}$ if $C \notin \mathcal{I}$, but any proper subset C' of C is independent. The family $\mathcal{C}$ of circuits of a matroid has the following properties.

$$\text{If } C, C' \in \mathcal{C} \text{ and } C' \subseteq C \text{ then } C = C'. \tag{3.7}$$

$$\begin{gathered}\text{If } C_1, C_2 \in \mathcal{C}, C_1 \neq C_2 \text{ and } e \in C_1 \cap C_2 \text{ then there is} \\ C \in \mathcal{C} \text{ such that } C \subseteq (C_1 \cup C_2) \setminus \{e\}\end{gathered} \tag{3.8}$$

Property (3.8) is called the *weak circuit-exchange axiom.* Finally, we define the *rank function* $r = r_{\mathcal{M}} : E \to \mathbb{N}$ of matroid $\mathcal{M}$ by $r(X) := \max\{|I| : X \supseteq I \in \mathcal{I}\}$. The rank function has the following properties:

$$r(\emptyset) = 0. \tag{3.9}$$

$$r(X \cup \{e\}) \leq r(X) + 1 \text{ for any } X \subseteq E \text{ and } e \in E \setminus X. \tag{3.10}$$

$$r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y) \text{ for any } X, Y \subseteq E \tag{3.11}$$

Properties (3.9,3.10) together are called the *subcardinality property*, and property (3.11) is referred to as the *submodularity* of the rank function.

Matroid bases, circuits and rank functions are characterized by the above properties.

Theorem 3.1. *If family $\mathcal{B} \subseteq 2^E$ has properties (3.4–3.6) and $\mathcal{I} = \{I \subseteq E : \exists B \in \mathcal{B}$ with $I \subseteq B\}$ then $\mathcal{M} = (E, \mathcal{I})$ is a matroid.*
If family $\mathcal{C} \subseteq 2^E$ has properties (3.7,3.8) and $\mathcal{I} = \{I \subseteq E : \nexists C \in \mathcal{C}$ with $I \supseteq C\}$ then $\mathcal{M} = (E, \mathcal{I})$ is a matroid.
If function $r : 2^E \to \mathbb{N}$ has properties (3.9–3.11) and $\mathcal{I} = \{I \subseteq E : r(I) = |I|\}$ then $\mathcal{M} = (E, \mathcal{I})$ is a matroid . □

In view of Theorem 3.1, we can also describe matroids by their bases, circuits or rank function. So when later we talk about a matroid $(E, \mathcal{B})$, $(E, \mathcal{C})$ or (E, r), then by these notations we implicitly mean that the particular matroid is given by its family of bases, circuits, or its rank function, respectively.

3.1 Matroid operations and special matroids

An interesting class of matroids are the class of linear matroids over some field $\mathbb{F}$. Matroid $\mathcal{M} = (E, \mathcal{I})$ is a *linear matroid* (over field $\mathbb{F}$), if there is a representing function $\phi : E \to \mathbb{F}^n$ such that $\mathcal{I}$ is the family of subsets of E that are mapped by ϕ into linearly independent subcollections of $\mathbb{F}^n$. Bases of the above linear matroid are the independent subsets I of E such that $\phi(I)$ spans $\phi(E)$ (over $\mathbb{F}$). The rank-function of $\mathcal{M}$ is $r(X) = \dim(\mathrm{span}_{\mathbb{F}}(\phi(X)))$. As we see, the terms "independence" and "basis" for matroids come from linear algebra. Vectors of $\mathbb{F}^n$ representing E are often arranged as columns of a matrix, and the rank function of column submatrices is the rank function of the linear matroid. The linear example is also the motivation for the notion of spanning: we say that element e of E is *spanned* by subset S of E in matroid $\mathcal{M} = (E, \mathcal{C})$ if there is a circuit C of $\mathcal{C}$ such that $e \in C \subseteq S \cup \{e\}$. The set of elements spanned by subset S of E in matroid $\mathcal{M}$ is denoted by $\mathrm{span}(S) = \mathrm{span}_{\mathcal{M}}(S)$.

A *binary matroid* is a linear matroid over the binary field $\mathbb{F}_2$. A special kind of binary matroid is the *graphic matroid* $\mathcal{M}(G) = (E, \mathcal{C})$, where $G = (V, E)$ is an undirected graph, and $\mathcal{C}$ is the family of edge-sets of cycles of G. (To see that graphic matroids are binary, we may represent each edge $e = uv$ of the graph by its binary characteristic vector $\chi^{\{u,v\}}$.) This example indicates that the name of matroid-circuit comes from graph theoretic notions.[2]

There are other well-known examples of matroids. Also, it is possible to construct matroids from other matroids through different matroid operations. We consider two most important ones: the minor and the dual operations. For disjoint subsets X, Y of E, we define the *matroid-minor*

$$\mathcal{M}/X \setminus Y := (E \setminus (X \cup Y), \mathcal{I}/X \setminus Y),$$

where

$$\mathcal{I}/X \setminus Y := \{J \subseteq E \setminus (X \cup Y) : J \cup X' \in \mathcal{I} \text{ for any } X \supseteq X' \in \mathcal{I}\}.$$

We say that $\mathcal{M}/X \setminus Y$ is obtained from matroid $\mathcal{M}$ by the *contraction* of X and the *deletion* of Y. The minor of a linear matroid is also linear over the same field and can

[2]At least we should recognize some relation between words *circuit* and *cycle*. Anyway, the union of disjoint circuits of a matroid is called a cycle. Never mind that a cycle of a graphic matroid does not necessarily correspond to a cycle of the underlying graph.

be represented by an appropriate submatrix of the representing matrix. For a graph G, we have $\mathcal{M}(G)/X \setminus Y = \mathcal{M}(G/X \setminus Y)$.

The *dual matroid* of matroid $\mathcal{M} = (E, \mathcal{B})$ is $\mathcal{M}^* = (E, \mathcal{B}^*)$ where $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$ is the family of bases of $\mathcal{M}^*$. If $\mathcal{M}$ is a matroid, then clearly $(\mathcal{M}^*)^* = \mathcal{M}$ and it is easy to check that $(\mathcal{M}/X \setminus Y)^* = \mathcal{M}^*/Y \setminus X$ holds for any disjoint subsets X and Y of E. As a convention, for a matroid $\mathcal{M}$ with set of independent sets, bases, circuits and the rank function $\mathcal{I}, \mathcal{B}, \mathcal{C}$ and r, respectively, we denote dual equivalents by $\mathcal{I}^*, \mathcal{B}^*, \mathcal{C}^*$ and r^*, and call them them the family of *dual independent sets, cobases, cocircuits* and *corank function* of $\mathcal{M}$, respectively.

The following theorem of Nash-Williams exhibits another matroid operation.

Theorem 3.2 (Nash-Williams [74]). *If matroids $\mathcal{M}_1 = (E, \mathcal{I}_1), \mathcal{M}_2 = (E, \mathcal{I}_2), \dots, \mathcal{M}_k = (E, \mathcal{I}_k)$ have rank functions $r_1, r_2, \dots, r_k$ and $\mathcal{I} := \{\bigcup_{j \in [k]} I_j : I_j \in \mathcal{I}_j\}$ then $\mathcal{M} = (E, \mathcal{I})$ is a matroid with rank function $r(X) = \min\{|X \setminus T| + \sum_{j \in [k]} r_j(T) : T \subseteq X\}$.* □

Matroid $\mathcal{M}$ in Theorem 3.2 is called the *sum* (or the *union*) of the above matroids $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$. Theorem 3.2 generalizes the matroid partition theorem of Edmonds.

Theorem 3.3 (Edmonds [23]). *For matroids $\mathcal{M}_1 = (E, \mathcal{I}_1), \mathcal{M}_2 = (E, \mathcal{I}_2), \dots, \mathcal{M}_k = (E, \mathcal{I}_k)$ there are independent sets $I_j \in \mathcal{I}_j$ for $j \in [k]$ such that $\bigcup_{j \in [k]} I_j = E$ if and only if $|A| \leq \sum_{j \in [k]} r_j(A)$ for any subset A of E.* □

It is not difficult to see that Theorem 3.2 is equivalent with the minmax formula of Edmonds on matroid intersection:

Theorem 3.4 (Edmonds [25]). *If $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ are matroids on the same groundset then $\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(X) + r_2(E \setminus X) : X \subseteq E\}$, where r_1 and r_2 are the rank functions of $\mathcal{M}_1$ and $\mathcal{M}_2$.* □

3.2 The greedy algorithm

Edmonds gave another alternative way of characterizing a matroid than Theorem 3.1.

Theorem 3.5 (Edmonds [26]). *For a finite set E and a family $\mathcal{B} \subseteq 2^E$, the following two statements are equivalent:*

- *$\mathcal{M} = (E, \mathcal{B})$ is a matroid.*
- *If $w : E \to \mathbb{R}$ is any weight function, $E = \{e_1, e_2, \dots, e_n\}$ such that $w(e_i) \leq w(e_{i+1})$ for $i \in [n-1]$ and $B(n, w)$ is defined recursively by $B(0, w) := \emptyset$ and*

$$B(i,w) = \begin{cases} B(i-1,w) \cup \{e_i\} & \text{if } \{e_i\} \cup B(i-1,w) \in B \\ & \text{for some } B \in \mathcal{B} \\ B(i-1,w) & \text{else.} \end{cases} \tag{3.12}$$

then $B(n, w)$ is a minimum weight element of $\mathcal{B}$, that is, for any $B \in \mathcal{B}$ we have $w(B(n, w)) := \sum_{e \in B(n,w)} w(e) \leq w(B) = \sum_{e \in B} w(e)$. □

We say that basis $B(n,w)$ has been selected by the greedy algorithm. (The name 'greedy' comes from the fact that in each step we select the most attractive element we can. The content of Theorem 3.5 is that matroids are exactly those structures where this simple method always gives an optimal result.) The minimum weight basis $B(n,w)$ in Theorem 3.5 has the following important properties:

$$\begin{gathered}\text{For any } e \in E \setminus B(n,w) \text{ there exists a circuit } C \in \mathcal{C} \text{ such that}\\ C \subseteq B(n,w) \cup \{e\} \text{ and } w(c) \le w(e) \text{ for any } c \in C.\\ \text{For any } e \in B(n,w) \text{ there exists a cocircuit } C^* \in \mathcal{C}^* \text{ such that}\\ C^* \subseteq (E \setminus B(n,w)) \cup \{e\} \text{ and } w(c) \ge w(e) \text{ for any } c \in C^*.\end{gathered}$$

(These properties follow directly from the greedy algorithm and basis axiom (3.6) for $\mathcal{B}$ and $\mathcal{B}^*$.)

The interested reader can find further material on matroids in the book of Welsh [103] Oxley [75], Truemper [98] or of Recski [81].

4 Submodular functions

Recall that if $L = (X, \wedge, \vee)$ is a lattice then we say that function $b : X \to \mathbb{R}$ is *submodular* if the submodular inequality

$$b(x) + b(y) \ge b(x \wedge y) + b(x \vee y) \tag{4.1}$$

holds for all elements x and y of X. A function $p : X \to \mathbb{R}$ is *supermodular* if $-p$ is submodular. If we also have a convex set P of L then we can speak about *crossing submodular* and *crossing supermodular* functions, which we define by requiring inequality (4.1) only for elements x and y that are crossing in P. The most common situation when we use submodular functions is in the case of subsetlattices. Here, setfunction $b : 2^X \to \mathbb{R}$ is submodular if

$$b(A) + b(B) \ge b(A \cap B) + b(A \cup B) \tag{4.2}$$

for any subset A and B of X. Subsets A and B of X are called *crossing* if none of sets $A \cap B, A \setminus B, B \setminus A, X \setminus (A \cup B)$ is empty, that is, if A and B are crossing in lattice $(2^X, \cap, \cup)$ with respect to the convex set that we get by deleting $\emptyset$ and X from this subsetlattice. If (4.2) is required only for crossing subsets A and B of X, then setfunction b is called *crossing submodular*. Setfunction p is *(crossing) supermodular* if $-p$ is (crossing) submodular.

We have already seen examples of submodular functions: the height function h of any modular lattice L is both sub- and supermodular (shortly modular), and the cut-function $d_G : V(G) \to \mathbb{N}$ of graph G defined in section 2 is submodular. (To prove this latter fact it is enough to observe that any edge of graph G contributes to the left hand side of (4.2) at least as much as its contribution to the right hand side.) The rank-function of a matroid is also submodular.

In a sense, sub- and supermodular functions can be regarded as discrete convex and concave functions for the following analogy. If $f \le g$ for some concave function

$f : \mathbb{R}^n \to \mathbb{R}$ and convex function $g : \mathbb{R}^n \to \mathbb{R}$ then there is a linear function $h : \mathbb{R}^n \to \mathbb{R}$ such that $f \le h \le g$. (The proof follows from the fact that two disjoint convex sets in $\mathbb{R}^{n+1}$ can be separated by a hyperplane.) For sub- and supermodular functions there is the following result of Frank:

Theorem 4.1 (Frank [36]). *Let X be a finite set and $b : 2^X \to \mathbb{Z}$ be a submodular and $p : 2^X \to \mathbb{Z}$ be a supermodular function with $b(\emptyset) = p(\emptyset) = 0$ and $p \le b$. Then there is a modular (both sub- and supermodular) function $m : 2^X \to 2^X$ such that $p \le m \le b$.* □

In most optimization problems involving submodular functions, these functions are used as upper bounds of some quantity to be maximized. A most natural problem of this type is the minimization of an integer valued submodular function. Special cases of this problem are the computation of the rank function of a matroid sum (i.e. the calculation of $r(A)$ in Theorem 3.2), the decision of partitionability of the common groundset of k matroids into independent sets (that is, the decision problem in Theorem 3.3 whether submodular function b is nonnegative, where $b(A) = -|A| + \sum_{i\in[k]} r_i(A)$ for $A \subseteq E$) or the problem of finding a maximum size common independent set of two matroids in Theorem 3.4. The general problem can be efficiently solved using involved tools of Combinatorial Optimization (like the ellipsoid method see [48, 49]). Cunningham [15] designed an efficient 'combinatorial' algorithm that minimizes an integer-valued submodular function in pseudopolynomial time (that is the running time of the algorithm depends polynomially on the maximum absolute value of the submodular function). Recently, using the ideas of Cunningham, Fleischer *et al.* [35] and Schrijver [94] independently found a strongly polynomial, 'combinatorial' algorithm for this problem.

We call function $f : 2^X \to \mathbb{R}$ *symmetric* if $f(A) = f(X \setminus A)$ for any subset A of X. If $b : 2^V \to \mathbb{R}$ is a symmetric submodular function then besides (4.2),

$$b(X) + b(Y) \ge b(X \setminus Y) + b(Y \setminus X) \tag{4.3}$$

holds for all subsets X, Y of V. The cut function $d_G : V(G) \to \mathbb{N}$ is an example of a symmetric submodular function. We remark that for a nonnegative weight function $w : E(G) \to \mathbb{R}_+$ the weighted cut function $d_{G,w}$ of graph G is also symmetric and submodular (where $d_{G,w}(U) := \sum_{e\in D_G(U)} w(e)$). Note that the minimization problem of the (nonnegative weighted) cut function is trivial, as clearly $\emptyset$ (and hence $V(G)$, too) minimizes d_G (and $d_{G,w}$). In fact it is true in general, that if b is a symmetric submodular function then by $2b(A) = b(A) + b(X \setminus A) \ge b(\emptyset) + b(X) = 2b(\emptyset)$, so $\emptyset$ minimizes b. But finding a *proper, nonempty* subset of $V(G)$ that minimizes the cut function of a graph or a symmetric submodular function is not trivial. There are several known methods for this problem in case of the cut function. To see that this problem can be solved in polynomial time, it is enough to observe that after fixing a source vertex s of G, the execution of $|V(G)| - 1$ maxflow-mincut algorithm between source s and sink v (where v is any other vertex of G than s) provides a minimum cut as the overall minimum cut of the $|V(G)| - 1$ outputs. (This is because some vertex of G is separated from s by a minimum cut.) There is a more efficient minimum cut algorithm of Nagamochi and Ibaraki [73] that has been extended by Queyranne

[79, 80] to minimize symmetric submodular functions (for a simple proof and further extensions see Rizzi [83]). There are also very fast nondeterministic algorithms solving the minimum cut problem for graphs.

The rest of this section is devoted to an important theorem of Frank and Jordán on covering crossing bisupermodular functions. Let us fix sets X_1 and X_2. An element (T, H) of $2^{X_1} \times 2^{X_2}$ we call a *pair*, and we say that T is the *tail* and H is the *head* of the pair. Two pairs (T, H) and (T', H') are said to be *tail-disjoint (head-disjoint)* if $T \cap T' = \emptyset$ ($H \cap H' = \emptyset$). A family $\mathcal{F} \subseteq 2^{X_1} \times 2^{X_2}$ is *half-disjoint* if any two pairs of $\mathcal{F}$ are tail- or head-disjoint. A function $p : 2^{X_1} \times 2^{X_2} \to \mathbb{R}$ is called *crossing bisupermodular*, if

$$p(T, H) + p(T', H') \leq p(T \cap T', H \cup H') + p(T \cup T', H \cap H') \tag{4.4}$$

for any $T, T' \subseteq X_1$ and $H, H' \subseteq X_2$, provided that $T \cap T' \neq \emptyset \neq H \cap H'$ and $p(T, H) \neq 0 \neq p(T', H')$. A vector z of $\mathbb{R}^{X_1 \times X_2}$ *covers* function $p : 2^{X_1} \times 2^{X_2} \to \mathbb{R}$ if $z(T, H) \geq p(T, H)$ for any $T \subseteq X_1$ and $H \subseteq X_2$, where $z(T, H) := \sum_{(x_1, x_2) \in T \times H} z(x_1, x_2)$. Finally, $\tau_p := \min\{z(X_1, X_2) : z \in \mathbb{Z}^{X_1 \times X_2} \text{ covers } p\}$ and $\nu_p := \max\{p(\mathcal{F}) : \mathcal{F} \subseteq 2^{X_1} \times 2^{X_2} \text{ is half disjoint}\}$, where $p(\mathcal{F}) := \sum_{(T,H) \in \mathcal{F}} p(T, H)$.

Frank and Jordán [37] proved the following minmax theorem:

Theorem 4.2 (Frank-Jordán [37]). *If $p : 2^{X_1} \times 2^{X_2} \to \mathbb{N}$ is crossing bisupermodular then $\nu_p = \tau_p$.* □

To indicate the depth of Theorem 4.2, we remark that the following theorem of Edmonds [21] on packing matroid bases is a consequence.

Theorem 4.3 (Edmonds [21]). *Let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k \subset 2^E$ be the sets of bases and $r_1, r_2, \ldots, r_k : 2^E \to \mathbb{N}$ be the rank functions of matroids $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$, respectively. There are pairwise disjoint bases $B_i \in \mathcal{B}_i$ if and only if $\sum_{i \in [k]} r_i(Z) - r_i(E \setminus Z) \leq |Z|$ for each subset Z of E.*

(With the help of a $(k+1)$st matroid for one direction and by adding appropriate new elements to E for the other, it is not difficult to prove that Theorem 4.3 is equivalent with Theorem 3.3.)

5 Polyhedral combinatorics

Let X be a subset of the d-dimensional space $\mathbb{F}^d$ for some field $\mathbb{F} \in \{\mathbb{R}, \mathbb{Q}\}$. For arbitrary subsets X and Y of $\mathbb{F}^d$, $X + Y := \{x + y : x \in X, y \in Y\}$ denotes the *Minkowski sum* of X and Y. In this section, when we talk about subsets of vectorspaces then (unless we do not state the opposite) we always mean a closed subset of the vectorspace. For subset X of $\mathbb{F}^d$ let

$$\begin{aligned}
\mathrm{span}(X) &:= \{\textstyle\sum_{i\in[k]} \lambda_i x_i : x_i \in X, \lambda_i \in \mathbb{F}, k \in \mathbb{N}\},\\
\mathrm{aff}(X) &:= \{\textstyle\sum_{i\in[k]} \lambda_i x_i : x_i \in X, \lambda_i \in \mathbb{F}, \sum_{i\in[k]} \lambda_i = 1, k \in \mathbb{N}\},\\
\mathrm{cone}(X) &:= \{\textstyle\sum_{i\in[k]} \lambda_i x_i : x_i \in X, 0 \leq \lambda_i \in \mathbb{F}, k \in \mathbb{N}\},\\
\mathrm{conv}(X) &:= \{\textstyle\sum_{i\in[k]} \lambda_i x_i : x_i \in X, 0 \leq \lambda_i \in \mathbb{F}, \sum_{i\in[k]} \lambda_i = 1, k \in \mathbb{N}\},\\
\dim(X) &:= \min\{|Y| - 1 : X \subseteq \mathrm{aff}(Y)\},
\end{aligned}$$

denote the *span*, the *affine hull*, the *cone*, the *convex hull* and the *dimension* of set X, respectively.

Subset X of $\mathbb{F}^d$ is called an *(affine) subspace* if $\mathrm{span}(X) = X$ ($\mathrm{aff}(X) = X$). Set X is a *cone* if $\mathrm{cone}(X) = X$ and X is *convex*, if $\mathrm{conv}(X) = X$. We can characterize (affine) subspaces in terms of hyperplanes. Subset H of $\mathbb{F}^d$ is an *affine hyperplane* if there is a vector a of $\mathbb{F}^d \setminus \{0\}$ and a number b of $\mathbb{F}$ such that $H = \{x \in \mathbb{F}^d : x^T a = b\}$. If 0 lies in affine hyperplane H (that is, if $b = 0$), then H is a *hyperplane*. (The proof of the following facts can be found in the book of Schrijver [93].)

Theorem 5.1. *Subset X of vectorspace $\mathbb{F}^d$ is a subspace if and only if X is an intersection of hyperplanes.X is an affine subspace if and only if X is an intersection of affine hyperplanes.* □

(Affine) subspaces and cones are clearly convex. There is a characterization of convex sets in terms of halfspaces. Subset H^+ of vectorspace $\mathbb{F}^d$ is an *affine halfspace* if and only if there is a vector a of $\mathbb{F}^d$ and a number b of $\mathbb{R}$ such that $H^+ = \{x \in \mathbb{F}^d : x^T a \geq b\}$. If $b = 0$ then H^+ is a *halfspace*. If $a \neq 0$ then the above halfspace is *bordered* by affine hyperplane $H = \{x \in \mathbb{F}^d : x^T a = b\}$.

Theorem 5.2. *Subset X of vectorspace $\mathbb{F}^d$ is convex if and only if X is the intersection of affine halfspaces. X is a cone if and only if X is the intersection of halfspaces.* □

A *polytope* is the convex hull of a finite set, a *polyhedron* is the intersection of finitely many affine halfspaces. A *polyhedral cone* is the intersection of finitely many halfspaces.

Theorem 5.3 (Farkas [28, 29], Minkowski [72], Weyl [104]). *Subset P of $\mathbb{F}^d$ is a polytope if and only if P is a bounded polyhedron. P is a polyhedral cone if and only if P is the cone of a finite set.* □

Hyperplane H *supports* convex subset C of vectorspace $\mathbb{F}^d$ if $H \cap C \neq \emptyset$ and there is a halfspace H^+ bordered by H that contains C. Element x of convex set C is an *extremal point* if $x \notin \mathrm{conv}(C \setminus \{x\})$. Each extremal point $\{x\}$ of a polyhedron P has the property that there is a supporting hyperplane H of P such that $H \cap P = \{x\}$. Extremal points of polyhedra are called *vertices*. For a polyhedron P, the set of vertices of P are denoted by $\mathrm{vert}(P)$. Subset F of polyhedron P is a *face* of P if $F = P$ or $F = \emptyset$ or $F = P \cap H$ for some supporting hyperplane H of P. (That is,

$F = \emptyset$, or $F = \{x \in P : c^T x \geq c^T y$ for all $y \in P\}$ for some $c \in \mathbb{F}^d$.) Face F is a *facet* if $\dim(F) = \dim(P) - 1$.

Theorem 5.4. *If polyhedron P is the convex hull of some subset X of $\mathbb{F}^d$ and F is a face of P then* $\mathrm{vert}(F) = F \cap \mathrm{vert}(P) \subseteq X$. *If P is a polytope then* $F = \mathrm{conv}(\mathrm{vert}(F))$.

If polyhedron $P = \bigcap_{i \in [k]} H_i^+$ is the intersection of finitely many halfspaces then the facets of P are of the form $P \cap H_{i(j)}$ for some $j \in [l]$ and $i(j) \in [k]$, where H_i is the hyperplane that borders H_i^+. If, furthermore P is d-dimensional then $P = \bigcap_{j \in [l]} H_{i(j)}^+$. □

Note, that any polytope P of $\mathbb{F}^d$ can be described as $\{Mx : 0 \leq x \in \mathbb{F}^d, \mathbf{1}^T x = 1\}$ by some d by n matrix M, with as columns the vertices of P. Similarly, any polyhedron P can be described with the help of an n by d matrix A and a vector b as $P = \{x \in \mathbb{F}^d : Ax \geq b\}$.

5.1 Integer programming

One basic idea in Combinatorial Optimization is the following. We have some family $\mathcal{F}$ of subsets of some finite set X (the family of 'solutions') and we have some cost function $c : X \to \mathbb{R}$ on X. We would like to choose a solution of minimum cost, that is, an element Y of $\mathcal{F}$ minimizing $c(Y) := \sum_{x \in Y} c(x)$. Problems of this type are finding a maximum size chain of a poset or a minimum cut in a graph. In the first case, the elements of $\mathcal{F}$ are linearly ordered subsets of elements of the poset and the cost of each element is -1. In the second case, solutions are certain edge-sets, and the cost of each edge is 1. Let $\chi : 2^X \to \mathbb{F}^X$ be the characteristic function, i.e. χ^Y is the characteristic vector[3] of subset Y of X, with x-coordinate

$$\chi^Y(x) = \begin{cases} 1 \text{ if } x \in Y \\ 0 \text{ if } x \notin Y. \end{cases}$$

Define $P_{\mathcal{F}} := \mathrm{conv}\{\chi^Y : Y \in \mathcal{F}\}$. Instead of selecting the optimal solution by brute force, checking possibly exponentially many vertices of a polytope, we try to exploit the machinery of linear programming to optimize an objective function c on $P_{\mathcal{F}}$. That is, we are looking for a supporting hyperplane of $P_{\mathcal{F}}$ of the form $H_c = \{x \in \mathbb{F}^d : c^T x = p\}$ and try to find in $H_c \cap P_{\mathcal{F}}$ a characteristic vector of an element of $\mathcal{F}$. To be able to do so, we should be able to characterize $P_{\mathcal{F}}$ in terms of halfspaces. If we find a good characterization then we can apply e.g. the ellipsoid method to solve the optimization problem.[4]

Often it is easy to describe a polyhedron $P = \{x \in \mathbb{F}^d : Ax \leq b\}$ for which we know that $P_{\mathcal{F}} \cap \mathbb{Z}^d = P \cap \mathbb{Z}^d$, hence $P_{\mathcal{F}} = \mathrm{conv}(P \cap \mathbb{Z}^d)$. It means of course that $P_{\mathcal{F}} \subseteq P$, and sometimes, only by standard methods of integer programming, it is possible to prove that the two polyhedra are the same.

As an example, consider the case of Theorem 1.1. For poset $(X, \preceq)$, a maximum size chain is nothing else but a minimum cost linearly ordered subset of X where the

[3]Note that sometimes the term *characteristic function* is used for the above vector.

[4]Note that the ellipsoid method is not practical. In practice, the simplex method (which is not a polynomial one) works much better.

cost of each element of X is -1. Let $\mathcal{F}$ be the set of chains in X. Clearly,

$$P_{\mathcal{F}} \subseteq P := \{x \in \mathbb{R}^{|X|} : x \geq \mathbf{0}, (\chi^A)^T x \leq 1 \text{ for any antichain } A \text{ of } X\} . \tag{5.1}$$

Moreover, any integer vector of P is a characteristic vector of a chain. But at this point, it is not clear whether $P = P_{\mathcal{F}}$ or not. This question can be formulated as deciding the integrality of polyhedron P, where polyhedron P is called an *integer polyhedron* if $P = \mathrm{conv}(P \cap \mathbb{Z}^d)$. To answer this question positively, we use two results.

Lemma 5.5 (Edmonds-Giles [27]). *Polyhedron P in space $\mathbb{F}^d$ is integer if and only if* $\max\{c^T x : x \in P\} \in \mathbb{Z}$ *for any integer vector c of $\mathbb{Z}^d$ for which the maximum is finite.* □

The other tool we need is linear programming duality.

Theorem 5.6 (von Neumann [101], Gale *et al.* [43]). *For any $n \times d$ matrix A over $\mathbb{F}$ and vectors $c \in \mathbb{F}^d$ and $b \in \mathbb{F}^n$,*

$$\max\{c^T x : \mathbf{0} \leq x, x \in \mathbb{F}^d, Ax \leq b\} = \min\{y^T b : \mathbf{0} \leq y, y \in \mathbb{F}^n, yA \geq c\}, \tag{5.2}$$

if both sets in (5.2) are nonempty. □

The pair of linear programming problems in Theorem 5.6 is called a primal-dual pair of linear programs. If one of them declared to be the "primal" problem then the other is referred to as the "dual" one.

In case of our poset example, the nonemptyness condition of Theorem 5.6 trivially holds, so we get that for $P_c^* := \{y : \mathbf{0} \leq y \in \mathbb{R}^{\mathcal{A}}, y(a) \geq c(a) \text{ for } a \in X\}$,

$$\max\{c^T x : x \in P\} = \min\{y^T \mathbf{1} : y \in P_c^*\}. \tag{5.3}$$

where $\mathcal{A}$ is the family of antichains of poset $(X, \preceq)$, and $y(a) := \sum_{a \in A \in \mathcal{A}} y(A)$. In light of Lemma 5.5, to show that there is an equality in (5.1), we have to prove that (5.3) is integer whenever $c \in \mathbb{Z}^d$. But this is a direct consequence of the following (trivial) generalization of Theorem 1.1.

Theorem 5.7. *If $(X, \preceq)$ is a poset and $c : X \to \mathbb{Z}$ then the minimum number of antichains such that each element a of X is contained in at least $c(a)$ antichains is the maximum of $\sum_{a \in C} c(a)$ for chains C of X.*

Proof. Apply Theorem 1.1 for partially ordered set $(X_c, \preceq_c)$, where $X_c := \{a(i) : a \in X, i \in [c(a)]\}$ and partial order $\preceq_c$ is defined by $a(i) \preceq_c b(j)$ if $a \prec b$ or $a = b$ and $i \leq j$. □

Recall that $\mathcal{A}$ denotes the family of antichains of poset $P = (X, \preceq)$, and let $\mathcal{C}$ be the set of chains of P. According to Theorem 5.7, for any integer vector $c \in \mathbb{F}^d$ there is a chain C and a family of antichains $\mathbf{A} \subseteq \mathcal{A}$ in such a way that $c^T \chi^C = (\chi^{\mathbf{A}})^T \mathbf{1}$. As $\chi^C \in P$ and $\chi^{\mathbf{A}} \in P_c^*$, the common value in (5.3) is integer, indeed. What we got is, that

$$\mathrm{conv}\{\chi^C \in \mathbb{F}^X : C \in \mathcal{C}\} = \{x \in \mathbb{F}^X : \mathbf{0} \leq x, x(A) \leq 1 \text{ for any } A \in \mathcal{A}\}, \tag{5.4}$$

where $x(A) = \sum_{a\in A} x(a)$.

Our next aim is to find a similar linear characterization for the convex hull of antichains. To achieve this, we shall use blocking theory of polyhedra.

Polyhedron $P \subseteq \mathbb{F}_+^d$ is a *blocking type polyhedron* if $P = P + \mathbb{F}_+^d$, and it is an *antiblocking type polyhedron* if $P = \mathbb{F}_+^d \cap (P + \mathbb{F}_-^d)$. Any finite subset H of $\mathbb{F}_+^d$ defines a blocking and an antiblocking polyhedron by

$$H^\uparrow := \mathrm{conv}(H) + \mathbb{F}_+^d \qquad \text{and} \qquad H^\downarrow := \mathbb{F}_+^d \cap (\mathrm{conv}(H) + \mathbb{F}_-^d),$$

respectively. For a polyhedron P

$$\begin{aligned} B(P) &:= \{x \in \mathbb{F}_+^d : x^T y \geq 1 \text{ for all } y \in P\} \text{ and} \\ A(P) &:= \{x \in \mathbb{F}_+^d : x^T y \leq 1 \text{ for all } y \in P\} \end{aligned}$$

are the *blocking* and *antiblocking polyhedron* of P, respectively. As suggested by the name, if P is a polyhedron then both $A(P)$ and $B(P)$ are polyhedra.

Theorem 5.8 (Fulkerson [39, 40, 41]). *If P is a blocking type polyhedron then $B(P)$ is a blocking type polyhedron and $P = B(B(P))$. If P is an antiblocking type polyhedron then $A(P)$ is an antiblocking type polyhedron and $P = A(A(P))$. Furthermore,*

$$B(\{x_1, x_2, \dots, x_n\}^\uparrow) = \{y \in \mathbb{F}_+^d : \quad y^T x_i \geq 1 \text{ for } i \in [n]\} \tag{5.5}$$

$$\begin{aligned} A(\{x_1, x_2, \dots, x_n\}^\downarrow) + C_M = \{y \in \mathbb{F}_+^d : \quad & y^T x_i \leq 1 \text{ for } i \in [n] \text{ and} \\ & y(m) = 0 \text{ for } m \in M\} \end{aligned} \tag{5.6}$$

for any $n \in \mathbb{N}$ and elements x_i ($i \in [n]$) of $\mathbb{F}_+^d$ where for subset M of $[d]$ cone $C_M := \{x \in \mathbb{F}^d : x \geq 0 \text{ and } x(m) = 0 \text{ for } m \in [d] \setminus M\}$ is the projection of the positive orthant to $\mathbb{F}^d$. □

Observe that the polytope described in (5.4) is of antiblocking type. Combining this with Theorem 5.8 we get the following.

Corollary 5.9. *If $(X, \preceq)$ is a poset and $\mathcal{C}$ and $\mathcal{A}$ is the family of its chains and antichains, respectively, then*

$$\mathrm{conv}\{\chi^A \in \mathbb{F}^X : A \in \mathcal{A}\} = \{x \in \mathbb{F}^X : \mathbf{0} \leq x, x(C) \leq 1 \text{ for any } C \in \mathcal{C}\} \quad \square$$

After this result, we discuss the integrality of a general polyhedron

$$P = \{x \in \mathbb{F}^d : Ax \leq b\} \tag{5.7}$$

(where $A \in \mathbb{F}^{n\times d}$). According to Lemma 5.5 and linear programming duality, P is integer if and only if for the optimum value of the dual

$$\min\{y^T b : \mathbf{0} \leq y \in \mathbb{F}^n,\ yA = c\} \notin \mathbb{F} \setminus \mathbb{Z} \tag{5.8}$$

holds for any d-dimensional integral vector c. (That is, if there is a finite optimum then the objective value must be integral.) If for each vector c of $\mathbb{Z}^d$ with finite optimum in (5.8) there is an *integer* optimum vector of (5.8), we say that $Ax \leq b$ in (5.7) is a *totally dual integral system*, or shortly a *TDI* system. So if $Ax \leq b$ is TDI and b is integral, then polyhedron P in (5.7) is integral. There is a characterization of TDI systems in terms of Hilbert bases. We say that subset H of $\mathbb{F}^d$ is a *Hilbert basis*, if for any element z of $\mathrm{cone}(H) \cap \mathbb{Z}^d$ there is a natural number k, a subset $\{h_i : i \in [k]\}$ of H and coefficients $\lambda_i \in \mathbb{N}$ such that $z = \sum_{i \in [k]} \lambda_i h_i$ (that is any integer vector in the cone of H is a nonnegative integer combination of H). Hilbert basis H is *integral* if $H \subset \mathbb{Z}^d$. It is well-known that for any rational cone C, there exists a finite, integral Hilbert basis H such that $\mathrm{cone}(H) = C$. Moreover, if C does not contain a 1-dimensional subspace then there is a unique inclusionwise minimal Hilbert basis of C.

Theorem 5.10. *System $Ax \leq b$ is TDI if and only if set $H_F := \{a_i : i \in [n], a_i f = b_i$ for each $f \in F\}$ is a Hilbert basis for any face F of $\{x \in \mathbb{F}^d : Ax \leq b\}$, where $a_1, a_2, \ldots, a_n$ are the rows of A and $b = (b_1, b_2, \ldots, b_n)^T$.* □

The question about a possible integer analogue of Carathéodory's theorem is an interesting problem about Hilbert bases. The following bound has been proved by Cook *et al.*[13] .

Theorem 5.11 (Cook *et al.* [13]). *If H is a Hilbert basis in $\mathbb{Q}^d$ and $C = \mathrm{cone}(H)$ does not contain a positive dimensional linear subspace then for any vector z of $C \cap \mathbb{Z}^d$ there is are elements h_i of H and coefficients λ_i of $\mathbb{N}$ for $i \in [2d-1]$ such that $z = \sum_{i \in [2d-1]} \lambda_i h_i$.* □

András Sebő proved the above theorem with bound $2d-2$ instead of $2d-1$. It was conjectured that this bound can be as small as d, but recently it has been disproved. Still, it is an interesting question to determine a better bound than that of Sebő.

5.2 Linear description of combinatorial polyhedra

In what follows, we give linear characterizations of integer polyhedra $P_{\mathcal{F}}$ in case of certain special families $\mathcal{F}$. We use the abbreviation $x(H) := \sum_{e \in H} x(e)$.

Theorem 5.12 (Birkhoff). *If $G = (V, E)$ is a finite bipartite graph then*

$$\begin{aligned} &\mathrm{conv}\{\chi^F : F \subseteq E \textit{ is a matching of } G\} = \\ &\qquad \{x : \mathbf{0} \leq x \in \mathbb{F}^d, x(D(v)) \leq 1 \textit{ for each } v \in V\}. \quad \square \end{aligned}$$

Edmonds [24] has characterized the matching polyhedra of nonbipartite graphs.

Theorem 5.13 (Edmonds [24]). *If $G = (V, E)$ is a finite graph and $E(U)$ is the set of edges of E spanned by U then*

$$\mathrm{conv}\{\chi^F : F \subseteq E \textit{ is a matching of } G\} =$$

$$\{x : \mathbf{0} \leq x \in \mathbb{F}^d, x(D(v)) \leq 1 \textit{ for } v \in V, x(E(U)) \leq \left\lfloor \frac{|U|}{2} \right\rfloor \textit{ for } U \subseteq V\}, \tag{5.9}$$

$$\mathrm{conv}\{\chi^F : F \subseteq E \textit{ is a perfect matching of } G\} = \tag{5.10}$$

$$\{x : \mathbf{0} \leq x \in \mathbb{F}^d, x(D(v)) = 1 \textit{ for } v \in V, x(D(U)) \geq 1 \textit{ for odd sets } U \subseteq V\}. \quad \square$$

A generalization of Theorem 5.12 to matroids is the description of the matroid intersection polytope by Edmonds [25]:

Theorem 5.14 (Edmonds [25]). *If $\mathcal{M}_1 = (E, r_1)$ and $\mathcal{M}_2 = (E, r_2)$ are matroids on the same groundset then*

$$\operatorname{conv}\{\chi^I : I \textit{ is independent both in } \mathcal{M}_1 \textit{ and in } \mathcal{M}_2\} =$$
$$\{x : \mathbf{0} \leq x \in \mathbb{F}^E, x(F) \leq r_i(F) \textit{ for any } i \in \{1,2\} \textit{ and } F \subseteq E\}.$$

5.3 The ellipsoid method

One use of linear descriptions of integer polyhedra is that by well-known methods of linear programming we can efficiently solve linear optimization problems over them. That is, we can find e.g. minimum cost (stable) matchings or maximum weight common independent sets of matroids. Moreover, by linear programming duality, one can prove the optimality of a certain solution of the primal program by finding a feasible solution of the dual program of the same cost. Based on the well-known simplex method, there are several ways to solve such optimization problems in practice. Although this approach often works very satisfyingly, usually there is no proof that the time the method takes is polynomial in the size of the input.

However, there is another approach to linear programming, namely the ellipsoid method, that is a polynomial one in the above sense. The significance of this does not lie in everyday applications (because in practice the simplex method performs much better) but rather in the fact that we *can* use it to prove that some problems can be solved in polynomial time.

With the help of the ellipsoid method, Grötschel *et al.* [48, 49] have shown that the problem of optimizing a linear function over a polyhedron P in $\mathbb{Q}^d$ and the problem of separating a vector y from P by a hyperplane are polynomially equivalent in the following sense. If P can be described by linear constraints of size at most m then there exists an optimization algorithm that for a vector $c \in \mathbb{Q}^d$ either computes an element $x \in P$ maximizing $\{c^T x : x \in P\}$ or proves that such a vector does not exist, in time polynomial in d, m, the size of c, and the running time of the separation algorithm. On the other hand, there exists a separation algorithm that for a vector y either computes a halfspace that contains P and is disjoint from y, or proves that $y \in P$ in time polynomial in d, m, the size of y, and the running time of the optimization algorithm. (The *size* of the above terms are roughly the number of bits we need to describe them.)

It is also proved that if there is a membership testing algorithm that decides whether a vector x belongs to polyhedron P or not, and an element x_0 of P is known, then there is an optimization algorithm that runs in time polynomial in d, m, the size of x_0, and the running time of the membership testing algorithm.

The interested reader is referred to the book of Schrijver [93] and Grötschel *et al.* [49], where besides the missing proofs (s)he can find further material and references on polyhedral optimization.

Chapter II

Crossing structures

Our discussion in this chapter is mainly about crossing sets, that are crossing elements of the family of nonempty proper subsets of a groundset. In 1.1, we have defined the notion of crossing elements of a partially ordered set. By specializing that notion of crossing to the poset of subsets of a groundset, we define subsets A and B of groundset X to be *crossing* if none of $A \cup B, A \setminus B, B \setminus A, X \setminus (A \cup B)$ is empty.

In this chapter, we discuss three crossing related problems. First, we describe an uncrossing algorithm by exhibiting a finite winning strategy for a certain uncrossing game. For this reason, we present a straightforward extension of a result of Hurkens *et al.* [54] from subsets to lattices. We shall use this extension later, in Section 20 to prove a crucial consequence of the lattice structure of so-called $\mathcal{FG}$-kernels. Next, based on the result of Hurkens *et al.* [54], we describe a winning strategy of a more involved uncrossing game motivated by Frank and Jordán's minmax relation for covering a crossing bisupermodular function (Theorem 4.2). This winning strategy implies an efficient algorithm to construct a half-disjoint family of pairs with maximum demand as in Theorem 4.2.

In 7, we prove a conjecture of Frank (Theorem 7.2) on "symmetric posets" that model crossing-related properties of sets. The result contains Dilworth's theorem and the edge-cover formula for graphs. Further, we indicate some consequences of this result for the l_1-embeddability problem of finite metric spaces.

We close the chapter with an extremal problem on 3-cross-free families, by solving a special case of a conjecture of Karzanov. We show that if a family $\mathcal{F}$ of subsets of X does not contain 3 pairwise crossing sets then the size of $\mathcal{F}$ is a linear function of the size of X. By this, we provide a short and straightforward proof for a strengthening of a complicated result of Pevzner [76].

In the rest of this introductory part of Chapter II, we mention some basic facts about the notion of crossing sets. First, observe that A and B are crossing if and only if A and $X \setminus B$ are crossing.

A family $\mathcal{F} \subseteq 2^X$ is *laminar* (or *cross-free*) if $\mathcal{F}$ does not contain two crossing elements. Let $\mathcal{F}$ be a cross-free family on finite groundset X and fix an element x of X. Consider family $\mathcal{F}' := \{A \subseteq X \setminus \{x\} : A \in \mathcal{F} \text{ or } X \setminus A \in \mathcal{F}\}$. It is straightforward to prove that the diagram of poset $(\mathcal{F}', \subseteq)$ is a tree. This gives rise to a so-called tree-representation of cross-free families. For any cross-free family $\mathcal{F} \subseteq 2^X$, there exists a superset W of X, a tree $T_{\mathcal{F}} = (W, E)$, and a subset E' of E in such a way that

$$\mathcal{F}' = \{C_e \cap V : e \in E' \text{ and } C_e \text{ is a component of } T_{\mathcal{F}} - e \text{ not containing } x\}.$$

It follows easily from the existence of this tree-representation that the size of a laminar family is at most $4|X|$.

A family $\mathcal{F}$ of subsets of groundset X is *crossing* if for crossing members A, B of $\mathcal{F}$ both $A \cap B$ and $A \cup B$ are members of $\mathcal{F}$. If $b : 2^X \to \mathbb{R}$ is a (crossing) submodular function then it follows directly from the submodular property that the set of minima of b is a crossing family. Cross-free families are clearly crossing. Moreover, similarly to the above tree-representation, symmetric crossing families can be represented by so-called hypercactus hypergraphs [58, 16, 34]). (A family is symmetric if the complement of any member of the family is a member of the family.)

6 Uncrossing in lattices

In this section we shall deal with uncrossing problems that are motivated by Theorem 4.2 of Frank and Jordán. We will formulate the results in terms of lattices. After this, we explain how they can be used in connection with Theorem 4.2. The main result of this section also appears in [32].

6.1 The uncrossing game

Let $P = (E, \leq)$ be a convex set of a lattice $L = (X, \wedge, \vee)$ (e.g. $L = (2^V, \cap, \cup)$), and assume that we want to optimize a function $f : E \to \mathbb{R}_+$ that is a feasible solution for our problem. Often the following exchange property is true. If f is a feasible solution and $f(x) > 0 < f(y)$ for some crossing elements x, y of P (that is, x and y are uncomparable and $x \wedge y, x \vee y \in E$) then function f' is also a feasible solution, where

$$f'(z) = \begin{cases} f(z) - \varepsilon \text{ if } z = x \text{ or } z = y \\ f(z) + \varepsilon \text{ if } z = x \wedge y \text{ or } z = x \vee y \\ f(z) \text{ otherwise,} \end{cases}$$

for $\varepsilon := \min\{f(x), f(y)\}$. Moreover, the value of the objective function on f' is not worse than on f. This means that if f was an optimal solution of the problem then the above *uncrossing step* on f along x and y results in another optimal solution f'. Solution f is called *cross-free* if no uncrossing step can be performed on it. Observe that after the above uncrossing step, $f'(x) = 0$ or $f'(y) = 0$.

Our aim is to determine from a solution f a cross-free solution f' by performing a sequence of uncrossing steps. By this, we can reach a twofold goal. On one hand, the existence of a cross-free optimal solution might imply a stronger min-max relation than linear programming duality. E.g., the proof of Theorem 4.2 is based on the fact that there is a cross-free optimal solution of a related linear programming problem.

On the other hand, if we have an efficient uncrossing rule then we can use it in an algorithm to solve the particular combinatorial optimization problem, in polynomial time. This is done in 6.4, for the special case we describe next.

Our main motivation to consider uncrossing problems is Theorem 4.2. There, a fractional optimal solution for the dual packing problem (which we can find efficiently by the ellipsoid method) is a function $f : 2^X \times 2^Y \to \mathbb{Q}_+$. It is easy to see that if $A \cap A' \neq \emptyset \neq B \cap B'$ for some fractional optimum f then function f' is an optimum as well, where

$$f'(T,H) = \begin{cases} f(T,H) - \varepsilon & \text{if } (T,H) = (A,B) \text{ or } (T,H) = (A',B') \\ f(T,H) + \varepsilon & \text{if } (T,H) = (A \cap A', B \cup B') \\ & \text{or } (T,H) = (A \cup A', B \cap B') \\ f(T,H) & \text{otherwise,} \end{cases}$$

for $\varepsilon := \min\{f(A,B), f(A',B')\}$. In [37], an essential step of the proof of Theorem 4.2 is to show that there exists a cross-free optimum f. Although, from the proof in [37], one can easily construct a *finite* uncrossing method, it is not clear how to finish the whole uncrossing process in polynomial time. In 6.4, we exhibit such an algorithm. A consequence of this uncrossing algorithm is that there exists a polynomial time algorithm that constructs a half-disjoint system of maximum demand, as in Theorem 4.2.

In [54], Hurkens *et al.* solved the uncrossing problem for the special case where lattice $L = (2^V, \cap, \cup)$ is the lattice of subsets of some groundset V, and the convex set P in which they uncross is the lattice itself. The idea of the method is that the uncrossing strategy depends only on the support of function f to be uncrossed (thus it cannot depend on the concrete values of f on the certain subsets). This leads to a more general problem on finding a winning strategy of the following uncrossing game.

The game is played by player 1 and player 2 on a convex subset $P = (E, \leq)$ of a lattice L. There is also given a finite subset F of E. The game consists of *generalized uncrossing steps.* In each such step the actual subset F of E is modified as follows. Player 1 selects two elements a and b of F that cross in P. According to the choice of player 2, F is changed into $F \setminus \{a,b\} \cup \{a \wedge b, a \vee b\}$ or into $F \setminus \{a\} \cup \{a \wedge b, a \vee b\}$ or into $F \setminus \{b\} \cup \{a \wedge b, a \vee b\}$. (We can view this such that player 1 chooses two elements to uncross, and player 2 salvages at most one of these two elements that player 1 tries to eliminate by uncrossing.) Player 1 wins the game if there is no pair of crossing elements in the actual subset F. Player 2 wins if the actual F contains an earlier one. Note that this game is a nontrivial generalization of the uncrossing problem for real functions on E above, as any uncrossing step in the original problem can be regarded as a generalized uncrossing step on the support F of f with an appropriate move of player 2.

Extending the result of Hurkens *et al.* [54], but using exactly the same ideas, we show in 6.2 that if the generalized uncrossing problem is played on a lattice (in other words, if $P = L$), then player 1 has a winning strategy. Next, in 6.3, we prove our main result on uncrossing problems, by exhibiting a winning strategy for player 1 in case of certain convex subsets of finite distributive lattices. We remark that player 1 does not have a winning strategy for all convex sets of all distributive lattices. We illustrate this on Figure 6.1, where a diagram of a poset P' is depicted, with labels on the elements. These labels show that P' can be embedded into a subsetlattice, i.e. there exists a distributive lattice L and a convex set P of L such that P' on Figure 6.1 is a restriction of P. Moreover, we can choose the convex set P such that the set F of those eight elements that are indicated by full dots constitute a cross-free set. If the game starts on $F \cup \{x\}$ where x is one of the remaining elements of P, then player 1 has a forced move and player 2 can choose her move so that $F \cup \{x\}$ turns into $F \cup \{y\}$ for some other element y indicated by an empty dot. So player 2 can win in this case. This example also shows that even when the convex set is the lattice itself, player 1 might loose if she choose her uncrossing steps arbitrarily. Moreover, if we extend convex set P with a zero- and a unit-element, then we get a lattice. This shows that player 2 might have a winning strategy for convex sets of nondistributive lattices that are unions of intervals between atoms and co-atoms.

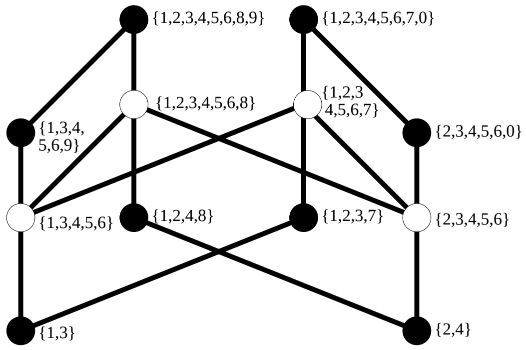

Figure 6.1: Essential part of convex set of distributive lattice where player 1 has no winning strategy.

6.2 The game on a lattice

In what follows we prove an easy extension of the result of Hurkens *et al.* [54].

Theorem 6.1. *If the above uncrossing game is played on a lattice $P = L = (X, \leq)$, then player 1 has a finite winning strategy. If the height of the lattice is bounded by n, and the game starts with subset $F = \{x_1, x_2, \ldots x_k\}$ of X then player 1 can win after $2nk$ generalized uncrossing steps.*

Proof. As in a lattice any two noncomparable elements are crossing, the aim of player 1 is to transform F into a chain. In order to achieve this goal, player 1 divides the game into stages, and in each stage he wins a subgame on a subset of the actual set F. After winning all stages, player 1 wins the entire game. In the ith stage player 1 starts an uncrossing game on a chain C_i ($C_1 := \emptyset$), together with element x_i. After player 1 has won the subgame, that is when $C_i \cup \{x_i\}$ is transformed into chain C_{i+1}, the $(i+1)$th stage starts.

In stage i, player 1 partitions the actual subset into three subsets C, D and U

with the property that

$$\begin{array}{l} C, D \text{ and } U \text{ are chains and} \\ c \vee d \leq u \text{ and } c \wedge d \leq d' \text{ for any } c \in C, d, d' \in D \text{ and } u \in U. \end{array} \tag{6.1}$$

To start the stage, player 1 chooses $C = C_i$, $D := \{x_i\}$ and $U_i := \emptyset$. As long as $C \neq \emptyset \neq D$, player 1 selects the maximal element c of the actual C and the maximal element d of the actual D. If they are comparable, player 1 redefines C, D and U by putting the greater element of c and d into U, and starts the uncrossing step anew. Observe that after this change, property (6.1) remains valid. If c and d are uncomparable (hence crossing), player 1 uncrosses them and puts $c \vee d$ into U and $c \wedge d$ into D. Then player 2 puts back at most one of c or d, and this element stays in the set C or D where it was before. Again, property (6.1) is preserved.

For an intermediate situation at stage i, let $a(|C|, |D|)$ denote the maximal number of further uncrossing steps that player 1 needs to win stage i if $c \wedge d \notin D$, and let $b(|C|, |D|)$ denote the same maximum when $c \wedge d \in D$ (c and d are the maxima of C and D that player 1 chooses to uncross in this situation). Clearly, $a(l, 0) = a(0, l) = 0$, and $b(l, 0) = b(1, 1) = b(0, l) = 0$, for any integer l. Using property 6.1, we get the following recursions.

$$\begin{array}{rcl} a(p,q) & \leq & 1 + \max\{b(p,q), a(p-1, q+1)\} \\ b(p,q) & \leq & 1 + \max\{b(p, q-1), a(p-1, q), b(p-1, q)\} \end{array}$$

It is easy to check that the solution of this recursion satisfies $a(p, q) \leq 2p + q - 1$ and $b(p, q) \leq 2p + q - 2$. Hence player 1 wins stage i after at most $2|C_i|$ uncrossing steps.

So far we described a finite winning strategy for player 1 to win a stage of the uncrossing game on a lattice. As this finite winning strategy works for all stages, player 1 can win all k stages (that is, the entire game), after finite number of steps. If n is the height of lattice L then one stage of the uncrossing game takes at most $2n$ time, and the winning strategy of player 1 requires at most $2nk$ uncrossing steps. $\square$

Remark. If the game is played on a modular lattice, we may assume that each stage is played on the free modular lattice L^m in Figure 1.2. Then, it is easy to prove that at any point of the stage, the sublattice generated by C and D is a homomorphic image of lattice $L^{m,k}$ in Figure 6.2. In the situation depicted in Figure 6.2, player 1 uncrosses c_{m-k} with d_k and puts $c_{m-k} \vee d_k$ into U. According to the move of player 2, the lattice generated by the new chains C and D will be either $L^{m-1,k-1}$, $L^{m-1,k}$ or $L^{m,k-1}$.

6.3 The distributive game

In 6.1, we mentioned that if P is a convex set of a distributive lattice, or if P is a convex set of a not necessarily distributive lattice between atoms and co-atoms, player 1 might not have a winning strategy. Next we show that if P is a convex set between a set of atoms and a set of co-atoms of a distributive lattice, then player 1 can win in finite time.

Thus, in what follows $L = (X, \wedge, \vee)$ is a finite distributive lattice, and poset $P = L_A^B$ is the convex set between some subset $A = \{a_1, a_2, \ldots, a_k\}$ of atoms and

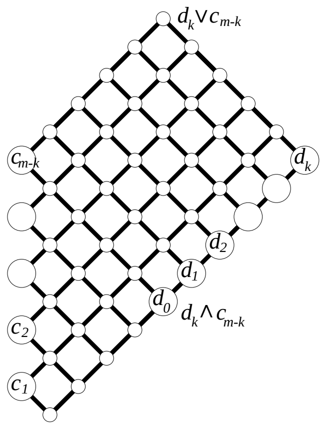

Figure 6.2: Lattice $L^{m,k}$ and the intermediate situation at a stage.

some subset $B = \{b_1, b_2, \dots, b_l\}$ of co-atoms of L. There is also given a set F of elements of P, which player 1 should uncross in P. The following lemma turns out to be crucial:

Lemma 6.2. *If $L = (X, \wedge, \vee)$ is a finite distributive lattice and a is an atom and b is a co-atom of L, then $x \vee y \geq a$ implies $x \geq a$ or $y \geq a$, and $x \wedge y \leq b$ implies $x \leq b$ or $y \leq b$ for any $x, y \in X$.*

Proof. If $x \vee y \geq a$ then $a = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$. As a covers $\mathbf{0}$, either $a \wedge x = a$ or $a \wedge y = a$, that is $a \leq x$ or $a \leq y$.

The above argument on $(X, \vee, \wedge)$ implies the analogous statement for the co-atoms, and Lemma 6.2 follows. □

So we have the following corollary.

Corollary 6.3. *Let $P = L_A^B$ be a convex set for a distributive lattice $L = (X, \wedge, \vee)$, a set of atoms A and set of co-atoms B. Then for any pair of elements x, y of P, for any minimal element a and maximal element b of P, we have that $x \vee y \geq a$ implies $x \geq a$ or $y \geq a$, and $x \wedge y \leq b$ implies $x \leq b$ or $y \leq b$.* □

Note that by Theorem 1.6, we know that distributive lattices are isomorphic to subsetlattices. In this sense, a winning strategy for our distributive uncrossing problem is not more general than a winning strategy for an uncrossing game where crossing sets can be uncrossed. The reason that we use the language of lattices unlike in [32], where the winning strategy is given in terms of sets, is twofold. On one hand, we can prove a more general theorem by relaxing distributivity, and on the other hand, for the application in 6.4, the lattice-form is more natural than the subset-language.

Theorem 6.4. *If for convex set $P = (E, \leq)$ of lattice $L = (X, \wedge, \vee)$ the consequence of Corollary 6.3 holds, then player 1 has a finite strategy to uncross a finite subset F of E. If the height of P is n, then player 1 can uncross F in $2kln(|F| + nkl)$ steps, where k and l are the number of minima and maxima of P, respectively.*

Proof. Let $A = \{a_1, a_2, \dots, a_k\}$ be the set of minima, and $B = \{b_1, b_2, \dots, b_l\}$ be the set of maxima of convex set P, that is $P = L_A^B$. Clearly, each interval of P is a lattice, so by to Theorem 6.1, player 1 can win the uncrossing game on $F_a^b := F \cap P_a^b$ for any interval P_a^b of P. In a so called *routine game* G_a^b, player 1 chooses an minimum a and a maximum b of P, and plays an uncrossing game on F_a^b until it is won. By this, player 1 transforms F into a set that we denote by $F[a, b]$. We shall show that if player 1 wins these routine games for each interval P_a^b in the lexicographic order of minimum-maximum pairs (a, b), then F becomes cross-free. Recall, that F is uncrossed if and only if F_a^b is a chain for each element a of A and b of B.

Claim 6.5. *For any $a \in A$, $b, b' \in B$, if F_a^b is cross-free then $(F[a, b'])_a^b$ is cross-free.*

Proof. From Corollary 6.3, $(F[a, b'])_a^b \setminus (F[a, b'])_a^{b'} = F_a^b \setminus F_a^{b'}$, which is a chain by assumption. $(F[a, b'])_a^b \cap (F[a, b'])_a^{b'}$ is a chain because $G_a^{b'}$ is won. In the beginning of G_a^b, every element of $F_a^b \setminus F_a^{b'}$ is less than any element of $F_a^{\{b,b'\}}$, and by Corollary 6.3, this property is preserved (6.3) after a generalized uncrossing step in G_a^b. □

For a set F and an element a_i of A, let $F[a_i]$ denote the set that we get after winning routine games $G_{a_i}^b$ for all $b \in B$. So $F[a_i] = (\dots((F[a_i, b_1])[a_i, b_2])\dots)$. We call this sequence of routine games the *ith phase of the uncrossing game*. Let $A(i) := \{a_j : 1 \le j \le i\}$.

Claim 6.6. *If $F_{A(i-1)}^B$ is cross-free, then $(F[a_i])_{A(i)}^B$ is cross-free.*

Proof. Claim 6.5 implies that $(F[a_i])_{a_i}^b$ is a chain for any $b \in B$. We have to show that the same holds for $(F[a_i])_a^b$ whenever $a \in A(i-1)$ and $b \in B$. By Claim 6.5, $(F[a_i])_a^b \cap (F[a_i])_{a_i}^b$ is a chain and by Lemma 6.3, so is $(F[a_i])_a^b \setminus (F[a_i])_{a_i}^b$. We will show that during the ith phase, for any intermediate set F' of the ith phase, for any $a \in A(i-1)$ and any $b \in B$

$$z \le x \text{ for any } z \in {F'}_a^b \setminus {F'}_{a_i}^b \text{ and } x \in {F'}_a^b \cap {F'}_{a_i}^b. \tag{6.2}$$

By assumption, property (6.2) is true for F.

If, indirectly, this is not the case at the end of the ith phase, then there is a first generalized uncrossing step which violates property (6.2). Suppose that it is first violated for $z \in {F'}_a^b \setminus {F'}_{a_i}^b$, right after a generalized uncrossing step in $G_{a_i}^{b'}$ ($b = b'$ is allowed), with elements $x, y \in {F'}_{a_i}^{b'}$, of some intermediate set F'.

If $x, y \in {F'}_a^b$ then by (6.2) $z \le x$ and $z \le y$, hence $z \le x \wedge y \le x \vee y$. Property (6.2) also holds when $x, y \notin {F'}_a^b$ as by Lemma 6.3 $x \wedge y, x \vee y \notin {F'}_a^b$. Thus by symmetry we may assume that $x \in {F'}_a^b \not\ni y$. If $a \not\le y \not\le b$ then by Lemma 6.3 $x \wedge y, x \vee y \notin {F'}_a^b$, so (6.2) cannot be violated. If $y \le b$ and hence $a \not\le y$ then $x \vee y \in {F'}_a^b \not\ni x \wedge y$, again by Lemma 6.3. As $z < x \le x \vee y$, property (6.2) holds after the uncrossing.

The remaining case is $a \le y \not\le b$. Again from Lemma 6.3, $x \vee y \notin {F'}_a^b$. From property (6.2) for ab, we have that $z < x \le b'$. Hence property (6.2) for ab' can be applied to z and it implies that $z < y$. So $z \le x \wedge y$, that is property (6.2) is preserved after a generalized uncrossing step. □

Let $F^{(0)} := F$ and for $1 \leq i \leq n$ let $F^{(i)} := F^{(i-1)}[a_i]$. By induction, using Claim 6.6, $(F^{(i)})^B_{A_i}$ is cross-free. In particular, $(F^{(n)})^B_A$ is cross-free, hence the above method is a finite winning strategy for the uncrossing game for player 1. If n is an upper bound on the length of chains of L then a routine game needs at most $2n|F|$ uncrossing steps and there are kl routine games. As the new elements of F that emerge after a routine game form a chain, there are at most nkl new elements in the final F. Hence, $2kln(|F|+nkl)$ is an upper bound on the number of uncrossing steps player 1 has to make. $\square$

6.4 An application to the set-pair cover problem

The motivation for solving the above particular uncrossing problem was to prove the existence of a polynomial-time algorithm for the dual problem of optimizing ν_p in Theorem 4.2 of Frank and Jordán.

To solve applications of Theorem 4.2, it is useful to have an efficient algorithm that provides both a half-disjoint family with maximum demand, and an optimal covering of p. The characteristic vector y of a half-disjoint family of maximum demand is an optimum of the integer program

$$\begin{aligned} max\{y^T p \quad : \quad & y \geq 0, y \text{ integer and} \\ & y(x_1, x_2) \leq 1 \text{ for every } (x_1, x_2) \in X_1 \times X_2\}, \end{aligned} \tag{6.3}$$

where $y : 2^{X_1} \times 2^{X_2} \to \mathbb{R}$ and $y(x_1, x_2) := \sum\{y(T, H) : x_1 \in T, x_2 \in H\}$. Consider the following integer program that solves the covering problem via an integer optimum z.

$$\begin{aligned} min\{z(X_1 \times X_2) \quad : \quad & z \geq 0, z \text{ integer and} \\ & z(T \times H) \geq p(T, H) \text{ for } (T, H) \in 2^{X_1} \times 2^{X_2}\}. \end{aligned} \tag{6.4}$$

It is easy to see that the linear relaxation of 6.3 is the dual linear programming problem of the linear relaxation of 6.4. Hence by linear programming duality (as both problems are feasible), the fractional optimum values of the two relaxations are equal. Theorem 4.2 can be interpreted such that for a crossing bisupermodular function p, these linear relaxations of (6.3) and (6.4) have integer optima. Frank and Jordán [37] also indicate a way to find an integer optimum z^* of (6.4) in polynomial time. What they do is that they compute a fractional primal optimum z using the ellipsoid method and with the help of this optimum they reduce the problem to another one where the bisupermodular function p' is "small". In this small problem, they can find "reducing" edges one by one, and this is sufficient for the efficient construction of an optimal solution z^*.

Let us introduce partial order $\preceq$ on $2^{X_1} \times 2^{X_2}$ by $(T, H) \preceq (T', H')$ if $T \subseteq T'$ and $H \supseteq H'$. This partial order defines a distributive lattice L on $2^{X_1} \times 2^{X_2}$ with lattice operations $(T, H) \wedge (T', H') = (T \cap T', H \cup H')$ and $(T, H) \vee (T', H') = (T \cup T', H \cap H')$. Define convex set $P = L^B_A$, where $A := \{(\{x_1\}, X_2) : x_1 \in X_1\}$ is a set of atoms and $B := \{(X_1, \{x_2\}) : x_2 \in X_2\}$ is a set of co-atoms of L. The bisupermodular property (4.4) of p yields that if y is an optimal solution of (6.3) and (T_1, H_1) and (T_2, H_2) are crossing pairs, then vector y' is also an optimal solution of (6.3), where

$$y'(T,H) = \begin{cases} y(T,H) - \varepsilon, \text{ if } (T,H) = (T_1,H_1) \text{ or } (T,H) = (T_2,H_2) \\ y(T,H) + \varepsilon, \text{ if } (T,H) \in \{(T_1,H_1) \wedge (T_2,H_2), (T_1,H_1) \vee (T_2,H_2)\} \\ y(T,H) \qquad, \text{ otherwise} \end{cases}$$

and $\varepsilon = \min\{y(T_1,H_1), y(T_2,H_2)\}$. That is, an uncrossing step transforms one optimum of (6.3) into another one.

Let y be an optimum of 6.3 found (say) by the ellipsoid method. It means that $|\mathcal{F}|$ is polynomial in the input size of the problem, where $\mathcal{F} := \text{supp } y$. By Theorem 6.4, we can uncross $\mathcal{F}$ in polynomial time so that we get cross-free family $\mathcal{F}^*$. This uncrossing procedure defines a sequence of transformations on the optima. This sequence transforms y into some rational optimum y^* in polynomial time such that $\mathcal{F}^* = \text{supp } y^*$. For the following proof, choose a positive integer M such that My^* is an integer vector.

Consider poset $(2^{X_1} \times 2^{X_2}, \preceq)$ and function $c(T,H) := My^*(T,H)$ on its elements. By (6.3), the total c-value of any chain of this poset is at most M. From Theorem 5.7, there are M antichains of $(2^{X_1} \times 2^{X_2}, \preceq)$ covering each pair (T,H) with multiplicity at least $c(T,H)$. The total p-weight of any antichain in this cover is at most $(y^*)^T p$, because any antichain corresponds to a half-disjoint family. On the other hand, the total p-value of the M antichains together is at least $c^T p = M(y^*)^T p$, as each element (T,H) is covered at least $c(T,H)$ times. It follows that all the M antichains in the cover above must have p-weight exactly $(y^*)^T p$, that is, all antichains in the optimal cover correspond to an optimal half-disjoint family. But we saw in the proof of Theorem 5.7 that we can choose the optimal antichain cover in such a way that one of the antichains is the set of $\prec$-minimal elements of supp y^*. Hence the $\prec$-minimal elements of any uncrossed optimum y^* is a half-disjoint family with maximum demand.

As a main application of Theorem 4.2, Frank and Jordán obtained a min-max formula for the unweighted directed node-connectivity augmentation problem. More precisely, they proved that the minimum number of arcs needed to augment a given directed graph to be k-node-connected equals the maximum total demand of a certain independent family. Here independent means that any edge can decrease the demand of at most one member of the family. By solving the primal problem (6.4), Frank and Jordán can find an optimum augmentation. A dual optimum that can be constructed via our uncrossing algorithm provides an independent family certifying the optimality of the primal solution.

7 Symmetric chain covers of symmetric posets

In this section we prove a min-max result on special partially ordered sets, conjectured by András Frank. It is a common generalization of Dilworth's theorem and of the well-known min-max formula for the minimum size edge cover of a graph. We also give an application of this result to the problem of minimum dimensional embedding of metric spaces into l_1-spaces. The results described in 7.1 and 7.2 also appear in [33].

To illustrate the method we are going to use, we show first a well-known construction that derives Dilworth's theorem from Kőnig's bipartite matching theorem (Theorem 2.2).

Theorem 7.1 (Dilworth [18]). *Let $P = (V, \prec)$ be a partially ordered set. The minimum number of chains of P that cover V equals the maximum size of an antichain of P.*

Proof. As the intersection of a chain and an antichain has at most one element, the size of any antichain of P is an lower bound on the size of any chain cover. In what follows, we construct a chain cover and an antichain such that the size of the cover is not more than the size of the antichain.

Define bipartite graph B on colour-classes V and V', where $V' = \{v' : v \in V\}$, by $E = E(B) := \{uv' : u \prec v\}$. Introduce perfect matching $M := \{vv' : v \in V\}$. Any chain $v_1 \preceq v_2 \preceq \ldots \preceq v_k$ of P corresponds to a path in $B + M$ with vertices $(v_1, v_1', v_2, v_2', \ldots, v_k, v_k')$ (a so-called *ME-alternating path*). Conversely, each ME-alternating path of $B + M$ is coming from a chain of P.

Hence the problem of covering P with the minimum number of (disjoint) chains can be equivalently formulated as the problem of covering $V(B)$ with the minimum number of (vertex-disjoint) ME-alternating paths. If we have a cover of B by k ME-alternating paths, then the E-edges of the paths in the cover form a matching of B of size $|V| - k$. On the other hand, if N is matching of B of size k, then $M \cup N$ is the edge-set of $|V| - k$ ME-alternating paths.

According to the above argument, the size of the minimum chain cover of P is $|V| - \nu(B)$. By Theorem 2.2, there is a subset U of $V \cup V'$ of size $\nu(B)$ that is adjacent to all edges of $E(B)$. But this means that $\{v \in V : v \notin U \not\ni v'\}$ is an antichain of P of size at least $|V| - |U| = |V| - \nu(B)$. □

In the following sections, we describe symmetric posets. With help of a nonbipartite auxiliary graph and the Edmonds-Gallai decomposition, we show a theorem similar to Theorem 7.1.

7.1 Symmetric posets

We prove the following result, conjectured by András Frank:

Theorem 7.2. *Let $P = (V, \preceq, M)$ be a symmetric poset. The minimum number of symmetric chains needed to cover P is equal to the maximum value of a legal subpartition of P.*

Here $P = (V, \preceq, M)$ is a *symmetric poset* if $(V, \preceq)$ is a finite poset and M is a perfect matching on V such that $u \preceq v$ and $uu', vv' \in M$ implies $u' \succeq v'$. By $u \prec v$ we mean that $v \neq u \preceq v$. A subset $\{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ of M is a *symmetric chain* in the symmetric poset $P = (V, \preceq, M)$ if $u_i \prec u_{i+1}$ for $1 \leq i < k$. Symmetric chains $S_1, S_2, \ldots, S_t$ *cover* symmetric poset P if $M = \bigcup_{i=1}^{t} S_i$.

$M_1, M_2, \ldots, M_l \subseteq M$ is a *legal subpartition of P* if

$$u_1v_1 \in M_i,\ u_2v_2 \in M_j \text{ and } u_1 \preceq u_2 \text{ implies } i = j \quad \text{and} \tag{7.1}$$

$$\text{there is no symmetric chain of length three contained in any } M_i. \tag{7.2}$$

The *value* of the legal subpartition $\mathcal{L}$ is $\sum_{M_i \in \mathcal{L}} \left\lceil \frac{|M_i|}{2} \right\rceil$.

We proceed with the proof of Theorem 7.2 and discuss the algorithmic aspects. Next, in 7.2 we derive from Theorem 7.2 two, in a sense extreme, cases: Dilworth's theorem and the well-known minmax relation for the minimum size edge covering of a graph. Also in 7.2, a generalization of Theorem 7.2 is proved with help of a node splitting construction. In 7.3, we explain how Theorem 7.2 applies in the problem of embedding metric spaces in minimum dimensional l_1-spaces. The discussion in 7.2 and 7.3 can be understood without mastering the proof of Theorem 7.2.

Proof of Theorem 7.2. By definition, any symmetric chain intersects at most one member M_i of a legal subpartition $\mathcal{L}$ and such an intersection contains at most two elements. So the value of a legal subpartition is a lower bound for the number of symmetric chains needed to cover M.

For the reverse inequality, we prove that there exists a special symmetric chain cover, in fact a symmetric chain partition, $\mathcal{S}$ of M and a legal subpartition $\mathcal{L}$ of P such that $|\mathcal{S}|$ is equal to the value of $\mathcal{L}$.

Define undirected graph $G = (V, E)$ by

$$E := \{uv' : \exists v \text{ such that } vv' \in M \text{ and } u \prec v\}. \tag{7.3}$$

E is well defined, as the equivalence of $u \prec v$ and $u' \succ v'$ (where $uu', vv' \in M$) implies that $uv' \in E$ if and only if $v'u \in E$.

Observe that $\{m_1, m_2, \dots, m_k\} \subseteq M$ is a symmetric chain if and only if there exist $e_1, e_2, \dots, e_{k-1} \in E$ such that $m_1 e_1 m_2 e_2 \dots m_{k-1} e_{k-1} m_k$ is an ME-alternating path. Observe moreover that transitivity of the partial order $\preceq$ means that if $vv' \in M$ and $u_1 v, u_2 v' \in E$ then $u_1 u_2 \in E$ (we refer to this property as the *transitivity* of E). Because the order is acyclic, there is no ME-alternating cycle (i.e. a closed ME-alternating path).

So now we are looking for the minimum number of ME-alternating paths covering V. If some family of k ME-alternating paths covers V, it contains a matching of G of size $|M| - k$. On the other hand, if $I \subseteq E$ is a matching of G then $M \cup I$ contains exactly $|M| - |I|$ ME-alternating paths and some ME-alternating cycles that together cover V. As there is no ME-alternating cycle, the minimum number of symmetric chains needed to cover is $|M| - \nu(G)$. Also, after contracting the edges of a maximum matching of G the M-components in the contracted graph will be a set of vertex-disjoint paths, defining an optimal symmetric chain partition $\mathcal{S}$ of P.

Thus it remains to construct a legal subpartition $\mathcal{L}$ of P with value $|\mathcal{S}|$. Using the fact that $|M| = \frac{|V|}{2}$ we get from (2.10) that

$$|M| - \nu(G) = \frac{1}{2}\left[o(G - A(G)) - |A(G)|\right]$$

Observe that every node $x \in A(G)$ is adjacent to at least two components of $G - A(G)$, as otherwise the only component hanging on x either would be completely covered by every maximum matching, or there would be a maximum matching that does not cover x. This contradicts the definition of the Edmonds-Gallai decomposition. We claim that for every $x \in A(G)$, if $xx' \in M$ then $\{x'\}$ is a component of $G - A(G)$. Indeed: if not, x' has a neighbour u in some component of $G - A(G)$ and x

must be adjacent to some v in another component of $G - A(G)$. From the transitivity of E, we see that $uv \in E$, a contradiction as two different components of $G - A(G)$ are non-adjacent.

Define $M^* := \{m \in M : m \text{ joins two different components of } G - A(G)\}$. If $vv' \in M^*$ then again by the transitivity of E, v or v' is an isolated vertex of $G - A(G)$. Thus by contracting the edges of E in $G - A(G)$ each component becomes a star of M^*-edges. Let $\mathcal{L} = \{M_i : 1 \leq i \leq l\}$ be the partition of M^* formed by these components. We claim that $\mathcal{L}$ is a legal subpartition of P with value $|\mathcal{S}|$.

We prove legality first. From the definition of $\mathcal{L}$, we see that there is no E-edge joining two different $M_i \in \mathcal{L}$, which proves (7.1). A symmetric chain of length three implies the existence of an ME-alternating path that contains three M^*-edges. The middle edge of this path must connect two nonisolated vertices from different components of $G - A(G)$, which is impossible. Hence $\mathcal{L}$ is a legal subpartition of P.

To calculate the value of $\mathcal{L}$, define $\mathcal{C}_i$ as the set of odd components of $G - A(G)$ that are incident with some edge in M_i. From the structure of $G - A(G)$, it is clear that either $\mathcal{C}_i$ consists of an even number of isolated vertices, each joined by M^* to a certain even component in $C(G)$, or $\mathcal{C}_i$ is an odd number of isolated vertices joined by M^* to an odd component of $D(G)$. In both cases $\frac{|\mathcal{C}_i|}{2} = \left\lceil \frac{|M_i|}{2} \right\rceil$. Hence

$$|\mathcal{S}| = |M| - \nu(G) = \frac{1}{2}\left(o(G - A(G)) - |A(G)|\right) = \sum_{i=1}^{l} \frac{|\mathcal{C}_i|}{2} = \sum_{i=1}^{l} \left\lceil \frac{|M_i|}{2} \right\rceil,$$

the equality we need. □

From the proof, it is clear that using any algorithm efficiently determining both a maximum matching and the Edmonds-Gallai decomposition we can construct in polynomial-time an optimal symmetric chain cover and a legal subpartition with maximum value. We remark that if X is an inclusionwise minimal subset of V that attains the maximum in the Tutte-Berge formula 2.9 then the structure of $G - X + M$ has all the properties of the structure of $G - A(G) + M$ that we needed in the above proof. Hence we can in fact determine an optimal legal subpartition of P from any subset X of V attaining the maximum in the Tutte-Berge formula.

7.2 Special cases of the symmetric chain cover formula

As corollaries we deduce two, in a sense extreme, special cases of Theorem 7.2. We prove again Dilworth's theorem and derive the well-known min-max formula for the minimum size edge cover of a graph. Note that by this we indicate the unifying nature of our result rather than provide a simple proof for these two consequences: in the first reduction instead of relying on Tutte's theorem we use only Theorem 2.2 of Kőnig (as the auxiliary graph is bipartite) and in the second case the edge cover formula itself is an immediate consequence of Tutte's theorem which has already been used in the proof of Theorem 7.2. We also prove a weighted generalization of Theorem 7.2.

Corollary 7.3 (Dilworth's theorem [18]). *Let $P = (V, \preceq)$ be a finite poset. Then the minimal number of chains that cover V equals the maximum size of an antichain of P.*

Proof. Define $V' := \{v' : v \in V\}$, $M := \{vv' : v \in V\}$, and $\preceq' := \preceq \cup \{u'v' : v \preceq u\}$. Thus $P' := (V \cup V', \preceq', M)$ is a symmetric poset, and $S \subseteq M$ is a symmetric chain if and only if the elements of V covered by S form a chain in P. Observe that for any antichain A, the system $\mathcal{L}_A := \{\{vv'\} : v \in A\}$ is a legal subpartition of P' with value $|A|$.

Thus it is sufficient to prove that for any legal subpartition $\mathcal{L}$ of P' there exists an antichain $A_{\mathcal{L}}$ of P with size not less than the value of $\mathcal{L}$. For a legal subpartition $\mathcal{L}$ let

$$\mathcal{L}^V = \{X \subseteq V : \{vv' : v \in X\} \in \mathcal{L}\} = \{V_i : 1 \leq i \leq l\}.$$

From the definition of legality, we see that if u and v belong to different members of $\mathcal{L}^V$ then u and v are $\preceq$-incomparable. Moreover, as no member of $\mathcal{L}^V$ contains a symmetric chain of length 3, we get that each part $V_i \in \mathcal{L}^V$ can be decomposed as $V_i = V_i^{max} \cup V_i^{min}$, the union of its $\preceq$-minimal and $\preceq$-maximal elements. Now define V_i^* as the set among V_i^{max} and V_i^{min} with greater cardinality. Clearly $|V_i^*| \geq \left\lceil \frac{|V_i|}{2} \right\rceil$, so $A_{\mathcal{L}} := \bigcup_{i=1}^{l} V_i^*$ is an antichain of P with size not less than the value of $\mathcal{L}$. □

Corollary 7.4. *Let $G = (V, E)$ be an undirected graph without isolated vertices. The minimum number of edges needed to cover V equals the maximum of $\frac{1}{2}[|V - X| + o(G - X)]$ for $X \subseteq V$.*

Proof. Define $V' := \{v' : v \in V\}$, $M := \{vv' : v \in V\}$, and $\prec := \{uv' : uv \in E\}$. Thus $P' := (V \cup V', \preceq, M)$ is a symmetric poset and $S \subseteq M$ is a maximal symmetric chain if and only if there is an edge $uv \in E$ such that $S = \{uu', vv'\}$. Thus a minimal symmetric chain cover of P' corresponds to an edge cover of G. Observe that for $X \subseteq V$ the system $\mathcal{L}_X := \{\{vv' : v \in C\} : C \text{ is a component of } G - X\}$ is a legal subpartition of P' with value $\frac{1}{2}[|V - X| + o(G - X)]$.

Thus it is sufficient to find for any legal subpartition $\mathcal{L}$ of P' a subset $X_{\mathcal{L}}$ of V such that $\frac{1}{2}[|V - X_{\mathcal{L}}| + o(G - X_{\mathcal{L}})]$ is not less than the value of $\mathcal{L}$. For a legal subpartition $\mathcal{L}$ let

$$\mathcal{L}^V = \{Y \subseteq V : \{vv' : v \in Y\} \in \mathcal{L}\} = \{V_i : 1 \leq i \leq l\}$$

be the natural subpartition of V corresponding to $\mathcal{L}$. Let $X_{\mathcal{L}} := V - \bigcup \mathcal{L}^V$. From the definition of legality we see that each part $V_i \in \mathcal{L}^V$ is the union of components $C_i^1, \ldots, C_i^{k_i}$ of $G - X_{\mathcal{L}}$. This means that

$$\frac{1}{2}[|V - X_{\mathcal{L}}| + o(G - X_{\mathcal{L}})] = \frac{1}{2}\left[o(G - X_{\mathcal{L}}) + \sum_{i=1}^{l}\sum_{j=1}^{k_i} |C_i^j|\right] =$$
$$= \sum_{i=1}^{l}\sum_{j=1}^{k_i}\left\lceil \frac{|C_i^j|}{2} \right\rceil \geq \sum_{i=1}^{l}\left\lceil \sum_{j=1}^{k_i} \frac{|C_i^j|}{2} \right\rceil = \sum_{i=1}^{l}\left\lceil \frac{|V_i|}{2} \right\rceil,$$

the value of $\mathcal{L}$. □

Finally, we use node splitting to prove a weighted generalization of the main result unifying the weighted version of Dilworth's theorem and to the so called b-matching problem.

We say that symmetric chains $S_1, S_2, \ldots, S_t$ *cover* symmetric poset P *with multiplicity* w if $\sum_{i=1}^{t} \chi^{S_i} \geq w$. The *$w$-value* of a legal subpartition $\mathcal{L}$ is

$$\sum_{M_i \in \mathcal{L}} w^*(M_i)\ , \qquad \text{where} \quad w^*(M_i) := \begin{cases} \lceil \frac{1}{2} \sum_{m \in M_i} w(m) \rceil & \text{if } |M_i| > 1 \\ w(m) & \text{if } M_i = \{m\} \end{cases} .$$

Corollary 7.5. *Let $P = (V, \preceq, M)$ be a symmetric poset and $w : M \longrightarrow \mathbb{N}$. The minimal number of symmetric chains needed to cover P with multiplicity w is equal to the maximal w-value of a legal subpartition of P.*

Proof. Define $V' := \{v_i : vv' \in M, 1 \leq i \leq w(vv')\}$, $M' := \{v_i v_i' : vv' \in M, 1 \leq i \leq w(vv')\}$ and $\prec' := \{u_i v_j : uu', vv' \in M, 1 \leq i \leq w(vv'), 1 \leq j \leq w(uu'), u \prec v\}$. Apply Theorem 7.2 to the symmetric poset $P' = (V', \preceq', M')$. Observe that if two copies $v_i v_i', v_j v_j' \in M'$ of a certain edge $vv' \in M$ lie in different parts of the optimal legal subpartition $\mathcal{L}'$ of P' then we can assume that each copy of the edge vv' forms a separate part of $\mathcal{L}'$ by itself. Thus the maximum value of a legal subpartition of P' equals the maximum w-value of a legal subpartition of P. $\square$

7.3 An application to the l_1-embeddability problem

In this section, we describe a connection between Theorem 7.2 and the problem of embedding a finite metric in an l_1-space with minimal dimension. The background of our discussion is explained in the book of Deza and Laurent [17]. Hereby I thank Jack Koolen for pointing out the connection between Theorem 7.2 and the above embedding problem.

We shall apply Theorem 7.2 to symmetric poset, $(\mathcal{F}, \subseteq, M^c(\mathcal{F}))$, where $\mathcal{F} \subseteq 2^X$ is a *symmetric family* of subsets of X. That is if $A \in \mathcal{F}$ then $A^c := X \setminus A \in \mathcal{F}$ and $M^c(\mathcal{F}) = \{\{A, X \setminus A\} : A \in \mathcal{F}\}$. Function $d : X^2 \to \mathbb{R}_+$ is a *distance on* X if it is *symmetric*, i.e. $d(x,y) = d(y,x)$ for $x, y \in X$, and $d(x,x) = 0$ for any element x of X. Distance d is called a *semimetric*, if the *triangle inequality*

$$d(x,y) \leq d(x,z) + d(z,y)$$

holds for every $x, y, z \in X$. A semimetric d, that satisfies $d(x,y) > 0$ whenever $x \neq y$ is called a *metric*. Observe that if d_1 and d_2 are semimetrics then $d_1 + d_2$ is a semimetric as well, and if at least one of d_1 and d_2 is a metric, then $d_1 + d_2$ is a metric. Pair (X, d) is a *distance space* (*metric space*) if d is a distance (metric) on X. For distance (metric) space (X, d) and subset Y of X, the distance (metric) space $(Y, d|_{Y^2})$ is a *subspace* of (X, d), where $d|_{Y^2}$ denotes the restriction of d to Y^2. We usually do not distinguish between d and $d|_{Y^2}$, so we say that (Y, d) is a distance (metric) subspace of (X, d).

A *split* of a groundset X is a two-partition $S := \{A, X \setminus A\}$ of X. We denote the set of splits of X by $\mathcal{S}(X)$. The *split semimetric* δ_S on X determined by split

$S = \{A, X \setminus A\}$ is defined by

$$\delta_S(x,y) := \begin{cases} 0, \text{ if } x, y \in A \text{ or } x, y \notin A \\ 1, \text{ else .} \end{cases}$$

An example of a metric space is $(V(G), \mathrm{dist}_G)$, where G is an undirected graph. Metric dist_G is called the *path metric* of graph G. Another example is $(\mathbb{R}^n, d_{l_p})$ for $1 \leq p \leq \infty$, where the l_p*-metric* d_{l_p} is defined by the l_p*-norm* $\|\cdot\|_p$ as

$$d_{l_p}(x,y) := \|x - y\|_p \ ,$$

for $x, y \in \mathbb{R}^n$. The l_p norm of $x = (x_1, x_2, \dots, x_n)$ is

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} .$$

If (X, d) and (X', d') are distance spaces and ϕ is a mapping from X to Y then ϕ is an *isometry* from (X, d) to (X', d') if

$$d(x,y) = d'(\phi(x), \phi(y))$$

holds for all $x, y \in X$. We say that distance space (X, d) can be *isometrically embedded* in distance space (X', d') if there is an isometry from (X, d) to (X', d').

There is an important connection between split semimetrics and metric subspaces of l_1-metric spaces:

Lemma 7.6. *For a finite metric space (X, d), the following are equivalent:*

1.

$$d = \sum_{S \in \mathcal{S}(X)} \lambda_S \delta_S \ , \qquad \textit{for some } \lambda_S \geq 0, \textit{ and} \tag{7.4}$$

2. (X, d) can be isometrically embedded to $(\mathbb{R}^k, l_1)$ for some $k \in \mathbb{N}$.

Proof. To see that 1. implies 2., fix an element a of X. For any split S of X, define function ϕ_S by

$$\phi_S(x) := \begin{cases} \lambda_S & \text{, if } a \in S_i \\ 0 & \text{, else.} \end{cases}$$

It is easy to check that $\phi := (\phi_{S_1}, \phi_{S_2}, \dots)$ is an isometry from X to $\mathbb{R}^{\mathcal{S}(X)}$, where $S_1, S_2, \dots$ are all the splits of X.

On the other hand, let $\phi : X \to \mathbb{R}^k$ be an isometry, and for $i \in [k]$ let $\phi_i : X \to \mathbb{R}$ be the ith coordinate function of ϕ. Define distance d_i on X by

$$d_i(x,y) := |\phi_i(x) - \phi_i(y)| \ .$$

Fix $i \in [k]$ and let $X = \{x_j : j \in [n]\}$, such that $\phi_i(x_1) \leq \phi_i(x_2) \leq \ldots \leq \phi_i(x_n)$. Define $\lambda_S^i := \phi_i(x_{j+1}) - \phi_i(x_j)$ if $S = \{\{x_1, x_2 \dots, x_j\}, \{x_{j+1}, x_{j+2}, \dots, x_n\}\}$, and

$\lambda_S^i := 0$ if S is not of the above form. It is easy to check that $d_i = \sum_{S\in\mathcal{S}(X)} \lambda_S^i \delta_S$ is a split-semimetric decomposition of d_i, hence there is a split-semimetric decomposition of d, namely

$$d = \sum_{i\in[k]} d(i) = \sum_{S\in\mathcal{S}(X)} \left(\sum_{i\in[k]} \lambda_S^i\right) \delta_S.$$

□

We say that isometry $\phi : X \to \mathbb{R}^k$ *corresponds to split-semimetric decomposition* $\sum_{S\in\mathcal{S}(X)} \lambda_S \delta_S$, if $\lambda_S = \sum_{i\in[k]} \lambda_S^i$ for all $S \subset X$, where λ_S^i is defined in the proof of Lemma 7.6.

Note that it is not true that if d is an l_1-embeddable metric, then it has a unique split-semimetric decomposition as in (7.4) above. An example to this phenomenon is the equidistant metric d on four points ($d(x, y) = 1$ iff $x \neq y$), which can be obtained as $d = \sum_{S\in\mathcal{S}(X)} \lambda_{S\in\mathcal{S}(X)} \delta_S = \sum_S \kappa_S \delta_S$, where $\lambda_S = 0$ and $\kappa_S = \frac{1}{4}$ if S is a split into parts of equal size, and $\lambda_S = \frac{1}{2}$ and $\kappa_S = 0$ if the parts of split S has unequal size. Hence different l_1-embeddings can correspond to different split-semimetric decompositions. Also, there might be substantially different l_1-embeddings that correspond to the same split-semimetric decomposition. In the proof of the first part of Lemma 7.6, we chose an in a sense maximal dimensional (and least economic) embedding. It is a natural problem, to determine for a metric space (X, d) the smallest dimension k such that (X, d) can be isometrically embedded into $(\mathbb{R}^k, l_1)$. This question seems to be a difficult one. Not only because it is already NP-complete to decide whether a metric space is l_1-embeddable at all (see [3][1]), but because even if we know already some l_1-embedding, there is no known formula for the minimum dimension of an l_1-embedding. To illustrate this problem, let us mention the problem of l_1-embedding of equidistant spaces (i.e. spaces of the form (X, δ), where δ is the "Kronecker delta": $\delta(x, y) = \delta_{x,y}$). It is easy to see that $(\{\pm\frac{e_i}{2} : i \in [k]\}, l_1)$ is an equidistant subspace of $(\mathbb{R}^k, l_1)$ on $2k$ elements, where $e_1, e_2, \ldots, e_k$ is the standard basis of $\mathbb{R}^k$. It is conjectured that this example has maximum size amongst isometrically $(\mathbb{R}^k, l_1)$-embeddable equidistant spaces, that is if (X, d) is equidistant then the minimum dimension of an l_1-embedding of it is $\left\lceil \frac{|X|}{2} \right\rceil$.

The problem to determine the lowest dimension k for a certain metric space (X, d) such that there is an isometry ϕ embedding (X, d) into $(\mathbb{R}^k, l_1)$ can be divided into two subproblems: the first problem is to determine an "optimal" split-semimetric decomposition (as in (7.4)) for d, which corresponds to *some* minimal dimensional l_1-embedding. The second problem would be to find a minimal dimensional l_1-embedding that corresponds to a *given* split-semimetric decomposition. By applying

[1] It is shown there that the problem of l_1-embeddability, i.e. the existence of a split-semimetric decomposition is equivalent with testing the membership of d in the cut-cone. But as we know some vector in the cut-cone, the optimization problem over the cut-cone can be polynomially reduced to membership testing (see e.g. [49]). As the MAX-CUT problem is NP-complete, the membership testing in the cut-cone is also NP-complete, hence our embedding problem turns out to be NP-complete, as well.

Theorem 7.2, we shall solve the second problem (indicating that the first problem is probably a difficult one).

When one wants to find an isometric l_1-embedding $\phi : X \to \mathbb{R}^k$ corresponding to a fixed split-semimetric decomposition $\sum_{S\in\mathcal{S}(X)} \lambda_S \delta_S$, one has to find coefficients λ_S^i for $i \in [k]$ and $S \in \mathcal{S}(X)$ in such a way that $\lambda_S = \sum_{i\in[k]} \lambda_S^i$ for all $S \in \mathcal{S}(X)$ and there is an isometry ϕ_i embedding metric space (X, d_i) to $(\mathbb{R}, l_1)$, where $d_i := \sum_{S\in\mathcal{S}(X)} \lambda_S^i \delta_S$. In this case, $\phi := (\phi_1, \phi_2, \dots, \phi_k)$ suffices. If a split-semimetric decomposition $d = \sum_{S\in\mathcal{S}(X)} \lambda_S \delta_S$ is given and we want to find coefficients λ_S^i, it is handy to have a characterization of split-semimetric decompositions that correspond to isometric embeddings into $(\mathbb{R}, l_1)$. This is done in the following easy observation:

Lemma 7.7. *For a finite metric space (X, d) and for split-semimetric decomposition $d = \sum_{S\in\mathcal{S}(X)} \lambda_S \delta_S$ the following two statements are equivalent:*

1. *there exists an isometric embedding $\phi : X \to \mathbb{R}$ of (X, d) into $(\mathbb{R}, l_1)$ corresponding to split-semimetric decomposition $\sum_{S\in\mathcal{S}(X)} \lambda_S \delta_S$*
2. *$\mathcal{F} := \{S \in \mathcal{S}(X) : \lambda_S > 0\}$ is a symmetric chain in symmetric poset $(2^X, \subseteq, M^c(2^X))$.*

Proof. In the proof of Lemma 7.6, we saw that $\phi = \phi_1$ (the first coordinate-function of ϕ) corresponds to a split-semimetric decomposition which has positive λ-coefficients only for splits of the form $\{\{x_1, x_2, \dots, x_j\}, \{x_{j+1}, \dots, x_n\}\}$. These splits form indeed a symmetric chain. This proves that 1. implies 2.

If we know 2. then there is an ordering $x_1, x_2, \dots, x_n$ of the elements of X in such a way that all splits in $\mathcal{F}$ are of the form $\{\{x_1, x_2, \dots, x_j\}, \{x_{j+1}, x_{j+2}, \dots, x_n\}\}$. Define $\phi(x_i) := \sum_{j<i} \lambda_{\{\{x_1,x_2,\dots,x_j\},\{x_{j+1},x_{j+2},\dots,x_n\}\}}$. It is easy to check that $\phi : X \to \mathbb{R}$ is indeed an isometry from (X, d) into $(\mathbb{R}, l_1)$. $\square$

From Lemma 7.7 it is clear that if we have given a split-semimetric decomposition $d = \sum_{S\in\mathcal{S}(X)} \lambda_S \delta_S$ then the minimum dimension of an l_1-space into which (X, d) can be embedded with corresponding split-semimetric decomposition $d = \sum_{S\in\mathcal{S}(X)} \lambda_S \delta_S$, equals the minimum number of symmetric chains covering $(\mathcal{F}, \subseteq, M^c(\mathcal{F}))$ for $\mathcal{F} := \{S \in \mathcal{S}(X) : \lambda_S > 0\}$.

To translate Theorem 7.2 to the language of l_1-embeddings of split-semimetric decompositions, we need a definition. By a *legal split-subpartition* of a family $\mathcal{F} \subseteq \mathcal{S}(X)$ of splits, we mean disjoint subsets $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_l$ of $\mathcal{F}$ (for some $l \in \mathbb{N}$) with the properties that

$$\{A, X \setminus A\} \in \mathcal{F}_i,\ \{B, X \setminus B\} \in \mathcal{F}_j \text{ and } A \subseteq B \text{ implies } i = j, \text{ and}$$
$$\text{there is no } i \in [l] \text{ and } \{A, X \setminus A\}, \{B, X \setminus B\}, \{C, X \setminus C\} \in \mathcal{F}_i \text{ such that } A \subseteq B \subseteq C.$$

The *value* of the above legal split-subpartition is $\sum_{i\in[l]} \left\lceil \frac{|\mathcal{F}_i|}{2} \right\rceil$. Now Theorem 7.2 yields the following minmax formula:

Theorem 7.8. *The minimum dimension of an isometric l_1-embedding of finite metric space (X, d) that corresponds to split-semimetric decomposition $d = \sum_{S \in \mathcal{F}} \lambda_S \delta_S$ (where $\mathcal{F} \subseteq \mathcal{S}(X)$ and $\lambda_S > 0$ whenever $S \in \mathcal{F}$) is equal to the maximum value of a legal split-subpartition of $\mathcal{F}$.* □

Although it is not known how to solve it efficiently, it is still hoped that deciding the $(\mathbb{R}^k, l_1)$-embeddability of a metric space is a polynomially solvable problem. More specifically, Malitz and Malitz have the following conjecture:

Conjecture 7.9 (Malitz-Malitz [70]). *For all $k \in \mathbb{N}$ there is a number $f_1(k) \in \mathbb{N}$ such that if metric space (Y, d) can be isometrically embedded into $(\mathbb{R}^k, l_1)$ for every subset Y of X of size at most $f_1(k)$ then (X, d) can also be embedded isometrically into $(\mathbb{R}^k, l_1)$.*

If this conjecture is true then checking the isometric embeddability into $\mathbb{R}^k, l_1)$ of all $f_1(k)$-tuples of X is an efficient way to decide isometric l_1-embeddability of X into $\mathbb{R}^k$.

The next observation is that the analogous statement to Conjecture 7.9 on the l_1-embedding of a split-semimetric decomposition is true. This follows again from Theorem 7.2. More specifically, we get

Theorem 7.10. *There is an isometric embedding of finite metric space (X, d) to $(\mathbb{R}^k, l_1)$ corresponding to split-semimetric decomposition $d = \sum_{S \in \mathcal{S}(X)} \lambda_S \delta_S$ if and only if for every subset $\mathcal{F}$ of $\mathcal{S}(X)$ of size at most $2k + 1$ there is an isometric embedding of $(X, d_{\mathcal{F}})$ to $(\mathbb{R}^k, l_1)$ corresponding to split-semimetric decomposition $d_{\mathcal{F}} = \sum_{S \in \mathcal{F}} \lambda_S \delta_S$.*

The proof of Theorem 7.10 follows immediately from the remark right after the proof of Lemma 7.7 and the next lemma:

Lemma 7.11. *Symmetric poset $P = (V, \preceq, M)$ can be covered by k symmetric chains if and only if any subset M' of M of size at most $2k+1$ can be covered by k symmetric chains of P.*

Proof. As the 'only if' part is trivial, we prove the 'if' part. If P cannot be covered by k symmetric chains then by Theorem 7.2 there is a legal subpartition $\mathcal{L} = \{M_1, M_2, \dots, M_l\}$ of value more than k. Let M' be an inclusionwise maximal subset of $\cup_{i \in [l]} M_i$ of size at most $2k + 1$. Consider legal subpartition $\mathcal{L}' = \{M_1' := M_1 \cap M, M_2' := M_2 \cap M, \dots, M_l' := M_l \cap M\}$. If $|\cup_{i \in [l]} M_i| < 2k + 1$ then $\mathcal{L} = \mathcal{L}'$, hence the value of $\mathcal{L}'$ is more than k. Otherwise $|M'| = 2k + 1$, and the value of $\mathcal{L}'$ is

$$\sum_{i \in [l]} \left\lceil \frac{|M_i'|}{2} \right\rceil \geq \left\lceil \sum_{i \in [l]} \frac{|M_i'|}{2} \right\rceil = \left\lceil \frac{|M'|}{2} \right\rceil = \left\lceil \frac{2k+1}{2} \right\rceil = k + 1,$$

showing that M' cannot be covered by k symmetric chains. □

8 3-cross-free families

We have seen that Dilworth's Theorem (Theorem 1.2 and Corollary 7.3) gives a formula for the minimum size of a chain cover of a poset. Theorem 7.2 did the same in the symmetric poset model. But Dilworth's Theorem can be also regarded as a formula for the maximum size of an antichain. In this sense, its symmetric counterpart would be a formula on the maximum size of a 'symmetric antichain' in a symmetric poset. A natural definition of a symmetric antichain could be a set of elements together with their mates that forms an antichain of the underlying order. We might also define a symmetric weak antichain the same way as above except for that we can allow comparability along mates.

The above problem of finding a maximum size symmetric (weak) antichain is certainly a difficult one. Using the construction in Corollary 7.4, one can polynomially reduce the NP-complete problem of finding the maximum size of an independent set of a graph to this maximum size symmetric (weak) antichain problem.[2]

In this section, we consider a certain inverse problem of the above one. The question is, what we can say about different parameters of the symmetric poset if we have a restriction on the size of its symmetric antichains. We consider this problem for symmetric poset used in 7.3, that is, for $(\mathcal{F}, \subseteq, M^c(\mathcal{F}))$, where $\mathcal{F}$ is a symmetric family of subsets of a groundset V. We shall show that if this symmetric poset does not have a symmetric antichain of size 3 (that is, $\mathcal{F}$ does not have 3 pairwise crossing elements), then $|\mathcal{F}|$, the size of the symmetric poset, is a linear function of the size of the groundset, $|V|$. Results of this section also appear in [31].

8.1 k-cross-free families

A family $\mathcal{F}$ of subsets of V is *k-cross-free* if $\mathcal{F}$ has no k pairwise crossing members. We shall prove that if $\mathcal{F} \subseteq 2^V$ is a 3-cross-free family then $|\mathcal{F}| \leq 10|V|$.

It was conjectured by Karzanov and Lomonosov that $|\mathcal{F}| = O(kn)$ if $\mathcal{F}$ is k-cross-free and $|V| = n$. For $k = 2$, this follows from the well-known tree representation of laminar families. Pevzner [76] gave a quite complicated and lengthy proof for the case $k = 3$. We will come back to this later. In 8.2, we present a direct and easy proof for this result. Actually, we prove a slightly more general theorem. We call a family $\mathcal{F} \subseteq 2^V$ *weakly k-cross-free with respect to $a \in V$*, if for every $b \in V \setminus \{a\}$ there are no k pairwise crossing members of $\mathcal{F}$ separating a from b. We say that X *separates a from b* if it contains exactly one of them. Family $\mathcal{F} \subseteq 2^V$ is *weakly k-cross-free* if $\mathcal{F}$ is weakly k-cross-free with respect to *some* element a of V. In 8.2, we show that the size of a weakly 3-cross-free family is at most 10 times greater than the size of the underlying groundset.

[2] It is interesting to observe that if we use the polar version of Dilworth's theorem (Theorem 1.1) instead of Dilworth's, then the symmetric counterpart of the minimum antichain cover contains the NP-complete problem of graph colouring (use the same construction). On the other hand, the maximum-size symmetric chain problem is not difficult. For example, if no element is comparable with its mate (like in all applications so far) then a maximum-size chain of the underlying partial order together with the chain of the mates is a maximum size symmetric chain. It is also not difficult to see that the general problem of finding a maximum size symmetric chain (where comparable mates can occur, as well) can be reduced to the problem of finding a maximum-weight union of two chains of a weighted poset.

As far as we know, Karzanov's conjecture is still open for $k > 3$ and the best known bound is $|\mathcal{F}| = O(kn \log n)$ due to Lomonosov. Recently, Dress *et al.* [19, 20] have found some new results concerning k-cross free families. They describe all maximum-size 3-cross-free families and check the conjecture of Karzanov for so-called cyclic 4-cross-free families. There, *cyclic* means that there is a cyclic order on V such that any element of $\mathcal{F}$ is an interval in it. Dress *et al.* also show that unlike maximum-size 3-cross-free families, maximum-size 4-cross-free families are not cyclic.

The background of the investigation of 3-cross-free families is the so called locking theorem of Karzanov and Lomonosov. Let G be a graph with a nonnegative capacity function c on the $E(G)$, and let f be a fractional path-packing (a so-called multiflow). That is, each edge is used by at most $c(e)$ paths in total. Let subset V of $V(G)$ be given. Subset X of V is *locked in G by f* if the total value of the paths of f connecting X to $V \setminus X$ equals the minimum capacity of an edge-cut of G separating X from $V \setminus X$. A family $\mathcal{F}$ of subsets of V is *lockable* if for any graph G with $V \subset V(G)$, and any nonnegative capacity function c, there is a multiflow f locking each member of $\mathcal{F}$. The locking theorem characterizes lockable families as follows.

Theorem 8.1 (Karzanov-Lomonosov [58, 67, 57]). *A family $\mathcal{F}$ of subsets of finite set V is lockable if and only if $\mathcal{F}$ is 3-cross-free.* □

For a shorter proof of the above result see also [38], where a stronger version is proved. To formulate it, we define an undirected graph G with a fixed subset V of its vertices to be *inner Eulerian* if the degree of any vertex of G outside V is even.

Theorem 8.2 ([58, 67, 38]). *A family $\mathcal{F}$ of subsets of finite set V is 3-cross-free if and only if for any inner Eulerian graph G with $V \subseteq V(G)$, there is a collection $\mathcal{P}$ of edge-disjoint paths of G such that for any member X of $\mathcal{F}$, $\mathcal{P}$ contains a maximum number of edge-disjoint paths connecting X to $V \setminus X$.* □

8.2 Bounding the size of weakly 3-cross-free families

Throughout this section we use the following notation:

$$\begin{aligned} \mathcal{F}/v &:= \{X \setminus \{v\} : X \in \mathcal{F}\} \\ \mathcal{F}(v) &:= \{X \in \mathcal{F} : v \in X \text{ and } X \setminus \{v\} \in \mathcal{F}\} \end{aligned}$$

Theorem 8.3. *If $\mathcal{F}$ is a weakly 3-cross-free family of a finite set V then $|\mathcal{F}| \leq 10|V|$.*

Proof. Assume to the contrary that $\mathcal{F}$ is a counterexample with $|V|$ minimal, that is, $|\mathcal{F}| > 10n$ and $\mathcal{F}$ is weakly 3-cross-free with respect to a. Let us define $\mathcal{F}' := \{X \in \mathcal{F} : a \notin X\} \cup \{V \setminus X : a \in X \in \mathcal{F}\}$. Clearly, $|\mathcal{F}'| > 5|V|$ with the property that

(1) *if $X, Y, Z \in \mathcal{F}'$ with $X \cap Y \cap Z \neq \emptyset$ then X, Y, Z cannot pairwise cross.*

Next we prove:

(2) *For each $x \in V \setminus a$, there exist $A_x, B_x \in \mathcal{F}'(x)$ such that $B_x \neq A_x \subset B_x$ and* $|A_x| \geq 3$.

If $\{x\} \neq P \subset Q \subset R$ is a chain of three different elements from $\mathcal{F}'(x)$, then $A_x = Q$, $B_x = R$ suffices. Otherwise each element of $\mathcal{F}'(x) \setminus \{x\}$ is either inclusionwise minimal or maximal. By (1), we see that $\mathcal{F}'(x) \setminus \{x\}$ contains at most two maxima and at most two minima, hence altogether $|\mathcal{F}'(x)| \leq 5$. As $|\mathcal{F}(x)| \leq 2|\mathcal{F}'(x)|$, we get that $|\mathcal{F}/x| = |\mathcal{F}| - |\mathcal{F}(x)| > 10|V| - 2|\mathcal{F}'(x)| \geq 10|V \setminus \{x\}|$. This contradicts to the minimality assumption as $\mathcal{F}/x$ is also a weakly 3-cross-free family with respect to a. So (2) follows.

Choose $x \in V \setminus a$, such that $|B_x|$ is as small as possible. Let $y, z \in A_x \setminus \{x\}$ be different elements. Observe that $y \in A_x \cap (B_x \setminus \{x\}) \cap B_y$ and that A_x crosses $B_x \setminus \{x\}$. By the choice of x, $|B_y| \geq |B_x|$. Hence, B_y must contain A_x or $B_x \setminus \{x\}$ by (1). In particular, we have that $z \in A_x \setminus \{x\} \subset B_y$. So $z \in A_x \cap (B_x \setminus \{x\}) \cap (B_y \setminus \{y\})$, and the three members $A_x, B_x \setminus \{x\}$ and $B_y \setminus \{y\}$ of $\mathcal{F}'$ are pairwise crossing, contradicting (1). □

8.3 Further comments on 3-cross-free families

As indicated, Karzanov's conjecture about the linear size of k-cross-free families is still open for $k > 3$. However Lomonosov's argument is also valid in our weakly k-cross-free setting. Indeed, let $\mathcal{F}^i := \{X \in \mathcal{F}' : |X| = i\}$ for $i = 0, 1, \ldots, n$, where $\mathcal{F}'$ is defined as in the proof of Theorem 8.3. Clearly, for every $v \in V \setminus a$ there are less than k sets in $\mathcal{F}^i$ covering v, hence $|\mathcal{F}| \leq 2|\mathcal{F}'| = 2\sum_{i=0}^{n} |\mathcal{F}^i| < 2\left(1 + \sum_{i=1}^{n} \frac{kn}{i}\right) = O(kn \log n)$.

Pevzner [76] published a paper about the linear size of 3-cross-free families. Although the proof contains several important observations on 3-cross-free families, it is not at all easy to understand. Beyond exploring some important properties of k-cross-free and 3-cross-free families, Pevzner [76] also had some interesting remarks that are worth citing. His question reads as follows (in our terminology, non-k-crossing is k-cross-free).

> Is it true that any non-k-crossing family on n elements can be decomposed into r non-$(k-1)$-crossing families (r is independent of $n, k > 3$)?

He also observes:

> It is possible to show that for $k = 3$ the answer to the above problem is negative (an example of an r-indecomposable non-3-crossing family is a family of stars in a graph without triangles with a chromatic number exceeding r).

It is interesting to see that for families weakly k-cross-free with respect to a fixed point, the answer to the above question is negative for all k. Let $[n] := \{i \in \mathbb{N} : 1 \leq i \leq n\}$; $\binom{[n]}{k} := \{X \subseteq [n] : |X| = k\}$ and define $\mathcal{F}([n], k) := \{\{X \in \binom{[n]}{k} : i \in X\} : i \in [n]\} \subseteq 2^{\binom{[n]}{k}}$. Clearly, for $n \geq 2k \geq 2$ and $X \in \binom{[n]}{k}$ the family $\mathcal{F}([n], k)_X := \{F \in \mathcal{F}([n], k) : X \notin F\}$ consists of pairwise crossing sets, $\mathcal{F}([n], k)$ is already weakly $(k+1)$-cross-free with respect to X. Moreover, any k elements of $\mathcal{F}([n], k)_X$ separate X from another element Y of $\binom{[n]}{k}$, hence for $n \geq (c+1)k$ it is not possible to partition $\mathcal{F}([n], k)_X$ into c families that are all weakly k-cross-free with respect to X.

Our last remark is that the bound in Theorem 8.3 is not very far from the best possible one: notice that $\mathcal{F}[n, k] := \{i + [j], [i] + j, [n] \setminus (i + [j]), [n] \setminus ([i] + j) : i + 1 \in [k], j \in [n - i]\} \subseteq 2^{[n]}$ (where $a + [b] := [a + b] \setminus [a]$) is a k-cross-free family with roughly $4(k-1)n$ members. In particular, there is a 3-cross-free family $\mathcal{F}[n, 3]$ with roughly $8n$ members.

Chapter III

Kernels and stable structures

This chapter is based on a connection between a lattice-theoretic fixed-point theorem and kernel-type results (like the stable marriage theorem of Gale and Shapley) in Graph Theory and Combinatorial Optimization. We start by recalling some well-known facts on graph-kernels.

9 Kernels and stable matchings in graphs

To motivate the notion of graph-kernels, we approach them from graph-colourings. Let $G = (V, E)$ be a graph. The *chromatic number* $\chi(G)$ of G is the smallest number k such that there exists k induced coclique subgraphs $G_1, G_2, \ldots G_k$ of G such that $V(G) = \cup_{i=1}^{k} V(G_i)$. In other words, $\chi(G)$ is the minimum number of colours needed to colour the vertices of G in such a way that no edge is spanned by vertices of the same colour. Observe that graph G is bipartite if and only if $\chi(G) \leq 2$. The *chromatic index* (or sometimes *edge-chromatic number*) $\chi'(G) := \chi(L(G))$ is the minimum number of colours needed to colour the edges such that no two adjacent edges receive the same colour. Obviously, $\omega(G) \leq \chi(G)$ and $\Delta(G) \leq \chi'(G)$, where $\omega(G) := \alpha(\overline{G})$, the size of the largest clique subgraph of G, and where $\Delta(G) := \max\{d(v) : v \in V\}$ denotes the maximum degree in G. By Vizing's theorem the latter inequality is a fairly good estimate:

Theorem 9.1 (Vizing [100]). *If G is a simple undirected graph then $\chi'(G)$ is not more than $\Delta(G) + 1$.* □

Moreover, for bipartite graphs these parameters are equal by Kőnig's theorem.

Theorem 9.2 (Kőnig [60]). *If G is a finite bipartite graph, then $\chi'(G) = \Delta(G)$.* □

Note that if we exchange the role of edges and vertices in the definition of $\chi(G)$ and $\omega(G)$ then we get $\chi'(G)$ and $\Delta(G)$, respectively. However, the translation of Theorem 9.1, namely that the chromatic number is close to the clique number, is not true. There exist graphs with arbitrarily large chromatic number and without a triangle (i.e. a K_3) subgraph[1]. Still, the class of graphs for which the translation of Theorem 9.2 is true (graphs that correspond to bipartite graphs in this vertex-edge exchange sense) is a very important one.

Graph G is called *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Bipartite graphs are clearly perfect, just like complete or empty graphs. Cycle C_n is perfect if and only if $n = 3$ or n is even. The "weak perfect graph theorem", an important result of Lovász, states that the class of perfect graphs is closed on complementation.

Theorem 9.3 (Lovász [68]). *Graph G is perfect if and only if its complement $\overline{G}$ is perfect.* □

From above, it follows that if a graph G spans a C_{2k+1} or a $\overline{C_{2k+1}}$ subgraph for some $k > 1$ (a so-called *odd hole* or *odd antihole*), then G is not perfect. According to the *strong perfect graph conjecture* of Berge, it is a characterization of perfect graphs:

Conjecture 9.4 (Berge). *Graph G is perfect if and only if G does not have an induced subgraph isomorphic to C_{2k+1} or to $\overline{C_{2k+1}}$ for $k > 1$.*

A well-known class of perfect graphs are *comparability graphs*, these are graphs $G = (V, E)$ for which there exist a partial order $\prec$ on V such that $E = \{uv : u \prec v \text{ or } v \prec u\}$. Indeed, Theorem 1.1, the polar of Dilworth's theorem, proves that any comparability graph is perfect. Dilworth's theorem (Theorem 1.2) justifies that the complement of a comparability graph is perfect.

For a digraph $D = (V, A)$, a subset K of V is a *kernel* of D if K spans no arc in A and for every vertex v of $V \setminus K$ there is a vertex k in K such that vk is an arc in A. Digraph D is said to be *normal* (or *clique-acyclic*), if every induced clique C of D has a kernel, that is, there is a vertex c of C such that $xc \in A(D)$ for each vertex x of C different from c. According to an important conjecture of Berge and Duchet, perfect graphs can be characterized in terms of kernels:

Conjecture 9.5 (Berge-Duchet). *Graph G is perfect if and only if every normal orientation of G has a kernel.*

Note that if the strong perfect graph conjecture (Conjecture 9.4) is true, then the 'if' part of Conjecture 9.5 follows as soon as we exhibit a normal orientation of C_{2k+1} and $\overline{C_{2k+1}}$ without a kernel for $k > 1$.

Boros and Gurvich [8] have proved the only if part of the above equivalence and obtained a partial result for the other direction. Aharoni and Holzman [1] showed how the first result of Boros and Gurvich follows more or less directly from a lemma of Scarf in [91]. Scarf's lemma can be considered as an algorithmic version of Sperner's lemma. So we can say that the only if part of Conjecture 9.5 follows from the topological fixed point theorem of Brouwer. We come back to this in Section 18.

An important special case of Conjecture 9.5 has been proved by Maffray [69]:

Theorem 9.6 (Maffray [69]). *The line-graph $L(G)$ of graph G is perfect if and only if every normal orientation of $L(G)$ has a kernel.* □

[1] This fact was used by Pevzner to observe indecomposability of 3-cross-free families (see 8.3).

Theorem 9.6 is a generalization of the well-known theorem of Gale and Shapley on stable marriages. To state it, we need some definitions. Let $G = (V, E)$ be a multigraph, and for each vertex v of V let $\prec_v$ be a linear order on $D(v)$. A matching $M \subseteq E$ is said to be *blocked by edge* $e = uv$ of E if there is no edge f of M with $f \preceq_v e$ or $f \preceq_u e$. Matching M is *stable* if it is not blocked by any edge of E, that is if for every edge $e = uv$ of E there is an edge f of M such that $f \prec_u e$ or $f \prec_v e$.

Theorem 9.7 (Gale-Shapley [42]). *If B is a finite bipartite (multi)graph with linear orders on the stars, then there exists a stable matching in B.*[2] □

Theorem 9.7 is often interpreted in such a way that colour-classes X and Y of B represent a set of women and men, respectively, and edges of E indicate possible marriages. A "marriage scheme" M is stable if there is no common interest of a man and of a woman to quit their marriages in favour of each other. It is easy to see that Theorem 9.7 is indeed a special case of Theorem 9.6: by Theorem 9.2, $L(B)$ is perfect and Theorem 9.7 is equivalent with the following kernel-result:

Theorem 9.8. *If B is a bipartite graph then every normal orientation of $L(B)$ has a kernel.* □

Gale and Shapley proved Theorem 9.7 by giving an algorithm that always terminates with a stable matching. To describe this algorithm, it is convenient to use the terminology of the "marriage model". So men and women represent vertices of the different colour-classes, and an edge between a man and a woman means that they both can agree on participating in a certain marriage with each other. (That is why there can be multiple edges between the same two persons, and they are not indifferent which edge to choose when marrying the other person.)

The *proposal* (originally *deferred acceptance*) *algorithm* of Gale and Shapley works in rounds. A round starts with each men proposing to his most preferable partner (that is, he chooses the edge of the bipartite graph he likes most). Then each women refuses all but the best proposals she received. That is, those edges of the bipartite graph along which a refused proposal arrived get deleted. After this, the next round starts. (So in this round each man who has not been refused proposes to the same woman in the hope that sooner or later she will accept it.) The algorithm terminates when each woman receives at most one proposal, hence, when no refusal takes place. Then each woman who received a proposal accepts it. It is not difficult to show that these marriages determine a stable marriage scheme. Gale and Shapley also show that the stable matching constructed by the proposal algorithm is so called *man-optimal*, that is, each man gets the best partner he can have in a stable marriage scheme. Of course, if we interchange the role of men and women in the algorithm then it will find the woman-optimal scheme.

Perhaps the main stream of stable matching studies is about its game theoretical context. On the other hand, we have already indicated a connection between stable

[2] Gale and Shapley proved this result only for complete bipartite graphs where both colour-classes are finite and have the same cardinality. However, their method (with natural modifications) works for the above extension, too.

matching and kernel problems[3]. There is yet another interesting link between stable matchings and Graph Theory, namely Galvin's theorem.

Theorem 9.9 (Galvin [47]). *If G is bipartite then its list-chromatic index $\chi'_l(G)$ equals its chromatic index $\chi'(G)$.*

The list-chromatic index of graph G is the smallest integer l for which no matter how we label each edge of G with a set of l natural numbers, it is always possible to find a proper edge-colouring of G by choosing the colour of each edge from its label. Theorem 9.9 settles the famous List Colouring Conjecture for the case of bipartite graphs[4]. The List Colouring Conjecture is the same as Theorem 9.9 for multigraphs.

Galvin used Theorem 9.6 to prove Theorem 9.9. Galvin's method can be described in terms of bipartite stable matchings: with the help of a Δ-colouring c of $E(G)$ (that exists by Theorem 9.2 of Kőnig), one can define linear orderings on the stars by

$$e \prec_x f \quad \begin{cases} \text{if } c(e) < c(f) \text{ and } x \in X \text{ or} \\ \text{if } c(e) > c(f) \text{ and } x \in Y. \end{cases}$$

for $e, f \in D(x)$, where X and Y are the colour-classes of G. In this model, it is relatively easy to prove that by iteratively colouring with colour i a stable matching of the subgraph of yet uncoloured i-colourable edges, one constructs a proper list-colouring.

Although the proof of Theorem 9.7 works only in the bipartite case, the notion of a stable matching makes sense for multigraphs as well. It follows from Theorem 9.6 that in the nonbipartite model stable matchings do not always exist, or, what is stronger, there is a normal orientation of any non-perfect line-graph that does not possess a kernel. Still, it is an interesting problem to decide whether a *given* normal orientation of a line-graph has a kernel or not. Or, a bit more generally, to decide the existence of a stable matching in the nonbipartite preference model. This question is often referred as the *stable roommates problem*. It has been settled by Irving in [56] (see also [50]), where he designed an efficient algorithm which either constructs a stable matching or proves that none exists. Later, Feder [30] and Subramanian [96] exhibited a different algorithm that essentially reduces the problem of finding a stable matching to finding a fixed point of a certain set-function.

Another important result on stable matchings is the description of the stable matching polyhedron. Vande Vate [99] did this for the original stable matching theorem, (Theorem 9.7 with graph $K_{n,n}$), and Rothblum extended it for bipartite multigraphs.

Theorem 9.10 (Vande Vate [99] and Rothblum [89]). *Let $G = (V, E)$ be a finite bipartite graph and for each $v \in V$ let $\prec_v$ be a linear order on $D(v)$. Define*

[3]The notion of kernel seem to come from Cooperative Game Theory. Although what is known as kernel there does not have too much to do with graph kernels. The connection is given by the concept of the so called *von Neumann-Morgenstern solution* [102] (which is also known as *stable set*). This is nothing else but a kernel of the domination digraph of a cooperative game. See 16.1 for the details.

[4]The List Colouring Conjecture seems to be conjectured independently by several people (for the history, see [7, 51, 2]).

$\phi(e) := \{f \in E : f \preceq_u e$ *or* $f \preceq_v e\}$ *for edge* $e = uv \in E$. *Then*

$$\operatorname{conv}\{\chi^F : F \subseteq E \text{ is a stable matching of } G\} =$$
$$\{x : \mathbf{0} \leq x \in \mathbb{F}^d, x(D(v)) \leq 1 \text{ for } v \in V, x(\phi(e)) \geq 1 \text{ for } e \in E\}. \quad \square$$

10 Tarski's fixed point theorem

In this section we describe the lattice-theoretic fixed point theorem of Tarski, our main tool to handle kernel-problems.

Lattice $L = (X, \wedge, \vee)$ is *complete* if there is both a meet and an join for any subset Y of X. These generalized meet and join operations on Y are denoted by $\bigwedge Y$ and $\bigvee Y$, respectively. Clearly, $\bigwedge X = \mathbf{0} \in X$ and $\bigvee X = \mathbf{1} \in X$. Let, by definition, $\bigwedge \emptyset := \mathbf{1}, \bigvee \emptyset := \mathbf{0}$. The following fixed-point theorem of Tarski is a most important result on complete lattices:

Theorem 10.1 (Tarski [97]). *If* $L = (X, \wedge, \vee)$ *is a complete lattice and* $f : X \to X$ *is a monotone function, then* $L_f := (X_f, \leq)$ *is a nonempty, complete lattice subset of* L*, where* $X_f := \{x \in X : f(x) = x\}$ *is the set of fixed points of* f.[5]

Proof. Let Y be a (possibly empty) subset of X_f. By monotonicity of f, $f(\bigwedge Y) \leq f(y) = y$ for any $y \in Y$, hence $f(\bigwedge Y) \leq \bigwedge Y$. Define

$$K := \{k \in X : k \leq f(k) \wedge \bigwedge Y\}$$

and $l := \bigvee K$. Clearly, if $x = f(x) \leq \bigwedge Y$ for a fixed point x of f, then $x \in K$ and $x \leq l$. Hence it is enough to show that $f(l) = l$.

By definition, $k \leq l \leq y$ for any $k \in K$ and $y \in Y$. Thus by monotonicity, $k \leq f(k) \leq f(l) \leq f(y)$. This means that $l = \bigvee K \leq \bigvee\{f(k) : k \in K\} \leq f(l) \leq \bigwedge Y$, hence that $l \leq f(l) \leq \bigwedge Y$. Again, by monotonicity, $f(l) \leq f(f(l))$, that is $f(l) \in K$. We got that $l \leq f(l) \leq \bigvee K = l$. Thus l is indeed the meet of Y in X_f.

Obviously, $L^{-1} = (X, \geq)$ is a complete lattice as well, and f is monotone on L^{-1}. According to the above argument, any subset Y of X_f has a $\geq$-meet in X_f, that is a $\leq$-join in X_f.

We conclude that L_f is indeed a nonempty, complete lattice subset of L. $\square$

We remark that in case of finite lattices (that are clearly complete) there is an algorithmic proof for the existence of a minimal and a maximal fixed point in Theorem 10.1. This is based on the observation that by monotonicity, $\mathbf{0} \leq f(\mathbf{0}) \leq$

[5]Theorem 10.1 seems to be proved first for $(2^X, \subseteq)$ by Knaster and Tarski in [62] as early as 1927. They used the notion increasing instead of monotone. Birkhoff published a weaker form of Tarski's Theorem (cf. [5, p. 54] where isotone refers to the monotone property). He proved only the existence of a fixed point and remarked later in an exercise that the set of fixed points is not necessarily a sublattice. In light of further applications of Theorem 10.1 for stable matching-type results, it is interesting to observe that in [64], Kolodner (only six years after the pioneering paper of Gale and Shapley [42], in the very same journal) announced essentially the same result with a modified condition. Also, a special case of Theorem 10.1 is proved in the book of Roth and Sotomayor (cf. Lemma 2.30. in [88]). It is even more interesting, that the same authors have even observed that this fixed-point theorem has to do with some properties of the core of a certain stable matching related assignment game [87].

$f(f(\mathbf{0})) \leq \ldots$ holds. This increasing chain has to stabilize after some iteration at (say) $x := f^{(k)}(\mathbf{0}) = f^{(k+1)}(\mathbf{0}) = f(x)$, providing the zero-element of L_f. Similarly, if we start to iterate f form $\mathbf{1}$, then we get a decreasing chain, that stabilizes at the unit-element of L_f. Note that this algorithmic proof can be extended to a transfinite induction proof of Theorem 10.1. An advantage of Tarski's proof above is that unlike transfinite induction, it does not lean on the axiom of choice.

We give another algorithm to find a fixed point of a monotone function on a finite lattice. Let x be any element of lattice L and consider the iterated images $f^{(i)}(x)$ of x, for $i \in \mathbb{N}$. As the lattice is finite, there are different indices i and $i+k$ such that $f^{(i)}(x) = f^{(i+k)}(x)$. Let

$$y := \bigwedge_{j \in [k]} f^{(i+j)}(x).$$

As $y \leq f^{(i+j)}(x)$, by monotonicity, $f(y) \leq f^{(i+j+1)}(x)$, hence $f(y) \leq y$. But then, sequence $y \geq f(y) \geq f(f(y)) \geq \ldots$ must stabilize at a fixed point of f. Similarly, if we define $z := \bigvee_{j \in [k]} f^{(i+j)}(x)$ then sequence $z \leq f(z) \leq f(f(z)) \ldots$ stabilizes at some other fixed point. From here we got that if f has only one fixed point then we can find it simply by iterating f, starting from any element of the lattice.

Next we recall a well-known set theoretical application of Theorem 10.1.

Theorem 10.2 (Cantor-Bernstein). *If $f : A \to B$ and $g : B \to A$ are injections between sets A and B then there is a bijection h between A and B.*[6]

Proof. Define function $f \star g : A \to A$ by $f \star g(X) := A \setminus (g(B \setminus f(X)))$ for $X \subseteq A$. Clearly $f \star g$ is monotone, hence there is a subset X of A such that $f \star g(X) = X$. But this means that f is a bijection between X and $f(X)$, and g is a bijection between $B \setminus f(X)$ and $A \setminus X$, hence

$$h(a) := \begin{cases} f(a) & \text{if } a \in X \\ g^{-1}(a) & \text{else} \end{cases}$$

defines a bijection $h : A \to B$. □

Theorem 10.2 justifies the notion of cardinality, as it can be equivalently stated such that $|A| \leq |B|$ and $|B| \leq |A|$ implies $|A| = |B|$. Theorem 10.2 is a special case of the following well-known result from Graph Theory.

Theorem 10.3 (Mendelsohn-Dulmage [71]). *If $G = (U \cup V, E)$ is a bipartite graph with colour classes U and V, and M_1 and M_2 are matchings in G, then there is a matching M of G that covers all vertices U' of U that are covered by M_1 and all vertices V' of V that are covered by M_2.*

To see that Theorem 10.3 implies Theorem 10.2, we may assume that A and B are disjoint, and we can define matchings M_1 and M_2 as the underlying undirected

[6] Note that Theorem 10.2 has several names. Sometimes, it is called Schröder-Bernstein or Bernstein-Schröder. According to Levy's account [66], it has been proved by Dedekind in 1887, conjectured by Cantor in 1895 and proved again by Bernstein in 1898. Other sources talk about Schröder, giving a wrong proof in 1896.

graph of $(A \cup B, f)$ and $(A \cup B, g)$, respectively. (Remember that functions f and g are sets of ordered pairs, i.e. arcs.) As $M_1 + M_2$ is bipartite, by Theorem 10.3, there is a matching M of $M_1 + M_2$ covering all vertices of A covered by M_1 and all vertices of B covered by M_2. Hence M is a perfect matching between A and B, exhibiting a bijection between these sets. We shall deduce Theorem 10.3 from the stable matching theorem, on page 69.

11 Monotone and comonotone set-functions

In [97], Tarski gave the following application of Theorem 10.1. (A *Boolean algebra* is a complemented lattice where complementation is unique. The notation $a - b$ means the meet of a and the complement of b.)

Corollary 11.1 (Tarski [97]). *Let $A = (X, \leq)$ be a complete Boolean algebra, $a, b \in X$ and $f : \{x : x \leq a\} \to X$ and $g : \{x : x \leq b\} \to X$ be monotone functions. Then there are elements a' and b' such that $f(a - a') = b'$ and $g(b - b') = a'$.* □

Note that the above theorem in case of subsetlattices was proved already in 1927 by Tarski and Knaster [62]. In what follows, we shall use only this latter result to formulate our tool on so called comonotone functions. A setfunction $f : 2^X \to 2^X$ is *monotone*, if $A \subseteq B \subseteq X$ implies $f(A) \subseteq f(B)$. We say that $\mathcal{F} : 2^X \to 2^X$ is *comonotone* if there is a monotone function $f : 2^X \to 2^X$ such that

$$\mathcal{F}(A) = A \setminus f(A) \quad \text{for } A \subseteq X.$$

In particular, if $\mathcal{F}$ is comonotone then $\overline{\mathcal{F}}$ is monotone, where

$$\overline{\mathcal{F}}(A) := A \setminus \mathcal{F}(A) = A \cap f(A), \text{ if } A \subseteq X. \tag{11.1}$$

The following statement gives equivalent reformulations of the comonotone property.

Proposition 11.2. *For a set-function $\mathcal{F} : 2^X \to 2^X$ the following conditions are equivalent:*

1. *$\mathcal{F}$ is comonotone.*
2. $$\mathcal{F}(Y) \subseteq Y \qquad \text{for any } Y \subseteq X, \text{ and} \tag{11.2}$$
$$\mathcal{F}(Y) \cap Y' \subseteq \mathcal{F}(Y') \qquad \text{whenever } Y' \subseteq Y \subseteq X. \tag{11.3}$$
3. *For each $x \in X$ there exists a family $\mathcal{H}_x \subseteq 2^X$ such that $\mathcal{F} = \mathcal{F}_{\mathcal{H}}$, where $\mathcal{F}_{\mathcal{H}}$ is defined by*

$$\mathcal{F}_{\mathcal{H}}(A) := \{a \in A : 2^A \cap \mathcal{H}_a = \emptyset\} \quad \text{for} \quad A \subseteq X. \tag{11.4}$$

Proof. If f is comonotone then (11.2) follows by definition, and (11.3) is equivalent with the monotonicity of $\overline{\mathcal{F}}$. So 1. implies 2.

To deduce 3. from 2. let

$$\mathcal{H}_x := \{H \subseteq X : x \notin \mathcal{F}(H \cup \{x\})\}.$$

By definition, $\mathcal{F}_{\mathcal{H}}(A) \subseteq \mathcal{F}(A)$ for all $A \subseteq X$. If $a \in A \setminus \mathcal{F}_{\mathcal{H}}(A)$, then there is a subset H of A such that $a \notin \mathcal{F}(H \cup \{a\})$. By (11.3), we see that $(H \cup \{a\}) \cap \mathcal{F}(A) \subseteq \mathcal{F}(H \cup \{a\}) \subseteq X \setminus \{a\}$, i.e. $\mathcal{F}_{\mathcal{H}}(A) \supseteq \mathcal{F}(A)$. Thus $\mathcal{F} = \mathcal{F}_{\mathcal{H}}$, indeed.

To show 1. from 3., we observe that by (11.4)

$$\overline{\mathcal{F}}(A) = \{a \in A : 2^A \cap \mathcal{H}_a \neq \emptyset\},$$

so $\overline{\mathcal{F}}$ is monotone. The rest follows from the identity $\mathcal{F}(A) = A \setminus \overline{\mathcal{F}}(A)$. □

To formulate the basic result of this section, a translation of Theorem 11.1 to comonotone language, we need further definitions. For $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ we call (A, B) an *$\mathcal{FG}$-stable pair* if

$$A \cup B = X \text{ and}$$
$$\mathcal{F}(A) = A \cap B = \mathcal{G}(B).$$

We say that a subset K of X is an *$\mathcal{FG}$-kernel* if there is an $\mathcal{FG}$-stable pair (A, B) such that $K = A \cap B$. We introduce the partial order $\preceq$ on $2^X \times 2^X$ as in 6.4, by

$$(A, B) \preceq (A', B') \text{ if } A \subseteq A' \text{ and } B \supseteq B'. \tag{11.5}$$

Note that $(2^X \times 2^X, \preceq)$ is a complete lattice with lattice operations

$$(A, B) \wedge (A', B') = (A \cap A', B \cup B') \text{ and } (A, B) \vee (A', B') = (A \cup A', B \cap B'). \tag{11.6}$$

Theorem 11.3. *If $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ are comonotone functions then the set of $\mathcal{FG}$-stable pairs is a nonempty complete lattice subset of $(2^X \times 2^X, \preceq)$.*

In the proof of Theorem 10.2 we saw an application of Theorem 10.1. Here we do essentially the same construction.

Proof. Define $f : 2^X \times 2^X \to 2^X \times 2^X$ by

$$f(A, B) := (X \setminus \overline{\mathcal{G}}(B), X \setminus \overline{\mathcal{F}}(A)). \tag{11.7}$$

Clearly, the $\mathcal{FG}$-stable pairs are exactly the fixed points of f.

If $(A, B) \preceq (A', B')$ then $X \setminus \overline{\mathcal{F}}(A) \subseteq X \setminus \overline{\mathcal{F}}(A')$ and $X \setminus \overline{\mathcal{G}}(B) \supseteq X \setminus \overline{\mathcal{G}}(B')$, because $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are monotone. Hence $f(A, B) \preceq f(A', B')$, so f is monotone.

From Theorem 10.1, the set of fixed points of f (that is the set of $\mathcal{FG}$-stable pairs) is a nonempty lattice subset of $(2^X \times 2^X, \preceq)$. □

Note that with natural modifications, the above proof works also for Corollary 11.1. Nevertheless, the part of Theorem 11.3 claiming the existence of a fixed point is clearly a special case of Corollary 11.1, and the lattice subset property is just an easy extra observation that Tarski probably did not think important enough to put into the corollary. Our only reason to call 11.3 a 'Theorem' is because this is the basis of all results in this chapter.

In case of a finite groundset X, we can construct the $\prec$-minimum and the $\prec$-maximum $\mathcal{FG}$-stable pair for comonotone functions $\mathcal{F}$ and $\mathcal{G}$. To do this, we find the $\prec$-maximum and $\prec$-minimum fixed points of f in (11.7) according to the algorithm that we described in Section 10. That is, we iterate f starting from $(\emptyset, X)$ and $(X, \emptyset)$, respectively. This observation leads to the following algorithm that generalizes the proposal algorithm of Gale and Shapley.

Define $A_0 := X$, $B_0 := \emptyset$ and let

$$B_{i+1} := X \setminus \overline{\mathcal{F}}(A_i) \text{ and } A_{i+1} := X \setminus \overline{\mathcal{G}}(B_i). \tag{11.8}$$

Then $(A^{max} := A_{|X|}, B^{min} := B_{|X|})$ is the $\preceq$-maximal $\mathcal{FG}$-stable pair. If we start the recursion with $A_0 := \emptyset$ and $B_0 := X$, then (11.8) will produce the $\preceq$-minimal $\mathcal{FG}$-stable set (A^{min}, B^{max}). Note that this algorithm (just like the iterative method for monotone functions) can be extended to a transfinite induction proof of Theorem 11.3. The advantage of the method we have followed is that it does not lean on the axiom of choice and indicates an unexpected connection with lattice theory. Here I would like to acknowledge András Biró for drawing my attention to the fixed-point theorem of Knaster and Tarski.

In Section 12, we give applications of Theorem 11.3. Although usually we state those consequences for infinite sets, in case of finite groundsets, the above algorithm can be easily translated, indicating that the corresponding structure can be efficiently constructed. The interested reader can find a detailed analysis of the proposal algorithm of Gale and Shapley (a special case of the above monotone function-iterating method) in the book of Knuth [63].

12 Stable antichains

Our first example of a comonotone set-function comes from partially ordered sets.

Observation 12.1. *Let $\prec$ be a partial order on X and $\mathcal{F}(A)$, the set of $\prec$-minimal elements of A. Then $\mathcal{F}$ is a comonotone function on X.*

Proof. $\overline{\mathcal{F}}(A)$ is the set of nonminimal elements of A, hence $\overline{\mathcal{F}}$ is a monotone function. □

Let $\prec_1$ and $\prec_2$ be fixed partial orders on X. A subset S of X is a *stable antichain* if it is a common antichain of $\prec_1$ and $\prec_2$ and bounds all other elements from below, i.e. if

$$\text{the elements of } S \text{ are pairwise both } \prec_1\text{- and } \prec_2\text{-incomparable, and} \tag{12.1}$$

$$\text{for each } x \in X \setminus S \text{ there exists an } s \in S \text{ such that } s \prec_1 x \text{ or } s \prec_2 x. \tag{12.2}$$

Note that if both comonotone functions $\mathcal{F}$ and $\mathcal{G}$ come from Observation 12.1 then any stable antichain is an $\mathcal{FG}$-kernel, but the converse not necessarily true. However, stable antichains and $\mathcal{FG}$-kernels are the same if both partial orders are partial well-orderings.

Theorem 12.2.

A *If $\prec_1$ and $\prec_2$ are partial orders on X, then there are subsets X_1 and X_2 of X such that*

$$X_1 \cup X_2 = X \text{ and} \tag{12.3}$$

$$X_1 \cap X_2 \text{ is the set of } \prec_i\text{-minima of } X_i \text{ for } i \in \{1,2\}. \tag{12.4}$$

B *Moreover, if $\prec_1$ and $\prec_2$ are partial well-orders then there exist stable antichains S^1 and S^2 such that if S is a stable antichain then there is no $s \in S$, $i \in \{1,2\}$, and $s^i \in S^i$ such that $s \prec_i s^i$.*

Proof. Define $\mathcal{F}(A)$ as the set of $\prec_1$-minima of A, and $\mathcal{G}(A)$ as the set of $\prec_2$-minima of A. By Observation 12.1, set-functions $\mathcal{F}$ and $\mathcal{G}$ are comonotone. Applying Theorem 11.3 to $\mathcal{F}$ and $\mathcal{G}$, we see that there are subsets X_1^{max}, X_2^{min} and X_1^{min}, X_2^{max} with properties (12.3, 12.4), such that if subsets X_1, X_2 of X have properties (12.3, 12.4) then $X_i^{min} \subseteq X_i \subseteq X_i^{max}$ for $i \in \{1,2\}$. This proves part A.

Define $S^1 := X_1^{max} \cap X_2^{min}$ and $S^2 := X_1^{min} \cap X_2^{max}$. We prove part B for S^1; the statement for S^2 follows by interchanging the role of $\prec_1$ and $\prec_2$.

Property (12.1) of S^1 follows directly from property (12.4) of X_1^{max} and X_2^{min}. By partial well-orderedness, for any element x of X_1^{max} and y of X_2^{min}, there are elements x' of $\mathcal{F}(X_1^{max})$ and y' of $\mathcal{G}(X_2^{min})$ such that $x' \preceq_1 x$ and $y' \preceq_2 y$. This proves (12.2) for S^1. Thus S^1 is a stable antichain, indeed.

For stable antichain S, define $X_1 := \{x \in X : \exists S \ni s \prec_1 x\}$ and $X_2 := (X \setminus X_1) \cup S$. Clearly X_1 and X_2 have properties (12.3, 12.4), so $X_i^{min} \subseteq X_i \subseteq X_i^{max}$ for $i \in \{1,2\}$. This implies the last part of Theorem 12.2 B. ☐

Stable antichains S^1 and S^2 in Theorem 12.2 are called the $\prec_1$- and $\prec_2$-*optimal stable antichains*, respectively. If we apply Theorem 12.2 B to partial orders that both consist of disjoint chains, then we obtain an infinite version of the stable marriage theorem:

Theorem 12.3 (Gale-Shapley [42]). *Let $G = (U \cup V, E)$ be a bipartite (not necessary simple) graph with colour-classes U and V and let for each $x \in U \cup V$, $\prec_x$ be a well-order on $D(x)$. Then there exists a matching $M \subseteq E$ of G such that*

$$\begin{aligned}&\text{for each edge } e \text{ of } E \setminus M, \text{ there is an endnode } x \text{ of } e\\ &\qquad\qquad \text{and edge } e_M \text{ of } M \text{ such that } e_M \prec_x e.\end{aligned} \tag{12.5}$$

Proof. Define $\prec_1$ and $\prec_2$ on E by $e \prec_1 f$ if $e \prec_u f$ for some $u \in U$, and $e \prec_2 f$ if $e \prec_v f$ for some $v \in V$. Apply Theorem 12.2 B. ☐

The above matching M with property (12.5) is often called a *stable marriage scheme*, or shortly a *stable matching*. The first name refers to the model in which each man and woman orders those partners with whom they possibly can get married. The assignment in Theorem 12.3 corresponds to the stable situation where no man-woman pair has the common interest to quit the scheme and marry each other. The $\prec_1$- and $\prec_2$-optimal stable antichains coming from the proof of Theorem 12.3 are usually called the *man-* and *woman-optimal* stable marriage schemes, as those are the ones where each person of the corresponding sex gets the best possible partner of all stable marriage schemes.

Note that if we specialize fixed point algorithm (11.8) for the stable matching case then we get exactly the proposal algorithm of Gale and Shapley.

As an application of the stable matching theorem of Gale and Shapley (Theorem 12.3), we deduce the Mendelsohn-Dulmage theorem (Theorem 10.3). Define linear order $\prec_u$ on $D(u)$ by $e \prec_u f$ for vertex u of U if e belongs to M_2 and f to M_1. Similarly, $e \prec_v f$ for vertex v of V if $e, f \in D(v)$ and $e \in M_1$ and $f \in M_2$. By Theorem 12.3, there is a stable matching M of G. As no edge of M_1 can be a blocking edge of M, each vertex of U covered by M_1 must be covered also by M. Similarly, no edge of M_2 blocks M, hence each vertex of V covered by M_2 must be covered by M. Thus M has the property required by Theorem 10.3.

Next we discuss a generalization of the stable matching theorem which also includes the so called college admission problem. Here the role of men are played by colleges, women are the students, and limited polygamy is allowed for men. In the classical example of this problem (see Roth [84] and Roth and Sotomayor [88]), medical students may spend their professional practice at a hospital from a selection list. Each hospital has a quota of residents that it can and would like to accept. The stable situation is, when no resident would be happy to change his or her appointment to another hospital that would be glad to fire somebody to get this particular person. This example emerged as a real life problem in the United States in the late forties. The desire of the competing hospitals for the best students turned the selection process into a mad rush, that was only eased after a centralized scheme had been introduced. It seems that up till recently this centralized scheme (the then called NIMP: National Intern Matching Program) was the only practical application of the Gale-Shapley algorithm. More about the background and origin of the stable matching research can be found in the book of Roth and Sotomayor [88]. In the book of Gusfield and Irving [50], a similar situation is described that corresponds to the original (non-polygamous) stable marriage problem. There, the role of greedy hospitals is played by federal judges and the bewildered students are their law clerk candidates.

In our model we will use a more general comonotone set-function than the one in Observation 12.1. This comonotone set-function turns out to be useful in special posets defined below. We say that partial order $\prec$ on finite set X is of *arborescence type* if $t \prec u$ and $t \prec v$ implies that u and v are $\prec$-comparable. (In the finite case, it means that the diagram of partial order $\prec$ is an in-arborescence).

Theorem 12.4. *For $i \in \{1, 2\}$ let $\prec_i$ be a partial well-order of arborescence type on X, and for subset A and element x of X define*

$$\tau_i(x, A) := |\{a \in A : a \prec_i x\}|.$$

If $t_1, t_2 : X \to \mathbb{N}_+$ are arbitrary functions, then there is a subset S of X such that

$$\tau_i(s, S) < t_i(s) \text{ for any } s \in S \text{ and } i \in \{1, 2\} \text{ and} \tag{12.6}$$

$$\text{for every element } x \in X \setminus S \text{ there is an } i \in \{1, 2\} \text{ with } \tau_i(x, S) \geq t_i(s). \tag{12.7}$$

Proof. For element x of X and $i \in \{1,2\}$ define

$$\mathcal{H}_x^i := \{T \subseteq X : \tau_i(x,T) = t_i(x) \text{ and } \tau_i(x',T) < t_i(x') \text{ for } x' \in T\}.$$

Functions $\mathcal{F} := \mathcal{F}_{\mathcal{H}^1}$ and $\mathcal{G} := \mathcal{F}_{\mathcal{H}^2}$ are comonotone by Lemma 11.2. Applying Theorem 11.3 to this $\mathcal{F}$ and $\mathcal{G}$, we get an $\mathcal{FG}$-stable pair (X_1, X_2). Let $S := X_1 \cap X_2$. Then (12.6) is true by definition. To see (12.7), define $X_i' := \{x \in X_i \setminus S : \tau_i(x,S) < t_i(x)\}$. We show that $X_i' = \emptyset$. If not, then there is a minimal element x of X_i' as $\prec_i$ is a pwo. By definition, there is a subset T of X_i such that $T \prec_i x$ and $|T| = t_i(x)$. Let $\{m_1, m_2, \dots, m_k\}$ be the set of $\prec_i$-maxima of $T \setminus S$ (as T is set, this is well-defined) and let $T' := \{t \in T : t \not\preceq_i m_j \text{ for } j \in [k]\}$. By the arborescence property of $\prec_i$:

$$\tau_i(x,S) \geq |T'| + \sum_{j=1}^{k} \tau_i(m_j, S) \geq |T'| + \sum_{j=1}^{k} t_i(m_j) \geq |T| = t_i(x),$$

a contradiction. ☐

It is worthwhile to make an observation about the algorithmic aspects of Theorem 12.4. The algorithm explained after Theorem 11.3 works without modification. However, since $\mathcal{H}_x^i$ is not explicitly defined, it is not that clear from the definition how to compute $\mathcal{F}_{\mathcal{H}^i}(A)$ efficiently. Justified by the above proof, a way of doing this is as follows: index elements of X along a linear extension of $\prec_i$ i.e. $X = \{x_1, x_2, \dots x_n\}$ such that if $x_j \prec_i x_k$ then $j \leq k$. Define $\mathcal{F}^0_{\mathcal{H}^i}(A) := \emptyset$ and for $1 \leq j \leq n$ let

$$\mathcal{F}^j_{\mathcal{H}^i}(A) := \begin{cases} \mathcal{F}^{j-1}_{\mathcal{H}^i}(A) & \text{if } x_j \notin A \text{ or} \\ & \text{if } \tau_i(x_j, \mathcal{F}^{j-1}_{\mathcal{H}^i}(A)) \geq t_i(x_j) \\ \mathcal{F}^{j-1}_{\mathcal{H}^i}(A) \cup \{x_j\} & \text{else.} \end{cases} \tag{12.8}$$

By this iterative definition we obtain $\mathcal{F}_{\mathcal{H}^i}(A) = \mathcal{F}^n_{\mathcal{H}^i}(A)$. Note that there is a conspicuous similarity between (12.8) and the greedy algorithm (3.12).

To illustrate Theorem 12.4, we prove a generalization of the college admission (sometimes called the many-to-one stable matching) problem. It concerns the 'many-to-many' problem, which in a sense is the stable counterpart of the so called 'b-matching' problem. Usually, the college admission problem is reduced to the stable matching problem by a node duplication construction, i.e. each vertex of the bipartite graph on the college side is responsible for one unit in the quota of some college. This construction does not work for the stable b-matching problem because an edge between a particular student and college can yield disjoint edges after node duplication. The proposal algorithm might select two disjoint copies of the same edge as different ones, assigning a student to a college with multiplicity more than one. Note that it is mentioned in [50] that a modified proposal algorithm also works for this case but the proof is left to the reader there. Our way of deducing the stable b-matching theorem from Theorem 12.4 is very much similar to the way of reducing the stable matching theorem to Theorem 12.2.

Theorem 12.5. *Let $G = (U \cup V, E)$ a bipartite (not necessary simple) graph between disjoint vertex-sets U and V and let $D(x)$ be well-ordered by $\prec_x$ for every vertex x of*

G. Then for every $b : U \cup V \to \mathbb{N}$ there is a subset M of E such that

$$d_M(x) \leq b(v) \text{ for all } x \in U \cup V \text{ and} \tag{12.9}$$

$$\begin{aligned} &\text{for each edge } e \text{ of } E \setminus M \text{ there is a endnode } x \text{ of } e \text{ such that} \\ &d_M(x) = b(x) \text{ and } f \prec_x e \text{ for each } f \in M \cap D(x). \end{aligned} \tag{12.10}$$

Proof. Define $\prec_1, \prec_2$ as in the proof of Theorem 12.3, and for edge $e = uv$ (with $u \in U$ and $v \in V$) let $t_1(e) := b(u)$ and $t_2(e) := b(v)$. Apply Theorem 12.4. $\square$

A matching M as above with properties (12.9,12.10) is called a *stable b-matching.*

13 Paths and stability

There is a generalization of Theorem 12.2 B that can be formulated by dropping the acyclic requirement for orders. This is in fact a theorem of Sands *et al.* [90] on monochromatic paths. Here we pose a bit weaker condition on the directed graphs involved than in [90].

Theorem 13.1 (Sands *et al.* [90]). *Let A_1 and A_2 be arc-sets on vertex-set V, such that there is no $i \in \{1, 2\}$ and vertices v_j of V (for $j \in \mathbb{N}$) such that*

$$\begin{aligned} &\text{there is a simple } A_i\text{-path from } v_j \text{ to } v_{j+1} \text{ and} \\ &\text{there is no simple } A_i\text{-path from } v_{j+1} \text{ to } v_j. \end{aligned} \tag{13.1}$$

Then there is a subset K of V such that

$$\begin{aligned} &\text{for each element } v \in V \text{ there is a simple path in } A_1 \text{ or in } A_2 \\ &\qquad\text{from } v \text{ to } K, \text{ and} \end{aligned} \tag{13.2}$$

$$\begin{aligned} &\text{there is neither a simple } A_1\text{-, nor a simple } A_2\text{-path} \\ &\qquad\text{between different elements of } K. \end{aligned} \tag{13.3}$$

Proof. Let $\prec$ be a well-ordering on V, i.e. $\prec$ is a linear order and every subset of V has a $\prec$-minimal element. The existence of such a well-order follows from the axiom of choice; this is actually the only place in our treatment where we use this axiom. For $i \in \{1, 2\}$ define $\prec_i$ such that $u \prec_i v$ if and only if

$$\text{there is a simple } A_i\text{-path from } v \text{ to } u,$$

and

$$u \prec v \text{ or there is no simple } A_i\text{-path from } u \text{ to } v.$$

Relation $\prec_i$ is transitive because if $x \prec_i y \prec_i z$ and there is a zx-path of A_i, then x, y and z are in the same strong A_i-component, so $x \prec y \prec z$ must hold.

If $x \preceq_i y \preceq_i x$ then x and y are in the same strong component. Thus $x \preceq y \preceq x$, that is $x = y$. It means that $\prec_i$ is antisymmetric. As $\prec_i$ is trivially reflexive, it is a partial order, indeed.

Next we check that $\prec_i$ is pwo, i.e. any subset U of V has a $\prec_i$-minimal element, for $i \in \{1, 2\}$. From (13.1), there is an element u of U with the property that if there is a simple A_i-path from u to some u' then there is a simple A_i-path from u' to u. Consider $U' := \{x \in U : \text{there is a simple } A_i\text{-path from } u \text{ to } x\}$. By definition, orders $\prec$ and $\prec_i$ are the same on U', so the $\prec$-minimal element of U' is a $\prec_i$-minimal element of U as well.

Theorem 13.1 directly follows from the application of Theorem 12.2 B to partial well-orders $\prec_1$ and $\prec_2$ as any stable antichain K of $\prec_1$ and $\prec_2$ has the kernel property described in (13.2, 13.3). $\square$

Remark. If in Theorem 13.1 we assume that A_1 and A_2 are acyclic then we arrive back to Theorem 12.2 B. Theorem 13.1 can also be considered as a generalization of the variant of the stable marriage theorem where no strict preference orders are required for men and women but indifference is allowed.

In what follows, we prove the so called 'linking theorem' of Pym [77, 78] as a special case of Theorem 12.2. Although, formally we prove an extension of Pym's result by showing the extra property (13.4), the proof that we give is essentially Pym's [78]. Our aim here is only to indicate that this result can also be viewed in the comonotone framework.

Theorem 13.2 (Pym [77, 78]). *Let $D = (V, A)$ be a directed graph and X, Y subsets of V. Let moreover $\mathcal{P}$ and $\mathcal{Q}$ be families of vertex-disjoint simple XY-paths. Then there exists a family $\mathcal{R}$ of vertex-disjoint simple XY-paths, such that*

any path of $\mathcal{R}$ consists of a (possibly empty) initial segment of a path of $\mathcal{P}$ and of a (possibly empty) end segment of a path of $\mathcal{Q}$, moreover (13.4)

$$In(\mathcal{P}) \subseteq In(\mathcal{R}) \subseteq In(\mathcal{P} \cup \mathcal{Q}) \tag{13.5}$$
$$End(\mathcal{Q}) \subseteq End(\mathcal{R}) \subseteq End(\mathcal{P} \cup \mathcal{Q}).$$

Proof. To prove Theorem 13.2, it suffices to find a set S of switching vertices. Knowing S, we can construct vertex-disjoint path family $\mathcal{R}$ the following way. Define vertex-disjoint path-family $\mathcal{P}'$ as the set of paths of $\mathcal{P}$ disjoint from S together with the set of initial segments of paths of $\mathcal{P}$ ending in S. Similarly, we define $\mathcal{Q}'$ as the set of paths of $\mathcal{Q}$ disjoint from S and the end segment of $\mathcal{Q}$-paths starting from S. To obtain $\mathcal{R}$, we merge paths in $\mathcal{P}' \cup \mathcal{Q}'$ that start and end in the same vertex of S.

To make this construction work, subset S of V must have the following properties:

1. any path p of $\mathcal{P} \cup \mathcal{Q}$ contains at most one vertex from S, and
2. if v is a common vertex of path p of $\mathcal{P}$ and of path q of $\mathcal{Q}$ then either $v \in S$ or there is a vertex s of S before v on p or after v on q.

Define $\prec_p$ on $V(\mathcal{P}) \cap V(\mathcal{Q})$ such that $u \prec_{\mathcal{P}} v$ if there is a uv-subpath of some path of $\mathcal{P}$. Define $\prec_{\mathcal{Q}}$ also on $V(\mathcal{P}) \cap V(\mathcal{Q})$ by $u \prec_{\mathcal{Q}} v$ if there is a vu-subpath of some path of $\mathcal{Q}$. Observe that properties 1. and 2. above are nothing else but the description of a stable antichain of $\prec_{\mathcal{P}}$ and $\prec_{\mathcal{Q}}$. As both relations are partial well-orders, Theorem 13.2 follows from Theorem 12.2 B. $\square$

Note that in the above proof we did not use Theorem 12.2 in full generality. For finite vertex-set V, what we actually need is the Gale-Shapley theorem for multigraphs. In that framework paths of $\mathcal{P}$ correspond to men, paths in $\mathcal{Q}$ are women, and each common vertex of a $\mathcal{P}$-path and $\mathcal{Q}$-path yields a possible marriage. Each man would like to switch to a woman-path from his path as soon as possible and each woman would like to receive a man-path as late possible. (So everybody strives to minimize the part of his/her path that is used in $\mathcal{R}$.) A stable marriage scheme in this model is exactly a set of switching vertices for some family $\mathbb{R}$ as in Theorem 13.2.

Brualdi and Pym proved a modified version of the linking theorem of Pym (Theorem 13.2 without (13.4)) where they require condition (13.6) but allow generalized paths [9]:

Theorem 13.3 (Brualdi-Pym [9]). *In digraph $D = (V, A)$, let $\mathcal{P}$ and $\mathcal{Q}$ be families of vertex-disjoint general paths. There exist a family $\mathcal{R}$ of vertex-disjoint general paths of D such that*

$$\begin{aligned} In(\mathcal{P}) \subseteq In(\mathcal{R}) \subseteq In(\mathcal{P} \cup \mathcal{Q}) \qquad & End(\mathcal{Q}) \subseteq End(\mathcal{R}) \subseteq End(\mathcal{P} \cup \mathcal{Q}) \\ V(\mathcal{P}) \cap V(\mathcal{Q}) \subseteq V(\mathcal{R}) \subseteq V(\mathcal{P} \cup \mathcal{Q}) \qquad & A(\mathcal{P}) \cap A(\mathcal{Q}) \subseteq A(\mathcal{R}) \subseteq A(\mathcal{P} \cup \mathcal{Q}). \end{aligned} \tag{13.6}$$

Note that although this theorem sounds similar to Theorem 13.2, it seems to be substantially different. To be able to prove condition (13.6), we must drop condition (13.4), as even if $\mathcal{P}$ and $\mathcal{Q}$ consists of finite simple paths, it might be necessary to use both circular and infinite paths in $\mathcal{R}$ (see [9]). For a simple proof of Theorem 13.3, based on node-splitting, see Ingleton and Piff [55].

The following corollary is also observed by others (see e.g. [12]) and provides an interesting application of Theorem 13.2 on families of edge-disjoint (rather than vertex-disjoint) paths. In [12], by Conforti *et al.*, this is deduced directly from the stable matching theorem on bipartite multigraphs, using the framework we described after the proof of Theorem 13.2.

Corollary 13.4. *Let $G = (V, E)$ be an undirected graph and x, y, z be different vertices of V. Let $\mathcal{P}$ be a set of k edge-disjoint xy-paths and $\mathcal{Q}$ be a set of k edge-disjoint yz-paths. Then there exist a set $\mathcal{R}$ of k edge-disjoint xz-paths such that each path of $\mathcal{R}$ is the union of a (possibly empty) initial segment of a path of $\mathcal{P}$ and of a (possibly empty) end segment of a path of $\mathcal{Q}$.*

To prove the above result, we apply Theorem 13.2 on the line-graphs of paths of $\mathcal{P}$ and $\mathcal{Q}$. (A line-graph of a path is a path again.) There still remain some small details to take care of. This is done in the following.

Proof. Let vertex-disjoint path-families $\mathcal{P}', \mathcal{Q}'$ be the collection of the line-graphs of the paths in $\mathcal{P}$ and in $\mathcal{Q}$, respectively. By applying Theorem 13.2 on $\mathcal{P}'$ and $\mathcal{Q}'$ we get a vertex-disjoint path collection $\mathcal{R}'$. Family $\mathcal{R}'$ is the set of line-graphs of a set of edge-disjoint walks (not paths, in general). Clearly, $|\mathcal{R}^* \cap \mathcal{P}| = |\mathcal{R}^* \cap \mathcal{Q}|$, so we can pair those paths and merge them via y. By this operation, $\mathcal{R}^*$ becomes a collection of edge-disjoint xz-walks. To obtain $\mathcal{R}$ as described in the corollary, we have to shortcut the possible circles on each element of $\mathcal{R}^*$. When no more shortcut is possible, we get edge-disjoint xz-paths switching exactly once, as stated. $\square$

Using Corollary 13.4 in [12], Conforti *et al.* describe a Gomory-Hu based maxflow-representing structure. For each edge uv of a Gomory-Hu tree of a graph G, they store a list of $\lambda_G(u,v)$ edge disjoint uv paths. They also do it for some other $|V(G)|$ pairs uv of vertices of G. Then, by applying the stable marriage algorithm $O(\alpha(n))$ times as in Corollary 13.4, they construct a collection of $\lambda_G(x,y)$ edge-disjoint xy-paths of G for any two vertices x and y of G (where $\alpha(n)$ is the inverse Ackerman-function of n that is regarded almost as good as a constant function).

14 Graph-kernels

The motivation for the name $\mathcal{FG}$-kernel (suggested by András Sebő) is that Theorem 13.1 can also be formulated as: any digraph which is the union of two transitive arc-sets, has a kernel. An arc-set is called *transitive* if $ab, bc \in A$ implies $ac \in A$. Recall, that a *kernel* is an independent subset K of vertex set V such that for any vertex v outside K there is a node k of K with $vk \in A$. In this context (as we have seen in Chapter I), the stable marriage theorem can be reformulated as: any normal orientation of the line-graph of a bipartite graph has a kernel [69], where we defined a normal orientation as one in which every spanned clique has a kernel.

As mentioned before, several kernel-problems in Graph Theory have been motivated by Conjecture 9.5 of Berge and Duchet which states that graph G is perfect if and only if any normal orientation of G has a kernel. Before the theorem of Boros and Gurvich [8] (the 'only if' part of Conjecture 9.5) was known, several special cases of the Berge-Duchet conjecture have been confirmed. Using theorems from Section 13, we can also obtain some results of this kind. A digraph is called *kernel-perfect* if any of its spanned subgraphs has a kernel.

Theorem 14.1. *Any orientation of a bipartite graph is kernel-perfect.*

Proof. Any subgraph of a bipartite graph is bipartite, hence it is sufficient to prove that any orientation of a bipartite graph has a kernel. Let $G = (U \cup V, A)$ be an orientation of a bipartite graph with colour-classes U and V and let $A_1 := D^+_G(U)$ and $A_2 := D^-_G(U)$. Clearly, $A = A_1 \cup A_2$, and neither A_1, nor A_2 contains a directed path of length 2. This means that the subset K of $U \cup V$ we get from Theorem 13.1 is a kernel of G. □

A generalization of this is the following result of Richardson.

Theorem 14.2 (Richardson [82]). *Any finite digraph G with no odd directed cycle is kernel-perfect.*

Proof. As no spanned subgraph of G has an odd directed cycle, it is enough to show that G has a kernel. Define an auxiliary graph on the strong components of G such that there is an arc from strong component C to strong component C' if there is a dipath of G from C to C'. This auxiliary graph must be acyclic by the definition of strong component, and hence there is a strong 'sink' component C, that has outdegree 0 in the auxiliary graph. As C is strongly connected and has no odd dicycle, it is bipartite. Hence the subgraph spanned by C has a kernel K_C. By induction,

$G - (C \cup D^-(C))$ has a kernel K'_C. It is easy to see that $K := K_C \cup K'_C$ is a kernel of G. $\square$

Champetier [10] proved that any normal orientation D of a comparability graph has a kernel if any directed 3-cycle of D has at most one arc uv such that $vu \notin A(D)$. We prove a weaker result and remark that the theorem of Champetier can also be proved with the help of the generalized Gale-Shapley algorithm.

Theorem 14.3 (see Champetier [10]). *If D is a normal orientation of a comparability graph such that $uv \in A(D)$ implies $vu \notin A(D)$, then D is kernel-perfect.*

Proof. Each spanned subgraph of D is a normal one-way orientation of a comparability graph, hence it is sufficient to prove that D has a kernel.

By definition, there is a a partial order $\preceq$ on $V(D)$ such that there is an arc between u and v in $A(D)$ if and only if u and v are $\prec$-comparable. Define $A_1 := \{uv : u \prec v\}$ and $A_2 := \{uv : v \prec u\}$. We claim that, for $i \in \{1, 2\}$, A_i is transitive, that is, $uv, vw \in A_i$ implies $uw \in A_i$. By symmetry, we only need to prove this for A_1. The condition means that $u \prec v \prec w$, i.e. there is an arc between u and w. If this arc is wu, then by lack of bioriented edges, uvw spans a clique without a kernel. Hence $uw \in A_1$, that is A_1 and A_2 correspond to partial orders $\prec_1$ and $\prec_2$. By Theorem 12.2 B, there is a stable antichain K of $\prec_1$ and $\prec_2$. Hence K is a kernel of D. $\square$

15 The stable roommates problem

Another kernel-type problem is the so-called stable roommates problem. In this problem we are given an undirected graph $G = (V, E)$ and a linear order $\leq_v$ of $D(v)$ for each vertex v of V. We may think that edges represent the possible roommates in a student hostel with only double rooms and $\leq_v$ is the preference order of person v on his possible roommates. The task is to find a matching M such that for any edge $e \in E \setminus M$ there is an edge m of M with $m \leq_v e$ for some vertex v of V. That is, we are looking for a stable scheme, where there are no two persons who would be roommates with each other rather than with their actual partners. If a matching has the above property, it is called a *stable matching*.

Observe that a nonbipartite stable matching, the solution of a stable roommates problem can also be seen as a kernel of an appropriate orientation of the line-graph $L(G)$ of G. Because not all cliques of $L(G)$ come from stars, the orientation we face here is not necessarily normal. Moreover, if G contains an odd cycle of length at least 5 then $L(G)$ is not perfect. So the Berge-Duchet conjecture does not seem to be relevant for this problem.

The stable roommates problem has been solved by Irving (see [56]) who described a two-stage algorithm to find a stable matching, if it exists. If the algorithm terminates without outputting a stable scheme then a stable matching does not exists. The first phase of the algorithm is similar to the proposal algorithm of Gale and Shapley, while in the second phase a new operation is used that reduces the problem but neither kills all stable matchings of the model, nor introduces a new one to the model.

By another approach, Feder [30] and Subramanian [96] solved the following *network stability problem*, an extension of the nonbipartite stable matching problem. A *network* is a directed graph $G = (V, A)$, where each vertex v of V is a logical gate on inputs corresponding to $D^-(v)$ with outputs corresponding to $D^+(v)$. The problem is to find a *stable configuration of the network*, that is an assignment of logical values to the arcs such that for each gate v, the values assigned to $D^+(v)$ are the output values of the gate on the input that is given by the assigned values on $D^-(v)$. An illustration of a network stability problem is the following puzzle.

Puzzle 15.1. Find decimal numbers $a_0, a_1, \ldots, a_9$ such that the following sentence becomes true:
In this sentence there are a_0 digit 0s, a_1 digit 1s, a_2 digit 2s, a_3 digit 3s, a_4 digit 4s, a_5 digit 5s, a_6 digit 6s, a_7 digit 7s, a_8 digit 8s, and a_9 digit 9s.

Here, the underlying network has ten vertices $(v_0, v_1, \ldots v_9)$ and from each vertex there are sufficiently many arcs (this case two suffices) to each other vertex and itself. The gate at vertex v_i works as follows. If it has j inputs with value 1 associated, then the output will be 1 on arcs that correspond to digits of $j+1$, and 0 elsewhere. (The multiple edges we need if the decimal form of $j+1$ uses more digits of the same kind.)

Feder showed that if all gates in V are *adjacency-preserving* (that is, at most one output value can change if one input is modified), then there is an efficient algorithm to decide the existence of a stable configuration, and if the answer is positive then the algorithm finds one. Note that this algorithm can be viewed as a method to find a fixed point of a certain map $f : 2^E \to 2^E$ if such a point exists. We come back to this reformulation in Section 19. Feder also showed how to transform a stable matching problem into a stability problem of a network with only adjacency-preserving gates.

Subramanian [96] formulated the stable matching problem in terms of 'X-networks' built up from 'X-gates'. A main property of an X-gate and of an X-network is that they are *scatter-free*. That is, no matter how we fix some set of inputs, the number of outputs that do not only depend on the fixed inputs is at most the number of non-fixed inputs[7]. Subramanian observed that stable configurations of a monotone scatter-free network have a natural lattice structure. (A network is monotone if the NOT gate can not be simulated by it.) Interestingly, in [96], Subramanian also cites the fixed point theorem of Tarski [97] (our Theorem 10.1), but does not see that the lattice structure of stable network-configurations would follow from the lattice structure of fixed points of a monotone function:

> Tarski's theorem [28][8] says that the set of fixed points of any monotone function from a complete lattice to itself forms a complete lattice. This theorem, together with Lemma 2.1[9], is sufficient to prove that the stable configurations (on a given input assignment) of any monotone scatter-free network form a lattice. However, this approach does not seem to yield the sublattice property. Nevertheless, and

[7]It is easy to see that any scatter-free gate is adjacency-preserving, but the converse is not true. Consider a gate with n inputs and 2^{n-1} outputs corresponding to odd subsets of the n inputs. For a given valuation of the n inputs, let all outputs of the gate be 0 except for the one (if any) that corresponds to the subset of the inputs with value 1. This gate is adjacency-preserving, but not scatter-free if $2^{n-1} > n$.

[8]See [97] and Theorem 10.1.

[9]This lemma states that a network built up from scatter-free gates is scatter-free.

> even though our proof is specific to the comparator, Theorem 6.4[10] does not extend to any monotone scatter-free network. An easy application of the ideas in [5][11] yields the desired proof.

We return to this question in Section 19, where with the help of the notion of increasing functions, we deduce the above sublattice property.

We end our discussion of the stable roommates problem by showing that any stable roommates problem can be reduced to a stable roommates problem on a 3-colourable graph. (In the terminology of Knuth [63], there are men, women and dogs with preferences on the others, and everybody is looking for a best partner, pet or owner with the restriction that married couples can not have a dog. In this model, we allow that a person prefers to have a specific dog rather than marrying a certain other person.)

Theorem 15.2. *For any graph $G = (V, E)$ and linear orders $\leq_v$ on $D_G(v)$ $(v \in V)$, we can construct in polynomial time a 3-colourable graph H on $|V| + 2|E|$ vertices with $4|E|$ edges and linear orders $\prec_v$ on $D_H(v)$ $(v \in V(H))$ such that there is a stable matching in G if and only if there is one in H.*

Proof. For any edge $e = uv$ of E introduce two new vertices u_e and v_e. Let $V(H) := V \cup \{v_e : v \in V, e \in D_G(v)\}$ and $E(H) := \{vv_e : e \in D_G(v)\} \cup \{e_{uv} = u_e v_e : e \in D_G(u) \cap D_G(v)\}$. Define linear order $\prec_v$ for $v \in V$ by $vv_e \prec_v vv_f$ if $e \leq_v f$, and for the new vertices by $e_{uv} \prec_{u_e} uu_e \prec_{u_e} e_{vu}$. (So we subdivide each edge of G by two new vertices, and duplicate the edge between them. The preference order of the old vertices does not change and each parallel edge will be the best choice of one new vertex and the worse for the other.)

For a stable matching M of G, let $L' := \{vv_e : e \in D_G(v) \cap M\} \cup \{e_{vu} \in E(H) : \exists\, f \in M \text{ such that } f \leq_u e\}$. Construct L from L' by deleting one copy of each pair of parallel edges of L'. It is easy to check that L is a stable matching of H.

On the other hand, let L be a stable matching of H. By stability, if $uu_e \in L$ for some edge $e = uv$ then $vv_e \in L$. Hence $M := \{e \in E : vv_e \in L \text{ for some } v \in V\}$ is a stable matching of G.

Clearly, V is an independent set in H and $H - V$ is a graph such that any two edge of it is either parallel or disjoint. Hence $H - V$ is 2-colourable and $\chi(H) \leq 3$. □

16 Stable matchings in Game Theory

When we talk about the stable matching problem, we cannot ignore its link to Game Theory. The stable matching model is an easy but powerful way to describe certain two-sided market economies. (For a well-written account on this connection, the reader (really) should consult the book of Roth and Sotomayor [88].) To illustrate this link to Game Theory and Mathematical Economics, earlier we have cited both the observation of Roth [84] on the problem emerged about the residentship of American

[10] A theorem, stating that the stable configurations of a comparator network (that is a network, built up from so-called comparator-gates) have the lattice property with bitwise AND and OR as lattice operations. This theorem generalizes the lattice property for bipartite stable matchings.

[11] See [30].

medical students and the report of Gusfield and Irving [50] on the cruel fight of federal judges for the best clerk-candidates.

András Lukács has pointed out a problem of similar nature to me. In Hungary, before entering a state university, candidates have to pass a more or less centralized entrance exam. According to these exams, universities assign a certain number of points to each candidate. After the university knows the points of all of its candidates, it has to declare a threshold. Knowing this thresholds, each candidate can choose any university that does not have a higher threshold than his/her number of points. When a university declares a threshold, it has two conflicting aims. On one hand, the ministry finances universities according to the number of their students. So universities would like to have as many students as possible, i.e. the threshold should be low. On the other hand, it can not be too low, because each university has a quota on students that they cannot exceed; and after all, the idea of an entrance exam is to have top candidates rather than second line ones. Some years ago, there emerged the idea of a centralized scheme to solve this problem, but this initiative has failed. After the ministry of education investigated the problem in details, they backed down because of the 'enormous computing capacity' the solution would demand. My guess is that they did not know too much about the Gale-Shapley algorithm.

Besides labour market situations, auction markets can also be viewed in the stable matching model. One issue about stable matching models is, what kind of rules will provide a 'fair' stable matching. (For example, the NIMP (nowadays NRMP: National Resident Matching Program) finds the hospital-optimal stable matching.) In an "English auction" (also called an open outcry, ascending bid auction), items are sold on the second highest price that some bidder is ready to pay. This results in a bidder-optimal stable matching between items and bidders. (We assume that each bidder has an estimate of each item, in such a way that he/she affords all items on his/her estimated price). In case of the auction model, the auctioneer-optimal stable matching would be the one in which each item is sold on the highest price that some bidder is ready to pay for it.

In the marriage model, it is a trivial observation that in case of a rule that constructs a men-optimal stable matching, each individual man has the interest to behave straightforwardly. However, in the above auction model, even in case of a bidder-optimal rule, bidders can form coalitions (rings) to try to keep prices low, and to divide the prey amongst themselves. On the other hand, auctioneers can operate with imaginary bids to push the price higher[12]. This real world practice relates to the fact that a stable matching is not necessarily strongly Pareto-optimal for the coalition of men, that is, there might be a matching in which no man receives a worse partner than the one in the men-optimal scheme but some men get strictly better wives. The book of Roth and Sotomayor [88] provides further details on the above market models.

16.1 Stable matchings in Cooperative Game Theory

In what follows, we describe stable matchings as solutions of an n-person game. Assume we have n players in a game, each of them has a selection of possible strategies

[12] According to Roth and Sotomayor ([88], p. 8), Rembrandt was bidding on his own paintings at an auction.

to choose from. The outcome v of the game is completely determined by the strategies the n players select. Each player p has a ranking $r_p : V \to \mathbb{R}$ on the set V of possible outcomes according to his/her preference. We say that outcome v is dominated by coalition C of players, if there exists a set of v-dominating strategies for players of C. A certain choice of strategies for players in C is a *set of v-dominating strategies* if no matter what strategies the other players select, each player in C is going to profit from the outcome. That is, for any outcome u for the above set of strategies of C-players, $r_p(u) < r_p(v)$ holds for any player p in C. We define the *domination graph* $D = (V, A)$ in such a way that A is the set of arcs vu where v is an outcome dominated by some coalition C and u is a possible outcome when players of C play a v-dominating strategy.

In terms of the above domination graph, we can define different solution concepts that result in an equilibrium situation on the market. The *core* of an n-person game is the set of undominated outcomes. These are those vertices of the domination graph that have outdegree 0. Clearly, once each player plays strategies that lead to an outcome v in the core, then no subset C of players has the interest to form a coalition and change strategies, because they cannot choose a v-dominating strategy. I.e., if coalition C plays other strategies then the remaining players can choose strategies in such a way that some player of C will not profit from the joint action. In spite of its clear advantages, the core of an n-person game can be empty.

The solution concept of von Neumann and Morgenstern is a more sophisticated one and it might be applicable for games with an empty core. We say that subset K of outcomes is a *von Neumann-Morgenstern solution* of the n-person game, if K is a kernel of the domination graph. That is, no outcome in K is dominated by another outcome in K and any outcome not in K is dominated by some outcome in K. To get an intuition about the idea behind this concept, von Neumann and Morgenstern propose to consider solution K as a “standard of behaviour” for the players. That is, if players accept that the outcome 'should be' in K then as soon as they have chosen strategies leading to some outcome k in K, no coalition C has the interest to depart from their strategies. This is because if each player of some coalition C profits from choosing other strategies, then the new outcome k_C must be outside K. So coalition C must face the 'danger' that some other coalition C' (not necessarily disjoint from C) can play strategies such that each member of C' will prefer the outcome k' (that might be in K) to k_C and *some* player in $C \setminus C'$ will be upset with k' to k_C.

Clearly, if a game has a nonempty core C then any von Neumann-Morgenstern solution K contains C . If core C has the extra property that it dominates any outcome of $V \setminus C$, then $K = C$ is the unique von Neumann-Morgenstern solution. There are games that have a von Neumann-Morgenstern solution in spite of having an empty core. (We shall see that some stable matching-related games are like that.) Also, there are games that have no von Neumann-Morgenstern solution, but for several interesting classes of games (even with an empty core) such a solution does exist.

From the definitions of core and von Neumann-Morgenstern solution, it might not be transparent that there is an essential difference between the two notions. Namely, as we have seen above, a core has a certain *local* stability property, that is, once an outcome is in the core, there is a good reason to think that it will stay there. However, starting from an outcome not in the core, it might be possible that players form again and again new coalitions to improve their profit, without ever reaching a core outcome, because the underlying walk on the domination graph never finds it. In case of a von-Neumann-Morgenstern solution, once a 'standard of behaviour' is accepted, and the outcome is not a solution, then players tend to form coalitions that move the outcome closer and closer to the solution. As soon as the outcome is a solution, it will stay there. So a von Neumann-Morgenstern solution has a *global* stability nature.

In case of a game in the simplest stable marriage model, the strategy of a person can only be to try to marry some specific person of the opposite sex. If this other person also chooses the strategy to marry this first one, then the edge between them will be in the outcome. (The proposal algorithm of Gale and Shapley can be regarded as an agreement protocol to select strategies leading to an equilibrium situation.) The possible outcomes are matchings of the underlying graph, and a person prefers one matching to another, if he/she has a more preferred partner in it. (This also means that each person prefers to have a partner rather than none at all.) Clearly, any nonstable matching in the marriage model is dominated by a 2-person coalition along the blocking edge. On the other hand, if a matching is dominated by some coalition, then this coalition must contain a man and a woman who chose to marry one another, and hence there is an edge blocking the matching. Thus stable matchings form the core of the above n-person marriage game. The content of the stable marriage theorem of Gale and Shapley is that the n-person marriage game has a nonempty core.

The previous construction of an n-person game can also be done for the stable roommates model. The same proof shows that the core of the game is the set of stable matchings. Unlike the bipartite stable matching problem, there are instances of the stable roommates problem that possess no stable matching, that is the core of the game may be empty. Such an example is the K_3 graph in Figure 16.1. The game described by this graph and preferences has neither a core nor a von Neumann-Morgenstern solution.

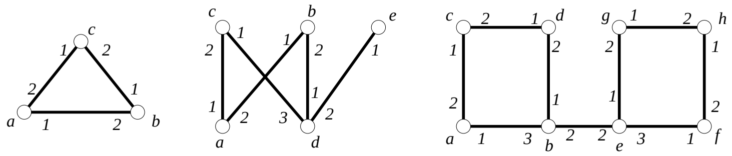

Figure 16.1: Examples of stable matching games.

The only stable matching (that is, the core of the game) in the second example of Figure 16.1 is $\{ac, bd\}$. Matching $\{ab, cd\}$ is blocked only by edge be, hence it is not dominated by any stable matching. In fact, $\{\{ac, bd\}, \{ad, bc\}\}$ is the unique von Neumann-Morgenstern solution of this game. This example shows that in the stable marriage game, the von Neumann-Morgenstern solution might be a proper superset of the core. The following observation is not very difficult to prove.

Claim 16.1. *Let graph $G = (V, E)$ and linear orders $\prec_v$ for each vertex v of V be given. If there is a stable matching M in this model (that is, the core of the related game is non-empty), and set $\mathcal{M}$ of matchings of G is a von Neumann-Morgenstern* *solution of the related game, then $M \in \mathcal{M}$, $V(M) = V(\mathcal{M})$, and $\mathcal{M}$ is exactly the set of stable matchings of the subgraph $(V, E(\mathcal{M}))$ of G.* □

The third example on Figure 16.1 comes from the idea of the second example, and shows a weak point of the von Neumann-Morgenstern solution concept. Here stable matchings are $\{ac, bd, eg, fh\}$, $\{ac, bd, ef, gh\}$ and $\{ab, cd, eg, fh\}$. If we delete edge be then all perfect matchings become stable, in particular matching $\{ab, cd, ef, gh\}$. It is easy to see that the set of perfect matchings is a von Neumann-Morgenstern solution of the related n-person game.

Assume now that matching $\{ab, cd, ef, gh\}$ is the outcome of the game. If, in this situation, players b and e agree on changing strategies to marry one another (and hence to deviate from the "standard of behaviour", as no von Neumann-Morgenstern solution matches them), then they become better off, as the outcome will be $\{cd, be, gh\}$. By definition, some players now can profit if they try to push the outcome back to the "standard of behaviour". In this case, players a, c and f, h can agree on marrying one another. This results in matching $\{ac, be, fh\}$. Now the deviant players b and e can form a coalition with d and g and they collectively profit if they move the outcome to $\{ac, bd, eg, fh\}$. This outcome is in the von Neumann-Morgenstern solution again, but both impertinent players b and e prefer the new situation to the one in the beginning. This shows that the stability that the "standard of behaviour" provides is based on the assumption that even if some players see that they can push the outcome to one that they all prefer, they respect the "standard" and do not attempt "dirty tricks".

The first graph on Figure 16.2 is an example of a game with an empty core, and a von Neumann-Morgenstern solution $\mathcal{M} := \{\{v_1v_2, v_3v_4\}, \{v_2v_3, v_1v_4\}\}$. Clearly, none of the two matchings in $\mathcal{M}$ dominates the other. We prove that any inclusionwise maximal matching M of G not in $\mathcal{M}$ is blocked by some edge of v_1v_2, v_2v_3, v_3v_4 and v_4v_1. If M covers neither a nor b, then M is a subset of some matching of $\mathcal{M}$. Else M covers both a and b. So let $bv_i \in M$. Edge $v_{i-1}v_i$ ($i-1$ is modulo 4) blocks M, proving that $\mathcal{M}$ is indeed a von Neumann-Morgenstern solution.

Obviously, any element of the core belongs to any von Neumann-Morgenstern solution. However, no matching in the above $\mathcal{M}$ is stable in G, justifying that the core of the related game is empty.

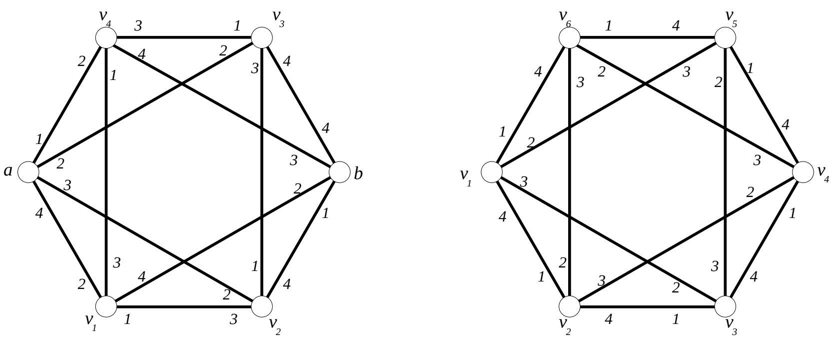

Figure 16.2: Examples of stable matching games on the octahedron graph with a core and without a von Neumann solution and vice versa.

The second graph on Figure 16.2 is an example of a game with a nonempty core that has no von Neumann-Morgenstern solution. It is easy to check that the core consists of stable matchings $\{\{v_1v_2, v_3v_4, v_5v_6\}, \{v_2v_3, v_4v_5, v_6v_1\}\}$. Any matching that consists of edges of the above stable matchings either belongs to the core or it is is dominated by some stable matching in the core. If an inclusionwise maximal matching M uses two diagonals of cycle $v_1v_2v_3v_4v_5v_6$ that do not intersect on Figure 16.2 then M is not perfect and is dominated by one of the stable matchings. Hence, the inclusionwise maximal nonstable matchings that are not dominated by any stable one are matchings $M_i := \{v_iv_{i+1}, v_{i+2}v_{i+4}, v_{i+3}v_{i+5}\}$ for $i \in [6]$, where addition is modulo 6. So if a von Neumann-Morgenstern solution exists, it is the union of the set of stable matchings and a kernel of the domination graph on matchings M_i ($i \in [6]$). It is easy to check that M_i dominates exactly M_{i+2} and M_{i+3}, so the domination graph on the M_i's does not have a kernel.

It is an interesting question whether for any stable marriage game (that is, for the bipartite stable matching game) there always exist a von Neumann-Morgenstern solution. My guess is not, but I could not construct an example showing this.

16.2 Generalizations of the stable marriage theorem

The stable matching theorem of Gale and Shapley has been generalized by several authors. For a fuller story than what we are going to present, the reader should consult especially Chapter 6 of the book of Roth and Sotomayor [88]. Here, we review those results that have a close connection to our topic.

Continuing on a paper of Crawford and Knoer [14], Kelso and Crawford [59] extended the hospital (or many-to-one) model to a model where workers are to be assigned to firms. Firms would like to have certain specific jobs to be done, and this is why they have a more sophisticated preference function on the workers than the plain ranking. Each firm f has choice function C_f that selects from any subset W' of workers a subset $C_f(W')$ of W' that firm f would hire if on the labour-market only workers in W' would be available. Each worker has an ordinary preference ranking on the firms.

An assignment of workers to firms is called *stable* if it is not blocked by a worker-firm pair. Worker-firm pair (w, f) *blocks* an assignment if w prefers f to his/her assignment and in the meanwhile firm f would take worker w if (s)he would be available (that is $w \in C_f(W_f \cup \{w\})$, where W_f is the set of workers assigned to firm f).

Not surprisingly, in the above model there might be no stable assignment. However, if the choice functions of the firms have the so-called substitutability property, then a stable assignment always exists. We say that the preferences of firm f have the *property of substitutability*, if $w \in C_f(W)$ implies $w \in C_f(W \setminus \{w'\})$ for any set W of workers and different workers w, w' of W. This means that if a firm would like to employ some worker, then it still would like to hire him/her if some other worker leaves the labour-market.

Theorem 16.2 (Crawford-Kelso [14]). *If firms have substitutable preferences in the worker-firm assignment model, then there is a stable assignment.*

The proof of Crawford and Kelso is via the accordingly modified Gale-Shapley algorithm. They observe that firm-proposing results in the firm-optimal assignment, and the worker-proposal based method leads to the worker-optimal situation.

Proof. Recalling the equivalent definition of comonotone functions by (11.2, 11.3), we notice that choice functions with the property of substitutability are comonotone functions over the set of workers. We can define a joint choice function for all the firms on the edges of the assignment graph in a natural way. Namely, let G be the complete bipartite graph between firms and workers. The joint choice function of firms maps subset E of $E(G)$ into $C_{firms}(E) = \{fw : f \text{ is a firm and } w \in C_f(\{w' : fw' \in E\})\}$. This joint choice function (being the union of comonotone functions on disjoint stars) is comonotone, and Theorem 16.2 follows directly from Theorem 11.3. ☐

Kelso and Crawford extended their result by involving salaries in the model. In our comonotone language, this is not much different from the original model: introduce parallel edges between the firm and the worker, each representing a certain salary, and define the preference functions accordingly. Kelso and Crawford need the "gross substitutes assumption" to formulate the comonotone rule in this situation, but in any case, the result is a special case of Theorem 11.3 again. (It does not cause any difficulty in the comonotone model that infinite graphs are involved, as the fixed point theorem works also for this case. However, as the partial order is not pwo here, in the proof we also should use the Dedekind property of $(\mathbb{R}, \leq)$.) We omit the details.

After this observation about comonotonicity and substitutability, one can foresee the many-to-many version of the stable assignment theorem, which indeed is a result of Roth [85, 86]. In [86], Roth studies three models: the one-to-one, the many-to-one and the many-to-many with substitutable preferences. He shows that for all three models there is a firm-optimal, 'worker-pessimal' and a worker-optimal, 'firm-pessimal' stable assignment. The name 'polarization of interests' refers to this property. Roth also observes the 'opposition of common interests' of workers and firms, which means that if all workers prefer some stable outcome at least as much as some other, then for the firms the opposite holds. In our language this means that if (A, B) and (A', B') are $\mathcal{FG}$-stable pairs and $A \subseteq A'$ then $B \supseteq B'$.

Further on, Roth introduces the notion of the *consensus property*, by which he means the following. If each agent on one side of the market chooses his/her favourite assignment from a set of stable assignments, then this way another stable assignment is constructed. This is a generalization of the lattice property for the marriage model, an observation attributed to John Conway: if each men chooses the better partner from two stable marriage schemes, then this yields a stable scheme in which each woman receives the worse partner from the two schemes.

Roth observed that the consensus property cannot be generalized to workers in the many-to-one model and reached the false conclusion that it holds for the firms there. His claim is that if each firm f chooses its workers from those workers that are assigned to f in at least one of two stable schemes, then a stable scheme is constructed again. Our next example shows a counterexample of this kind of firm-consensus.

Example 16.3. There are one firm f and three workers a, b and c. The firm would like to hire the maximum number of workers for at most 9 units of salary. Worker a

can be hired for 1 or 2 units, worker b for 3 or 4, and worker c for 5. Both the $1+4+0$ and the $2+3+0$ are stable schemes. From these two assignments, the choice of the firm would be $1+3+0$, but this is blocked by worker c, as in this situation, f and c would like to contract with each other.

In [86], Roth wrote the following.

> It remains an open question whether the set of stable outcomes might nevertheless always be a lattice, with some suitably defined meet and join.

Blair [6] gave a positive answer to this question by showing that stable assignments in the above models still have a lattice structure. He wrote

> In the monogamous case, the lattice is obtained by defining one stable matching as $\geq$ another if every man is at least as happy in the first as in the second. A partial ordering on multi-partner matchings could be obtained by replacing "man" by "firm" ... we show that this ordering is not a lattice.
>
> Instead, we will define $f \geq g$ only if each firm wishes to keep its partners in f, even if all the partners in g were also made available, and would not wish to add any new partners. We will show that this more restrictive partial ordering is a lattice. Since it clearly specializes to the standard definition in the monogamous case, it seems to be the appropriate generalization. ...

That is, instead of lattice operations (that correspond to the choice of the agents on one side of the market from two stable assignments), Blair defined the lattice through its partial order (i.e. a stable assignment a is not less than stable assignment b if all agents on (say) the firm-side of the market would choose a if they would have had all choices given in a or b). Clearly, this result has to do with the fact that fixed points of a monotone function constitute a lattice subset which is not necessarily a sublattice. However, neither Blair, nor the above authors observed that these stable assignment results have to do with the fixed point theorem of Knaster and Tarski.

We come back to the lattice property of stable assignments in Section 19. In his paper [86], Roth has some very good insight:

> Alternatively, it may be necessary to explore quite different kinds of structural properties of the set of stable outcomes. For example, the bipartite nature of the matching problem makes it possible to speculate that the set of stable outcomes might possess some matroid properties that would allow the existence of optimal stable outcomes to be explained in terms of the kind of optimization results associated with matroids.

We fulfill this prophecy in the next sections.

17 Matroid-kernels

There is a matroid generalization of the Mendelsohn-Dulmage theorem (Theorem 10.3) by Kundu and Lawler [65].

Theorem 17.1 (Kundu-Lawler [65]). *Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be two matroids on the same groundset, and let $I_1, I_2 \in \mathcal{I}_1 \cap \mathcal{I}_2$ be two common independent sets. Then there is a common independent set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ such that $\mathrm{span}_{\mathcal{M}_1}(I_1) \subseteq \mathrm{span}_{\mathcal{M}_1}(I)$ and $\mathrm{span}_{\mathcal{M}_2}(I_2) \subseteq \mathrm{span}_{\mathcal{M}_2}(I)$.*

While in case of matchings, the Mendelsohn-Dulmage theorem was more or less natural to prove in the comonotone framework, here it is not that clear how the fixed point theorem of Tarski can be applied. However, if we approach matroids from the greedy property, then a comonotone function emerges immediately. For this reason, we review some properties of the greedy algorithm (3.12) for the deletion minors of a matroid.

Fact 17.2. *Let $\mathcal{M} = (E, \mathcal{C})$ be a matroid on groundset E and let $c : E \to \mathbb{R}_+$ be a cost function on $E = \{e_1, e_2, \ldots, e_n\}$ such that $c(e_i) \leq c(e_{i+1})$ for $1 \leq i < n$. Then for any subset E' of E, set $K_n(E')$ is a minimum cost subset of E' that spans E', where $K_0(E') = \emptyset$ and for $0 \leq i \leq n$*

$$K_i(E') = \begin{cases} K_{i-1}(E') & \text{if } e_i \notin E' \text{ or} \\ & \text{if there is a subset } C \text{ of } K_i(E') \\ & \text{such that } \{e_i\} \cup C \in \mathcal{C} \\ K_{i-1}(E') \cup \{e_i\} & \text{else.} \end{cases} \tag{17.1}$$

Moreover, $K_n(E') = \mathcal{F}_{\mathcal{H}}(E')$ where $\mathcal{F}_{\mathcal{H}}$ is a comonotone function defined as in Proposition 11.2 with $\mathcal{H}_{e_i} := \{C \subseteq \{e_j : 1 \leq j < i\} : \{e_i\} \cup C \in \mathcal{C}\}$. Finally, if c is injective then the minimum cost spanning set is unique. □

For matroids $\mathcal{M}_1 = (E, \mathcal{C}_1)$ and $\mathcal{M}_2 = (E, \mathcal{C}_2)$ and cost functions $c_1, c_2 : E \to \mathbb{R}$, we say that (E_1, E_2) is an *$\mathcal{M}_1\mathcal{M}_2$-stable pair of E* if $E_1 \cup E_2 = E$ and $E_1 \cap E_2$ is a minimum c_i-cost spanning set of E_i in $\mathcal{M}_i$ for $i \in \{1, 2\}$. We call subset K of E an *$\mathcal{M}_1\mathcal{M}_2$-kernel* if it is a common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$ and if for every $e \in E \setminus K$ there is an $i \in \{1, 2\}$ and a subset C_e of K such that $\{e\} \cup C_e \in \mathcal{C}_i$ and $c_i(c) \leq c_i(e)$ for every $c \in C_e$. Set K is called a *dual $\mathcal{M}_1\mathcal{M}_2$-kernel* if it spans both $\mathcal{M}_1$ and $\mathcal{M}_2$ and for every element k of K there exists an $i \in \{1, 2\}$ and a subset C^*_k of $E \setminus K$ such that $C^*_k \cup \{k\}$ is a cocircuit of $\mathcal{M}_i$ with $c_i(k) \leq c_i(c)$ for all $c \in C_k$. Observe that if $\mathcal{M}_1 = \mathcal{M}_2$ and $c_1 = c_2$ then both an $\mathcal{M}_1\mathcal{M}_2$-kernel and a dual $\mathcal{M}_1\mathcal{M}_2$-kernel is a minimum cost basis of $\mathcal{M}$, so it can be constructed with the above greedy algorithm as $K_n(E)$. In this sense, we can regard matroid kernels and dual kernels as generalizations of minimum cost spanning sets.

Theorem 17.3. *Let $\mathcal{M}_1 = (E, \mathcal{C}_1)$ and $\mathcal{M}_2 = (E, \mathcal{C}_2)$ be matroids and $c_1, c_2 : E \to \mathbb{R}_+$ be cost functions on their common groundset. Then there is an $\mathcal{M}_1\mathcal{M}_2$-stable pair (E_1, E_2) of E and an $\mathcal{M}_1\mathcal{M}_2$-kernel K.*

Proof. For $A \subseteq E$ let $\mathcal{F}(A)$ be the minimum c_1-cost $\mathcal{M}_1$-spanning set $K_n(A)$ of A, constructed according to (17.1), and $\mathcal{G}(A)$ be the similarly constructed minimum c_2-cost $\mathcal{M}_2$-spanning set of A. From Fact 17.2, $\mathcal{F}$ and $\mathcal{G}$ are comonotone. So by Theorem 11.3 we have subsets E_1 and E_2 of E such that (E_1, E_2) is an $\mathcal{F}\mathcal{G}$-stable pair. Define $K := E_1 \cap E_2$. By the mincost spanning property, for each $i \in \{1, 2\}$ and for each $e \in E_i \setminus K$, there exists a subset C_e of K such that $\{e\} \cup C_e \in \mathcal{C}_i$ and $c_i(e) \geq c_i(x)$ if $x \in C_e$. As minimum cost spanning sets are independent, K is indeed an $\mathcal{M}_1\mathcal{M}_2$-kernel. □

Next we prove the Kundu-Lawler theorem (Theorem 17.1). By Theorem 17.3, there is a $\mathcal{M}_1\mathcal{M}_2$-kernel I corresponding to some $\mathcal{M}_1\mathcal{M}_2$-stable pair (A, B) for cost-functions

$c_1 := \chi^{E\setminus I_2}$ and $c_2 := \chi^{E\setminus I_1}$. As I_1 is independent in $\mathcal{M}_2$, the 0-cost elements of $\mathcal{M}_2$ cannot span any element of $I_1 \cap (B \setminus A)$. Thus $I_1 \subseteq A \subseteq \mathrm{span}_{\mathcal{M}_1}(I)$, and by symmetry $I_2 \subseteq B \subseteq \mathrm{span}_{\mathcal{M}_2}(I)$. Theorem 17.1 follows.

As another application of Theorem 17.3, we prove the existence of a dual $\mathcal{M}_1\mathcal{M}_2$-kernel for two matroids on the same groundset.

Theorem 17.4. *Let $\mathcal{M}_1 = (E, \mathcal{C}_1)$ and $\mathcal{M}_2 = (E, \mathcal{C}_2)$ be matroids and $c_1, c_2 : E \to \mathbb{R}_+$ be cost functions on their common groundset. Then there is a dual $\mathcal{M}_1\mathcal{M}_2$-kernel K.*

Proof. Let K^* be a $\mathcal{M}_1^*\mathcal{M}_2^*$-kernel with respect to cost functions $M - c_1$ and $M - c_2$, where we choose constant function $M > 0$ such that $M - c_i \geq 0$ for $i \in \{1, 2\}$. Define $K := E \setminus K^*$. As K^* is independent in both $\mathcal{M}_1^*$ and $\mathcal{M}_2^*$, K spans both $\mathcal{M}_1$ and $\mathcal{M}_2$. The kernel property of K^* implies the dual kernel property of K. $\square$

Next we deduce the stable matching theorem (Theorem 12.3) as a special case of Theorem 17.3, by applying it to partition matroids defined by the stars in one colour class of the bipartite graph. The stable b-matching theorem (Theorem 12.5) can be proved similarly by applying Theorem 17.3 to the direct sum of uniform matroids.

The uniform matroid of rank k on n element is $U_{n,k} = ([n], \mathcal{B}(U_{n,k}))$, where the set of bases $\mathcal{B}(U_{n,k}) := \binom{[n]}{k}$ is the set of k-element subsets of the groundset. Matroid $\mathcal{M}$ is a direct sum of matroids $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$ if $\mathcal{M}$ is the union of matroids $\mathcal{M}_1, \mathcal{M}_2 \dots, \mathcal{M}_k$, and these matroids have pairwise disjoint groundsets.

To prove the stable marriage theorem (Theorem 12.3) of Gale and Shapley for bipartite graph B with colour-classes X and Y, consider partition matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ that are the direct sums of uniform matroids of rank 1 on the stars of vertices in X and in Y, respectively. Define cost functions c_1 and c_2 on $E(B)$ such that for edge $e = xy$ ($x \in X, y \in Y$), $c_1(e)$ is the height of e in linearly ordered set $(D(x), \prec_x)$ and $c_2(e)$ is the height of e in linearly ordered set $(D(y), \prec_y)$. If we apply Theorem 17.3 to these matroids and weights, we arrive back to Theorem 12.3. If we keep the above weights c_1 and c_2, but in the direct sum in the definition of $\mathcal{M}_i$ we use a uniform matroid of rank $b(v)$ (instead of rank 1) on star $D(v)$ of vertex v of $V(B)$, then we obtain the stable b-matching theorem (Theorem 12.5).

18 Fractional kernels and Scarf's lemma

In this section we elaborate on the connection between Scarf's lemma (Lemma 18.6, see also [91]), the Boros-Gurvich theorem (Theorem 18.7, see also [8]) and special comonotone fractional kernels. The new results of this section are outcomes of a joint work with Ron Aharoni.

We approach Scarf's lemma from simplicial complices. An (*abstract*) *simplicial complex* is a family $\mathcal{C}$ of subsets of a finite groundset X with the property that $S' \subseteq S \in \mathcal{C}$ implies $S' \in \mathcal{C}$. By a *topological simplicial complex* we mean a simplicial complex $\mathcal{C}$ in which the inclusionwise maximal members have the same cardinality k and any member of $\mathcal{C}$ of size $k-1$ is contained in an even number of maximal members of $\mathcal{C}$. A *dual* (topological) simplicial complex is the family $\mathcal{C}^*$ of complements of a (topological) simplicial complex $\mathcal{C}$ on groundset X, that is $\mathcal{C}^* = \{Y \subset X : X \setminus Y \in \mathcal{C}\}$. A most natural example of a topological simplicial complex is the family of vertex sets of simplices of a triangulation of a smooth manifold. Here the 'even' number in the definition is always 2. It is also not very difficult to see that for subset X of $\mathbb{R}^n$ and for vector $x \in \mathbb{R}^n$ family $\{Y \subseteq X : x \in \mathrm{conv}(Y)\}$ is a dual topological simplicial complex. Here too, the 'even' number in the definition is always 2.

The following observation (communicated by Ron Aharoni) turns out to be extremely useful.

Lemma 18.1. *If $\mathcal{C}'$ is a topological simplicial complex and $\mathcal{C}^*$ is a dual topological simplicial complex on the same groundset X then the number of common members of $\mathcal{C}'$ and $\mathcal{C}^*$ that are inclusionwise maximal in $\mathcal{C}'$ and inclusionwise minimal in $\mathcal{C}^*$ is even.*

Proof. By definition, the inclusionwise maximal members of $\mathcal{C}'$ and the inclusionwise minimal members of $\mathcal{C}^*$ have the same size k and l, respectively. If $k \neq l$, then the lemma is trivial as 0 is an even number. So we may assume that $k = l$.

Fix an element x of groundset X and consider families

$$\mathcal{A} := \{Y \in \binom{X}{k} : x \in Y \in \mathcal{C}^*\} \qquad \text{and} \qquad \mathcal{B} := \{Y \in \binom{X}{k} : x \notin Y \in \mathcal{C}'\}.$$

Define bipartite graph G with colour classes $\mathcal{A}$ and $\mathcal{B}$ by $AB \in E(G)$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$ if $A \setminus B = \{x\}$. By the definition of topological simplicial complices, the degree of vertex Y of G is odd if and only if Y is a common element of $\mathcal{C}'$ and $\mathcal{C}^*$. As the degree sum of the vertices is twice the number of edges, the number of odd degree vertices of G is even. $\square$

We remark that if topological simplical complices $\mathcal{C}$ and $\mathcal{C}'$ in Lemma 18.1 have the extra property that the 'even' number in the definition of topological simplical complex is always 2 then graph G in the above proof has maximum degree 2, that is G is a union of disjoint paths. This means that common maximal members of $\mathcal{C}'$ and $\mathcal{C}^*$ are the end vertices of the nontrivial paths building up G. That is, if we can compute $E(G)$ efficiently (i.e. we can efficiently find the maximal members of $\mathcal{C}'$ that contain an almost maximal one, and the minimal members of $\mathcal{C}^*$ that are contained in an almost minimal one), then starting from a common member Y, of $\mathcal{C}'$ and $\mathcal{C}^*$ we can find another one, simply by following the path of G starting at Y until it ends. This algorithmic method is the basis of Scarf's proof for Lemma 18.6 (see [91]).

As a short detour we recall some connections between the above Lemma 18.1, the well-known Sperner's lemma and Brouwer's topological fixed point theorem. Define $\Delta_n := \{x \in \mathbb{R}^{n+1}_+ : \mathbf{1}^T x = 1\}$ the unit n-dimensional simplex in the $(n+1)$-dimensional space. We assume that the notion of triangulation is well-known by the reader.

Lemma 18.2 (Sperner). *Consider a triangulation of Δ_n and colour each vertex v of the triangulation with a coordinate in* supp v. *Then there is a simplex in the triangulation that receives all the n colours on its vertices.*

A strengthening of this result is that the number of such simplices is odd. Or, equivalently, the following is true.

Lemma 18.3. *If we colour the vertices of a triangulation of the n-dimensional sphere with n colours then the number of simplices that have all n colours assigned to its vertices is even.*

A generalization of the above lemma is the following.

Lemma 18.4 (Shapley). *If we assign to each vertex of a triangulation of the n-dimensional sphere a vector in $\mathbb{R}^m$ in such a way that no $m-1$ of the assigned vectors lie in the same hyperplane, then the number of simplices whose assigned vectors contain $\mathbf{0}$ in their convex hull is even.*

To reduce Lemma 18.3 to Lemma 18.4, let $m = n-1$ and instead of colouring, assign the $n-1$ unit vectors together with $-\mathbf{1} \in \mathbb{R}^{n-1}$ to the vertices of the triangulation. This proves Lemma 18.3.

On the other hand, Lemma 18.4 is a straightforward application of Lemma 18.1 to the topological simplicial complex $\mathcal{C}'$ defined by the vertex sets of simplices of the triangulation, and the dual topological simplicial complex $\mathcal{C}^*$ is given by the vertex sets of simplices that have assigned vectors containing $\mathbf{0}$ in their convex hull. Observe that in the above argument, we did not use that the triangulation was of the sphere; in fact, Lemma 18.4 also holds for any triangulation of any smooth manifold.

It is also worthwhile to mention that Sperner's lemma is equivalent with the following topological fixed point theorem of Brouwer.

Theorem 18.5 (Brouwer). *Any continuous map from the unit ball to itself has a fixed point.*

To deduce Brouwer from Sperner, it is enough to prove that any continuous map $\phi : \Delta_n \to \Delta_n$ has a fixed point. Let ϕ be such a map. We can choose a colour from $[n]$ for each vector x of Δ_n such that if colour i is assigned to vector x then $x_i \geq \phi(x)_i$ (where x_i denotes the ith coordinate of x). Further, we may assume that all vectors receive a colour that correspond to a coordinate in its support. By Lemma 18.2, any triangulation of Δ_n has a simplex with all n colours assigned to its vertices. By further triangulating such a simplex we can construct an infinite chain of simplices intersecting in a unique point. By continuity, this must be a fixed point of ϕ.

For the other direction, define map $\phi : \Delta_n \to \Delta_n$ the following way. For a vertex v of the triangulation with colour i let $\phi(v)$ be the $(i+1)$st (mod n) unit vector. Extend ϕ linearly on each simplex of the triangulation. This extension is a continuous map. By Theorem 18.5, there is a fixed point x of ϕ, so the simplex containing x must be one with all n colours assigned to its vertices.

After this detour we return to our main topic and prove Scarf's lemma and some corollaries on fractional kernels.

Lemma 18.6 (Scarf [91]). *Let $n < m$ be positive integers, $b \in \mathbb{R}^n_+$ and matrices $B = (b_{i,j}), C = (c_{i,j})$ be of size $n \times m$ with the following property. The first n columns of B form an $n \times n$ identity matrix (i.e. $b_{i,j} = \delta_{i,j}$ for $i, j \in [n]$) and $c_{i,i} \geq c_{i,k} \geq c_{i,j}$ for any $i \in [n]$, $i \neq j \in [n]$ and $k \in [m] \setminus [n]$.*

Then there is a subset J of $[m]$ such that the columns of B indexed by J span a cone containing vector b (we say that J spans b in B) and the columns of C indexed by J forms a dominating set, that is, for any column $i \in [m]$ there is a row $k \in [n]$ of C such that $c_{k,i} \geq c_{k,j}$ for any $j \in J$.

We sketch the proof of the above lemma; the interested reader can consult the paper of Aharoni and Holzman [1] for further details.

Proof. By a general position argument, one can perturb B and b, so that no $n-1$ elements of $[m]$ span the perturbed b in the perturbed B so that if some n elements span the perturbed b in the perturbed B then it also does it in the original setting. Also, we may assume that all entries in the same row of C are different.

If, in this situation, a set K of size $n+1$ spans b in B then there are exactly two n-tuples in K that span b in B. This means that the family of subsets of $[m]$ that can be extended to a set of size n that span b in B is a dual topological simplical complex $\mathcal{C}^*$ on $[m]$.

One can check that for any dominating set of columns of C of size $n-1$ there are exactly two ways to extend that set by one extra column to get a dominating set of columns or to get the first n columns. In other words, the family of subsets of $[m]$ that define a dominating sets of columns of C together with $[n]$ forms a topological simplicial complex $\mathcal{C}'$ on $[m]$.

As $[n]$ is a common element of $\mathcal{C}'$ and $\mathcal{C}^*$, by Lemma 18.1, there is an odd number of subsets J of $[m]$ of size n that defines a dominating set of columns in C and determines a set of columns of B that span a cone that contains b. In particular, there is at least one such J. □

Based on Lemma 18.6, Aharoni and Holzman gave a simple proof for the following direction of Conjecture 9.5 of Berge and Duchet.

Theorem 18.7 (Boros-Gurvich [8]). *If G is a perfect graph then any normal orientation D of G has a kernel.*

To prove Theorem 18.7, Aharoni and Holzman showed the following result.

Theorem 18.8 (Aharoni-Holzman [1]). *If G is a graph then any normal orientation D of G has a strong fractional kernel.*

A *strong fractional kernel* of digraph D is a nonnegative vector x on $V(D)$ with the following property. Vector x is *fractionally independent*, that is $\sum_{c\in V(C)} x(c) \leq 1$ for any clique C of the underlying undirected graph and x is also *fractionally dominating*, i.e. for any vertex v of D there is a clique C of D containing v such that $vc \in A(D)$ for any other vertex c of C than v and $\sum_{c\in V(C)} x(c) = 1$.

Proof. Introduce for each inclusionwise maximal clique K of G a new vertex v_K, and define digraph D' by adding these new vertices to D together with arcs $v_K v$, for $v \in V(K)$. Clearly, D' is normal. Let B be the adjacency matrix of the underlying undirected graph of D' after deleting the rows corresponding to $V(G)$. Let, moreover b be the all-one vector of the same size as the columns of B have.

To define matrix $C = (c_{K,v})$, fix for each inclusionwise maximal clique K of G a linear order $\prec_K$ on $V(K) \cup \{v_K\}$ such that $x \prec_K y$ implies $yx \in A(D')$. Because of the normality of D', this is possible. For inclusionwise maximal clique K and vertex v of D', let $c_{K,v}$ be the height of v in $\prec_K$ if v is in the groundset of $\prec_K$ (i.e. $v \in K$ or $v = v_K$), otherwise let $c_{K,v} = 0$.

By Lemma 18.6 of Scarf, there is a nonnegative vector x on the vertices of D' such that $Bx = \mathbf{1}$ (that is, $x(v_K) + \sum_{v\in K} x(v) = 1$ for any inclusionwise maximal clique of G), and for any vertex v of G there is an inclusionwise maximal clique K of G such that $x(v)+\sum\{x(u) : u \in K, u \prec_K v\} = 1$, that is, $x(v)+\sum\{x(u) : u \in K, vu \in A(D)\} = 1$. In other words, the restriction of x to $V(G)$ is a strong fractional kernel of D'. □

To prove Theorem 18.7 from Theorem 18.8, we only have to recall the well-known linear description of the independent set polytope of perfect graphs.

Theorem 18.9 (Chvátal [11]). *If G is a perfect graph then*

$$\operatorname{conv}\{\chi^I : I \subseteq V(G) \textit{ is independent}\} = \{x \in \mathbb{R}^{V(G)} : x \geq 0, x(K) \leq 1 \textit{ for any clique } K \textit{ of } G\}. \quad \square$$

For the proof of Theorem 18.7, it is enough to observe that the strong fractional kernel x provided by Theorem 18.8 is a convex combination of characteristic vectors of independent vertex-sets of G, and by the fractional dominance of x, each independent set in this convex combination must be a kernel of D.

In what follows, by using different constructions, we deduce fractional versions of kernel-type theorems we proved so far. Let G be a finite graph, such that for each vertex v of $V(G)$, a linear order $\prec_v$ is fixed on the edges incident with v. A *strong fractional stable matching* of G is a nonnegative vector x on the edges of G such that x is a *fractional matching* (that is, $\sum_{e\in D(v)} x(e) \leq 1$ for any vertex v of G), and x is *fractionally dominating*, that is for any edge e of G, there is a vertex v adjacent to e with $x(e) + \sum_{f \prec_v e} x(f) = 1$.

Theorem 18.10. *If G is a graph with linear orders $\prec_v$ on $D(v)$ for all vertices v of $V(G)$, then there is a strong fractional stable matching of G.*

Proof. Let l_v be a loop on vertex v of G, and construct graph G' by adding these loops to G for all vertices of G. Extend linear order $\prec_v$ so that l_v is the greatest element in the order. Let B be the $V(G') \times E(G')$ incidence matrix of G', and let vector b be the all-one vector of dimension $|V(G)|$. Define $V(G) \times E(G')$ matrix $C = (c_{v,e})$ by $c_{v,e} = 0$ if v is not incident to e, otherwise $c_{v,e}$ is the height of e in $\prec_v$. According to Lemma 18.6, there is a nonnegative vector x on the edges of G', such that x is both a fractional (perfect) matching and x is fractionally dominating in G'. By the choice of the extension of $\prec_v$ to G', the restriction of x to $E(G)$ is a strong fractional stable matching of G. □

To continue the exercise with Scarf's lemma, for partial orders $\prec_1, \prec_2, \ldots, \prec_k$ on groundset V we define vector x in $\mathbb{R}^X$ a *strong fractional stable antichain* if x is a *fractional antichain* (i.e. $\sum_{c\in C} x(c) \leq 1$ for any chain C of any of the partial orders $\prec_i$) and x is a *fractional lower bound* for any element of V, that is for each element

v of V there is a chain $v \succ_i v_1 \succ_i v_2 \succ_i \dots \succ_i v_l$ of some partial order $\prec_i$ with $x(v) + \sum_{j=1}^{l} x(v_i) = 1$.

Theorem 18.11. *If $\prec_1, \prec_2, \dots, \prec_k$ are partial orders on groundset V then there is a strong fractional stable antichain.*

Proof. Let $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ be the set of inclusionwise maximal chains of partial orders $\prec_1, \prec_2, \dots, \prec_k$ with multiplicity, that is if chain C is maximal in more than one partial order, then C is listed so many times. Let B' be the $V \times \mathcal{C}$ incidence matrix of of those chains in $\mathcal{C}$. Let $B := [I_n, B']$ be obtained by adding an $n \times n$ identity matrix I_n in front of B'.

For $v \in V$ and chain $C_i \in \mathcal{C}$ let c_{v,C_i} be 0 if $v \notin C_i$, otherwise the $\prec_j$-height of v in C_i, if C_i is listed in $\mathcal{C}$ with because of $\prec_j$. Let $V \times \mathcal{C}$ matrix $C' := (c_{v,C})$ (for $v \in V, C \in \mathcal{C}$), and let $C := [I_n, C']$ be obtained by adding an $n \times n$ identity matrix I_n in front of C'.

Applying Lemma 18.6 to the above matrices B, C and vector $b = \mathbf{1}_n$, we get a nonnegative vector $x \in \mathbb{R}^{\mathcal{C} \cup V}$. The restriction of x on $\mathbb{R}^V$ is a strong fractional stable antichain, by definition. □

In what follows we consider a similar model in which we define matroid-kernels. So for matroids $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$ on the same groundset E and for cost functions $c_1, c_2, \dots, c_k : E \to \mathbb{R}_+$ we call a vector $x \in \mathbb{R}_+^E$ a *strong fractional kernel for* $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$ if x is *fractionally independent* (that is, $\sum_{e \in E'} x(e) \leq r_i(E')$ for any subset E' of E and for the rank-function r_i of any of the matroids $\mathcal{M}_i$), and each element of E is *fractionally optimally spanned* in some of the matroids, i.e., for any element e of E, there is a subset E' of E and a matroid $\mathcal{M}_i$, such that $c_i(e) \geq c_i(e')$ for any $e' \in E'$, and $\sum_{e \in E'} x(e) = r_i(E')$.

Theorem 18.12. *If $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$ are matroids on the same groundset E and cost functions $c_1, c_2, \dots, c_k : E \to \mathbb{R}_+$ are given then there is a strong fractional kernel for $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$.*

Proof. We may assume that $c_i(e) \leq 1$ for any $i \in [k]$ and $e \in E$.

We applying Scarf's lemma again, but now with vector b different from $\mathbf{1}$. The rows of matrix B' are indexed by (E', i) pairs, where $E' \subseteq E$ and $1 \leq i \leq k$. The columns of B' correspond to E, and the entry at position $(E', i), e$ is 1 if $e \in E'$, otherwise it is 0. Let $B := [I, B']$, by adding an identity matrix of appropriate size in front of B'. Define vector b of the same size as the columns of B by $b_{(E',i)} := r_i(E')$.

Matrix C' has the same size as B', and its entry at position $(E', i), e$ is $c_i(e)$ if $e \in E'$, otherwise it is 0. Just like in case of the above B, we append an identity matrix of appropriate size as the first columns to C' to get matrix C.

The reader is so experienced already in applying Scarf's lemma, that (s)he can check without any further argument that the restriction to $\mathbb{R}^E$ of vector x provided by Lemma 18.6 for matrices B, C and vector b, is indeed a strong fractional matroid-kernel, as claimed. □

We have seen that by some polyhedral argument the existence of a strong fractional graph-kernel implies the existence of an integral kernel. We can do the same in our setting, that is we show an alternative proof for the existence of stable matchings and matroid-kernels using the above fractional results.

So if graph G in Theorem 18.10 is bipartite, then by Theorem 5.12 fractional stable matching x given by Theorem 18.10 is a convex combination of matchings: $x = \sum \lambda_i \chi^{M_i}$. Because x is fractionally dominating, all vectors χ^{M_i} must also be fractionally dominating, that is, all those M_i's are stable matchings. In case of the more general matroid-kernel model, if $k = 2$ in Theorem 18.12, then by Theorem 5.14 fractional matroid-kernel x given by Theorem 18.12 is a convex combination of common independent sets: $x = \sum \lambda_i \chi^{K_i}$. From the optimal spanning property of x we see that all vectors χ^{K_i} must have the same property, in other words all those K_i's must be matroid-kernels.

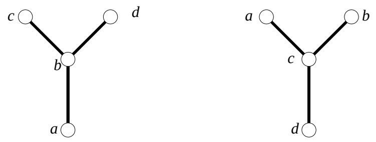

Figure 18.1: The diagram of two partial orders for a counterexample.

In contrast to the above argument, there is no similar proof for the existence of a stable antichain using Theorem 18.11. Namely, it can happen that in case of two partial orders $\prec_1$ and $\prec_2$ on the same groundset, a strong fractional stable antichain is not a convex combination of stable antichains. Figure 18.1 shows the diagram of two partial orders on four elements. As any two elements of the common groundset are comparable in one of the partial orders, a stable antichain contains exactly one element. However, it is easy to check that the all-$\frac{1}{3}$ vector is a strong fractional antichain of weight $\frac{4}{3}$.

19 The kernel lattice

In what follows, we focus on two well-studied aspects of the stable matching problem: first, we show a generalization of the so called 'lattice structure' of stable matchings, and then, in Section 20, we deduce from it a linear description of certain kernel polyhedra. By this, we characterize among others the matroid-generalization of the stable matching polytope described by Vande Vate [99] and Rothblum [89] (see Theorem 9.10).

To handle the lattice property, we go back to our general model, where we worked with set-functions. For a finite groundset X, we call a function $f : 2^X \to 2^X$ *strongly monotone* if f is monotone and $|f|$ has the subcardinal property (3.10) of rank functions of matroids. Recall, that $|f|$ is subcardinal if

$$|f(A \cup \{x\})| \leq |f(A)| + 1 \tag{19.1}$$

for any subset A and element x of X. Property (19.1) implies

$$|f(B) \setminus f(A)| \leq |B \setminus A| \tag{19.2}$$

for any $A \subseteq B \subseteq X$. Function f is *increasing* if

$$A \subseteq B \subseteq X \text{ implies } |f(A)| \leq |f(B)| \ . \tag{19.3}$$

Note that if comonotone function $\mathcal{F} = K_n$ is coming from Fact 17.2 then $|\mathcal{F}(A)| = \text{rank}(A)$, hence $\mathcal{F}$ is increasing. Comonotone functions coming from arborescence type partial orders in Observation 12.1 are also increasing, just like the ones in Theorem 12.4. We shall exhibit a link between the sublattice structure of fixed points, strongly monotone functions and increasing comonotone functions.

First we give a sufficient condition for a monotone function on subsetlattices so that the lattice subset of its fixed points is a sublattice. (Thus for subsetlattices lattice subsets are sublattices. A second jawbreaker, after symmetric semimetrics, as so far we did not use common and comonotone after one another.)

Theorem 19.1. *If $f : 2^X \to 2^X$ is a strongly monotone function for a finite set X, then the fixed points form a nonempty sublattice of $(2^X, \cap, \cup)$.*

Proof. Assume that $f(A) = A$ and $f(B) = B$. By monotonicity, $A \cap B = f(A) \cap f(B) \supseteq f(A \cap B)$ and $A \cup B = f(A) \cup f(B) \subseteq f(A \cup B)$. By property (19.2), $|A \setminus (A \cap B)| \geq |f(A) \setminus f(A \cap B)| \geq |A \setminus A \cap B|$ and $|(A \cup B) \setminus A| \geq |f(A \cup B) \setminus f(A)| \geq |(A \cup B) \setminus A|$, hence there must be equality throughout. In particular, $f(A \cap B) = A \cap B$ and $f(A \cup B) = A \cup B$. $\square$

Knowing this theorem, we can fulfill our promise and return to the remark of Subramanian in [96] about the lattice structure of stable configurations of monotone scatter-free networks (cf. page 76). Recall that a network is monotone, if the NOT gate cannot be simulated with it. It means that by setting some input from 0 to 1, the only thing that can happen is that some outputs change from 0 to 1. With the help of a network N, we can define a setfunction on the set of arcs A of N. Namely, $f_N(A') = A''$ means that A'' is the set of arcs with output 1 of the gates of N if the set of arcs with 1-inputs is A'. A monotone network N determines a monotone setfunction f_N. As we have seen, scatter-freeness of N implies the adjacency-preserving property, which in turn yields strong monotonicity of f_N. By Theorem 19.1, the fixed points of a strongly monotone function form a sublattice, and this means for the network that the stable configurations have the lattice-property for the arcwise AND and OR operations.

The following link between strongly monotone and increasing comonotone functions is crucial for the lattice property of $\mathcal{FG}$-kernels.

Lemma 19.2. *If function $\mathcal{F} : 2^X \to 2^X$ is increasing and comonotone then $\overline{\mathcal{F}}$ is strongly monotone.*

Proof. We have seen in (11.1) that function $\overline{\mathcal{F}}$ is monotone. If $A \subseteq B$ then

$$\begin{aligned} |\overline{\mathcal{F}}(B) \setminus \overline{\mathcal{F}}(A)| = |\overline{\mathcal{F}}(B)| - |\overline{\mathcal{F}}(A)| = |B \setminus \mathcal{F}(B)| - |A \setminus \mathcal{F}(A)| = \\ |B| - |\mathcal{F}(B)| - |A| + |\mathcal{F}(A)| \leq |B| - |A| = |B \setminus A|. \end{aligned}$$

Choice $B = A \cup \{x\}$ implies property (19.1) of $\overline{\mathcal{F}}$. $\square$

Based on Lemma 19.2, we give a sufficient condition for the stable pairs to form a sublattice in Theorem 11.3. Recall that on $2^X \times 2^X$ we have defined a lattice order $\preceq$ by (11.5) and lattice operations $\wedge$ and $\vee$ by (11.6).

Theorem 19.3. *If X is a finite groundset, $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ are increasing comonotone functions then $\mathcal{FG}$-stable pairs form a nonempty, complete sublattice of $(2^X \times 2^X, \wedge, \vee)$.*

Proof. We use the construction in the proof of Theorem 11.3. There we saw that $\mathcal{FG}$-stable pairs are exactly the fixed points of $f(A,B) := (X \setminus \overline{\mathcal{G}}(B), X \setminus \overline{\mathcal{F}}(A))$, defined in (11.7). It means that (A,B) is $\mathcal{FG}$-stable if and only if $B = X \setminus \overline{\mathcal{F}}(A)$ and $A = f'(A) := X \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(A))$. Hence it is enough to prove that the fixed points of f' form a nonempty sublattice of $(2^X, \cap, \cup)$.

If $A \subseteq B \subseteq X$, then $\overline{\mathcal{F}}(A) \subseteq \overline{\mathcal{F}}(B)$ by monotonicity of $\overline{\mathcal{F}}$. Hence $X \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(A)) \subseteq X \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(B))$, by monotonicity of $\overline{\mathcal{G}}$. So f' is monotone.

From the subcardinal property (19.2) of $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$

$$\begin{aligned}
|f'(B) \setminus f'(A)| &= |[X \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(B))] \setminus [X \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(A))]| = \\
&= |\overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(A)) \setminus \overline{\mathcal{G}}(X \setminus \overline{\mathcal{F}}(B))| \leq \\
&\leq |[X \setminus \overline{\mathcal{F}}(A)] \setminus [X \setminus \overline{\mathcal{F}}(B)]| = \\
&= |\overline{\mathcal{F}}(B) \setminus \overline{\mathcal{F}}(A)| \leq |B \setminus A|.
\end{aligned}$$

Choosing $B = A \cup \{x\}$, we see that f' is strongly monotone, and that its fixed points form a nonempty, complete sublattice of $(2^X, \cap, \cup)$. That is, $\mathcal{FG}$-stable pairs determine a nonempty, complete sublattice of $(2^X \times 2^X, \preceq)$. □

By Theorem 19.3, we can define a partial order on $\mathcal{FG}$-kernels. Namely, we say that $K_1 \preceq_{\mathcal{FG}} K_2$ if there are $\mathcal{FG}$-stable pairs $(A_1, B_1) \preceq (A_2, B_2)$ corresponding to $\mathcal{FG}$-kernels K_1 and K_2. From (11.3), it is easy to check that $\prec_{\mathcal{FG}}$ is indeed a partial order. Using this notion, Theorem 19.3 is equivalent with saying that if $\mathcal{F}$ and $\mathcal{G}$ are increasing comonotone functions, then $\prec_{\mathcal{FG}}$ defines a lattice $L_{\mathcal{FG}}$ on $\mathcal{FG}$-kernels. The lattice operations of $L_{\mathcal{FG}}$ are given by $K_1 \vee_{\mathcal{FG}} K_2 := (A_1 \cup A_2) \cap (B_1 \cap B_2)$ and $K_1 \wedge_{\mathcal{FG}} K_2 := (A_1 \cap A_2) \cap (B_1 \cup B_2)$. Observe, that K is an $\mathcal{FG}$-kernel if and only if it is a $\mathcal{GF}$-kernel, and note that $\prec_{\mathcal{GF}} = \prec_{\mathcal{FG}}^{-1}$. It also means that for $\mathcal{FG}$-kernels K_1, K_2 we have $K_1 \vee_{\mathcal{FG}} K_2 = K_1 \wedge_{\mathcal{GF}} K_2$. In what follows, we might omit the subscript in the lattice operations or in the partial order when it does not cause ambiguity and clearly comes from $\mathcal{FG}$-stability.

Corollary 19.4. *Let $\mathcal{F}$ and $\mathcal{G}$ be increasing comonotone functions. If $K_1, K_2, \ldots, K_n$ are $\mathcal{FG}$-kernels then $|K_i| = |K_j|$ and $\bigvee_{i \in [n]} K_i = \mathcal{F}(\bigcup_{i \in [n]} K_i)$ and $\bigwedge_{i \in [n]} K_i = \mathcal{G}(\bigcup_{i \in [n]} K_i)$.*

Proof. Assume that $\mathcal{FG}$-stable pairs (A_i, B_i) and (A_j, B_j) correspond to $\mathcal{FG}$-kernels K_i and K_j, respectively. From the increasing property (19.3) of $\mathcal{F}$ and $\mathcal{G}$, we get that $|K_i| = |\mathcal{F}(A_i)| \leq |\mathcal{F}(A_i \cup A_j)| = |K_i \vee K_j| = \mathcal{F}(B_i \cap B_j)| \leq |\mathcal{G}(B_i)| = |K_i|$. Hence $|K_i| = |K_i \vee K_j| = |K_j|$.

Let $A := \bigcup_{i \in [n]} A_i$ and $K := \bigcup_{i \in [n]} K_i$. Clearly, $\bigvee_{i \in [n]} K_i \subseteq K \subseteq A$. Property (11.3) of $\mathcal{F}$ yields $\bigvee_{i \in [n]} K_i = \mathcal{F}(A) \cap K \subseteq \mathcal{F}(K)$. On the other hand, the increasing property of $\mathcal{F}$ implies that $|\mathcal{F}(K)| \leq |\mathcal{F}(A)| = |\bigvee_{i\in[n]} K_i|$. Thus $\mathcal{F}(\bigcup_{i\in[n]} K_i) = \bigvee_{i\in[n]} K_i$. Similarly it follows that $\mathcal{G}(\bigcup_{i\in[n]} K_i) = \bigwedge_{i\in[n]} K_i$. $\square$

From now on, we use k to denote the common size of $\mathcal{FG}$-kernels for increasing comonotone functions $\mathcal{F}$ and $\mathcal{G}$. Theorem 19.3 implies the following observation on matroids:

Corollary 19.5. *If $\mathcal{M}_1, \mathcal{M}_2$ are matroids on the same groundset, c_1, c_2 are injective cost functions, and K_1, K_2 are $\mathcal{M}_1\mathcal{M}_2$-kernels, then* $\mathrm{span}_{\mathcal{M}_i}(K_1) = \mathrm{span}_{\mathcal{M}_i}(K_2)$ *for $i \in \{1, 2\}$.*

Proof. Choose $\mathcal{M}_1\mathcal{M}_2$-stable pairs (A_1, B_1) and (A_2, B_2) such that $K_i = A_i \cap B_i$ for $i \in \{1, 2\}$. By Theorem 19.3, $K_1 \vee K_2 = (A_1 \cup A_2) \cap (B_1 \cap B_2)$ is both a minimal c_1-cost independent set spanning $A_1 \cup A_2$ and a minimal c_2-cost independent set spanning $B_1 \cap B_2$. Clearly, $\mathrm{span}_{\mathcal{M}_1}(K_1) \subseteq \mathrm{span}_{\mathcal{M}_1}(K_1 \vee K_2) \supseteq \mathrm{span}_{\mathcal{M}_1}(K_2)$ and $\mathrm{span}_{\mathcal{M}_2}(K_1) \supseteq \mathrm{span}_{\mathcal{M}_2}(K_1 \vee K_2) \subseteq \mathrm{span}_{\mathcal{M}_2}(K_2)$. From $|K_1| = |K_1 \vee K_2| = |K_2|$ there must be equality everywhere. $\square$

Corollary 19.5 explains the phenomenon which is known as the rural hospital syndrome in the problem of assigning medical students to hospitals. The name comes from the experience that 'rural hospitals' (the ones that are less favorable amongst students) often cannot fill their quota, and if it is so, then no matter which stable configuration is used, they receive the very same residents. The explanation of this fact is, that as we remarked earlier, Theorem 12.5 is a special case of Theorem 17.3 when both $\mathcal{M}_1$ and $\mathcal{M}_2$ are direct sums of uniform matroids: in $\mathcal{M}_1$ the partition is defined by the U-stars, in $\mathcal{M}_2$ it consists of the V-stars and the rank of a part is the corresponding b-value. So, if for stable b-matching M we have $b(u) > d_M(u)$, then in the star of u only the edges of M are spanned by M, hence $D(u) \cap M = D(u) \cap M'$ for any stable b-matching M'. Note also that as a special case, we also see that in the original marriage model no matter which stable marriage scheme is chosen, always the same persons get married.

20 Kernel polyhedra

Another corollary of Theorem 19.3 has to do with the so called stable matching polytope. Usually, from an $\mathcal{FG}$-kernel K it is straightforward to construct an $\mathcal{FG}$-stable pair (A, B) with $K = A \cap B$. This is why our primal interest will be the *$\mathcal{FG}$-kernel polytope*

$$\begin{aligned} \mathcal{P}_{\mathcal{K}_{\mathcal{FG}}} &:= \mathrm{conv}\{\chi^K : K \in \mathcal{K}_{\mathcal{FG}}\}\ , \text{ where} \\ \mathcal{K}_{\mathcal{FG}} &:= \{K \subseteq X : K \text{ is an } \mathcal{FG}\text{-kernel}\}. \end{aligned} \tag{20.1}$$

We will find a linear description of polytope $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}$ from the one of its dominant polyhedron $\mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}}$. We shall also examine the *$\mathcal{FG}$-kernel cone*,

$$\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}} := \mathrm{cone}\{\chi^K : K \in \mathcal{K}_{\mathcal{FG}}\} = \{\sum_{i\in I} \lambda_i \chi^{K_i} : \lambda_i \geq 0, K_i \in \mathcal{K}_{\mathcal{FG}}\}$$

When we want to optimize the $\mathcal{FG}$-kernel then it is handy to have a linear description of $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}$. The result of Vande Vate [99] and its extension by Rothblum [89] exhibits this in case of the stable matching model (see Theorem 9.10). Our aim here is to extend this result to a description of $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}$ in case of increasing comonotone functions $\mathcal{F}$ and $\mathcal{G}$. The following uncrossing lemma shows in particular that $\{\chi^K : K \in \mathcal{K}_{\mathcal{FG}}\}$ is a Hilbert basis of $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$. For a vector $x \in \mathbb{R}^X$, supp $x := \{e \in X : x(e) \neq 0\}$ denotes the support of x.

Lemma 20.1. *If $\mathcal{F}$, $\mathcal{G}$ are increasing comonotone functions and K, L are $\mathcal{FG}$-kernels then $\chi^K + \chi^L = \chi^{K \vee L} + \chi^{K \wedge L}$.*

Proof. If $\mathcal{FG}$-stable pairs $(K_{\mathcal{F}}, K_{\mathcal{G}}), (L_{\mathcal{F}}, L_{\mathcal{G}})$ correspond to K and L, then by Theorem 19.3, $(K_{\mathcal{F}} \cup L_{\mathcal{F}}, K_{\mathcal{G}} \cap L_{\mathcal{G}})$ and $(K_{\mathcal{F}} \cap L_{\mathcal{F}}, K_{\mathcal{G}} \cup L_{\mathcal{G}})$ are also $\mathcal{FG}$-stable pairs that correspond to $K \vee L$ and $K \wedge L$, respectively. From $K_{\mathcal{F}} \cup K_{\mathcal{G}} = X = L_{\mathcal{F}} \cup L_{\mathcal{G}}$, it is easy to check that $\chi^{K_{\mathcal{F}} \cap K_{\mathcal{G}}} + \chi^{L_{\mathcal{F}} \cap L_{\mathcal{G}}} = \chi^{(K_{\mathcal{F}} \cup L_{\mathcal{F}}) \cap (K_{\mathcal{G}} \cap L_{\mathcal{G}})} + \chi^{(K_{\mathcal{F}} \cap L_{\mathcal{F}}) \cap (K_{\mathcal{G}} \cup L_{\mathcal{G}})}$. $\square$

Lemma 20.2. *If $\mathcal{F}$, $\mathcal{G}$ are increasing comonotone functions and $x \in \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$, then there is a decomposition $x = \sum_{1 \leq j \leq m} \lambda_j \chi^{K_j}$ such that $\lambda_j \geq 0$ and*

$$K_1 \prec_{\mathcal{FG}} K_2 \prec_{\mathcal{FG}} \ldots \prec_{\mathcal{FG}} K_m \ . \tag{20.2}$$

Proof. The above Lemma 20.1 allows us to transform a positive decomposition $x = \sum_{i=1}^{l} \lambda_i^* \chi^{K_i^*}$ into another one. Namely, we can execute an uncrossing step along $\prec_{\mathcal{FG}}$-uncomparable kernels K_i^* and K_j^*. That is, we decrease the λ^*-coefficients of $\chi^{K_i^*}$ and $\chi^{K_j^*}$ by ϵ and simultaneously we increase also by ϵ the coefficients of $\chi^{K_i^* \vee K_j^*}$ and of $\chi^{K_i^* \wedge K_j^*}$, where $\epsilon := \min\{\lambda_i^*, \lambda_j^*\}$. We have described in Section 6.2 (see Theorem 6.1) how it is possible to transform a decomposition into another one by uncrossing steps such that the decomposition becomes cross-free, that is no uncrossing step can be made any more. But in that situation, any two $\mathcal{FG}$-kernel in the decomposition are $\preceq_{\mathcal{FG}}$-comparable, that is $\preceq_{\mathcal{FG}}$ induces a linear order on them. $\square$

For increasing comonotone functions $\mathcal{F}$ and $\mathcal{G}$, using Corollary 19.4 and Lemma 20.2, we can test membership in $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$. That is, we can efficiently decompose a vector $x \in \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$ as a positive combination of $\mathcal{FG}$-kernels. Namely, let $x \in \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$ with

$$x = \sum_{1 \leq i \leq m} \lambda_i \chi^{K_i}$$

a decomposition provided by Lemma 20.2. Define

$$x_j := \sum_{1 \leq i \leq j} \lambda_i \chi^{K_i}.$$

From Corollary 19.4 and Lemma 20.2,

$$K_j = \bigvee_{i \in [j]} K_i = \mathcal{F}(\bigcup_{i \in [j]} K_i) = \mathcal{F}(\text{supp } x_j).$$

Clearly, $\lambda_j = \min\{x_j(e) : e \in K_j\}$, and

$$x_{j-1} = x_j - \lambda_j \chi^{K_j}. \tag{20.3}$$

So from knowing x_j, we can calculate K_j, λ_j and x_{j-1}. As $x_m = x$, we can find the decomposition by iterating 20.3 until we arrive to $x_0 = 0$.

Observe that for any nonnegative x, iteration 20.3 terminates with some decomposition of x because $|\text{supp } x_{j+1}| < |\text{supp } x_j|$. If in this decomposition there are only $\mathcal{FG}$-kernels, then we have a proof that $x \in \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$. So if nonnegative vector x is outside $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$, then there is an iteration j of 20.3 such that K_j is not an $\mathcal{FG}$-kernel. If this happens, we have a proof that $x \notin \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$.

Hence, to obtain a linear description of $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}} = \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}} \cap \{x \in \mathbb{R}^X : \mathbf{1}^T x = k\}$ (where k is the size of $\mathcal{FG}$-kernels), it is enough to find a set of valid linear constraints for $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$ with the property that least one of them is violated whenever 20.3 fails to find a proper positive combination. The problem with this approach is, that though from iteration 20.3 we always see when a vector is not in $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$, it is not immediate to find a "linear" reason for the failure at a certain iteration j. Thus instead, we shall concentrate on finding a linear description of $\mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}}$, and in turn we are going to describe $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}$ and $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$.

For set-functions $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$, we define the *blocker* of $\mathcal{FG}$-kernels by

$$\mathcal{B}_{\mathcal{FG}} := \{B \subseteq X : B \cap K \neq \emptyset \text{ for any } K \in \mathcal{K}_{\mathcal{FG}}\}$$

and the dominant of the blocker polytope by

$$\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}} := \{\chi^B : B \in \mathcal{B}_{\mathcal{FG}}\}^{\uparrow}.$$

Theorem 20.3. *For increasing comonotone functions $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}}$ is described by*

$$\{x \in \mathbb{R}^X : x \geq \mathbf{0} \text{ and } x(K) \geq 1 \text{ for } K \in \mathcal{K}_{\mathcal{FG}}\}. \tag{20.4}$$

Proof. Let P denote the polyhedron determined by (20.4). We shall prove that (20.4) is TDI description of P. Let $c \in \mathbb{Z}^X_+$, and consider the linear programming problem $\min c^T x$ subject to (20.4). The dual of this problem is

$$\max\{y^T \mathbf{1} : y \geq \mathbf{0}, y \in \mathbb{R}^{\mathcal{K}_{\mathcal{FG}}}, \sum_{K \in \mathcal{K}_{\mathcal{FG}}} y(K)\chi^K \leq c\}, \tag{20.5}$$

As all $\mathcal{FG}$-kernels have size k, for $z := \sum_{K \in \mathcal{K}_{\mathcal{FG}}} y(K)\chi^K$ (20.5) can be reformulated as

$$\max\{\frac{1}{k}\mathbf{1}^T z : c \geq z \in \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}\} \tag{20.6}$$

Let z be an optimum of (20.6). From Lemma 20.2, there are $\mathcal{FG}$-kernels $K_1 \prec_{\mathcal{FG}} K_2 \prec_{\mathcal{FG}} \dots \prec_{\mathcal{FG}} K_m$ and positive coefficients λ_i such that $z = \sum_{i \in [m]} \lambda_i \chi^{K_i}$. We may assume that z and j are chosen so that $\lambda_i \in \mathbb{N}$ for any $i > j$ and j is as small as possible. If $j = 0$, then

$$y(K) = \begin{cases} \lambda_i \text{ if } K = K_i \\ 0 \text{ else} \end{cases}$$

is an integer optimum of (20.5).

If $j > 0$ then

$$z' := \sum_{i\in[j-2]} \lambda_i \chi^{K_i} + (\lambda_{j-1} - \varepsilon)\chi^{K_{j-1}} + (\lambda_j + \varepsilon)\chi^{K_j} + \sum_{i\in[m]\setminus[j]} \lambda_i \chi^{K_i}$$

is another optimum of (20.6) if $\varepsilon \leq \lambda_{j-1}$ and there is no integer between λ_j and $\lambda_j + \varepsilon$. (This is because c is integer.) If we choose the maximal possible ε, we get an optimum z with integer coefficients $\lambda'_j, \lambda'_{j+1}, \dots$, contradicting the choice of z and j.

Hence for any integer c, there is an integer optimum y of (20.5). So by Lemma 5.5, P is an integer polyhedron. But all integer vertices of P are of the form χ^B for some $B \in \mathcal{B}_{\mathcal{FG}}$, so $\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}} = \operatorname{conv}\{P \cap \mathbb{Z}^X\} = P$. $\square$

Applying Theorem 5.8 to blocking type polyhedron $\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}}$, we get the linear description of $\mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}}$ and the other kernel-related polyhedra.

Corollary 20.4. *Let $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ increasing comonotone set-functions and let k be the size of $\mathcal{FG}$-kernels. Then*

$$\mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}} = \{x \in \mathbb{R}^X : x \geq \mathbf{0},\ x(B) \geq 1 \ \ \textit{for } B \in \mathcal{B}_{\mathcal{FG}}\}\ , \tag{20.7}$$

$$\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}} = \{x \in \mathbb{R}^X : x \geq \mathbf{0},\ \mathbf{1}^T x \leq k,\ x(B) \geq 1 \ \ \textit{for } B \in \mathcal{B}_{\mathcal{FG}}\}\ , \tag{20.8}$$

$$\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}} = \{x \in \mathbb{R}^X : x \geq \mathbf{0},\ k \cdot x(B) \geq \mathbf{1}^T x \ \textit{for } B \in \mathcal{B}_{\mathcal{FG}}\}\ . \tag{20.9}$$

Proof. By Theorem 5.8, (20.7) is an immediate consequence of Theorem 20.3. As all $\mathcal{FG}$-kernels have the same size k, (20.8) follows.

Clearly, the right hand side of (20.9) describes a cone C that contains $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$. Let $x \geq \mathbf{0}$ be a vector outside $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$, and $\lambda = \frac{1}{\mathbf{1}^T x}$. Then $\mathbf{1}^T(\lambda \cdot x) = k$ and $\lambda \cdot x \notin \mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}}$, hence there is a transversal B of $\mathcal{B}_{\mathcal{FG}}$ such that $\lambda \cdot x(B) < 1$, that is $x(B) < \frac{1}{\lambda} = \mathbf{1}^T x$. Thus $x \notin C$, proving (20.9). $\square$

Knowing these linear descriptions, the next natural question is whether we can efficiently optimize over these polyhedra. By the ellipsoid method, it is enough to solve the separation problem efficiently. As a first step towards this goal, we show how to optimize $\max\{\mathbf{1}^T x' : x' \leq x \text{ and } x' \in \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}\}$ for any nonnegative vector x, based on Lemma 20.2.

For increasing comonotone set-functions $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ and for vector $x \in \mathbb{R}^X_+$ let us define

$$K^{\vee}_x := \bigvee\{K : K \subseteq \operatorname{supp} x,\ K \in \mathcal{K}_{\mathcal{FG}}\},$$
$$K^{\wedge}_x := \bigwedge\{K : K \subseteq \operatorname{supp} x,\ K \in \mathcal{K}_{\mathcal{FG}}\},$$

where we define (for the moment) $\bigvee \emptyset = \bigwedge \emptyset = \emptyset$. By Theorem 19.3, $K^{\vee}_x$ and $K^{\wedge}_x$ are the $\prec_{\mathcal{FG}}$-maximal and the $\prec_{\mathcal{FG}}$-minimal $\mathcal{FG}$-kernels in supp x. Let moreover

$$\begin{aligned} \lambda^{\vee}_x &:= \min\{x(e) : e \in K^{\vee}_x\}, \\ \lambda^{\wedge}_x &:= \min\{x(e) : e \in K^{\wedge}_x\}. \end{aligned} \tag{20.10}$$

Next we define $x^\vee$ and $x^\wedge$ recursively. If $K_x^\vee = \emptyset$ then $x^\vee := 0$, and if $K_x^\wedge = \emptyset$ then $x^\wedge := 0$. Else let

$$\begin{aligned} x^\vee &:= \lambda_x^\vee \chi^{K_x^\vee} + (x - \lambda_x^\vee \chi^{K_x^\vee})^\vee, \\ x^\wedge &:= \lambda_x^\wedge \chi^{K_x^\wedge} + (x - \lambda_x^\wedge \chi^{K_x^\vee})^\wedge. \end{aligned} \tag{20.11}$$

The recursion is proper, as we use the definition only for vectors that have a strictly smaller support than the currently defined has. This recursive definition also provides us with a decompositions to a positive combination of characteristic vectors of $\mathcal{FG}$-kernels:

$$x^\vee = \sum_{i=1}^{m} \lambda_i^\vee \chi^{K_i^\vee} \qquad \text{and} \qquad x^\wedge = \sum_{j=1}^{l} \lambda_j^\wedge \chi^{K_j^\wedge}\ , \tag{20.12}$$

where

$$K_x^\vee = K_1^\vee \succ K_2^\vee \succ \ldots \succ K_m^\vee \text{ and } K_x^\wedge = K_1^\wedge \prec K_2^\wedge \prec \ldots \prec K_l^\wedge.$$

Note that if x is in $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$ then $x = x^\vee = x^\wedge$, and the above decomposition of $x^\vee$ is exactly the one that we have constructed right after Lemma 20.2 on page 96.

One consequence of the next lemma is that $x^\vee$ maximizes $\{\mathbf{1}^T x' : x' \le x \text{ and } \in \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}\}$.

Lemma 20.5. *Let $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ increasing comonotone set-functions and let k be the size of $\mathcal{FG}$-kernels. For any nonnegative vector x of $\mathbb{R}^X$ there exists a transversal $B_x \in \mathcal{B}_{\mathcal{FG}}$ such that $x(B_x) = \frac{1}{k} \cdot \mathbf{1}^T x^\vee$.*

Proof. We proceed by induction on $\operatorname{supp} x$. If $K_x^\vee = \emptyset$ then $B_x := X \setminus \operatorname{supp} x$ suffices. Let x be such that $K_x^\vee \neq \emptyset$ and assume that the lemma is true for x' whenever $\operatorname{supp} x' \subset \operatorname{supp} x$.

Let Y be the set of elements that determine the coefficient $\lambda_m^\vee$ in (20.12) according to (20.10). If $Y \cap K_1^\wedge = \emptyset$, then from $K_1^\wedge \preceq K_m^\vee$ we get that $\mathcal{FG}$-kernel $K_1^\wedge$ is still a subset of the support of $x - x^\vee$, which is a contradiction. So let $e \in Y \cap K_1^\wedge$ and $x' := x - x(e)\chi^e$. As $\operatorname{supp} x' \subset \operatorname{supp} x$, by induction we have a set $B_{x'} \in \mathcal{B}_{\mathcal{FG}}$. We claim that $B_x := B_{x'}$ suffices.

By (11.3) and (20.2) there is an $s < l$ such that $e \in \bigcap_{i=s+1}^m K_i^\vee \setminus \bigcup_{j=1}^s K_j^\vee$. Consider the vector $\bar{x} := x - \sum_{j=1}^s \lambda_j^\vee \chi^{K_j^\vee}$. By definition, $K_{\bar{x}}^\vee = K_{s+1}^\vee \ni e \in K_1^\wedge = K_{\bar{x}}^\wedge$. This means that all $\mathcal{FG}$-kernel in $\operatorname{supp} \bar{x}$ contains e, hence $x'^\vee = \sum_{j=1}^s \lambda_j^\vee \chi^{K_j^\vee}$. As $x(e) = \sum_{j=s+1}^m \lambda_j^\vee = \frac{1}{k} \cdot \mathbf{1}^T \left(\sum_{j=s+1}^m \lambda_j^\vee K_j^\vee \right)$, we get that

$$\begin{aligned} \frac{1}{k}\mathbf{1}^T x^\vee \le x(B_x) &\le x(e) + x'(B_{x'}) = x(e) + \frac{1}{k}\mathbf{1}^T x'^\vee = \\ &= \frac{1}{k} \cdot \mathbf{1}^T \sum_{j=s+1}^{m} \lambda_j^\vee K_j^\vee + \frac{1}{k} \cdot \mathbf{1}^T \sum_{j=1}^{s} \lambda_j^\vee K_j^\vee = \\ &= \frac{1}{k} \cdot \mathbf{1}^T \sum_{j=1}^{m} \lambda_j^\vee K_j^\vee = \frac{1}{k}\mathbf{1}^T x^\vee. \end{aligned}$$

$\square$

The use of the transversal B_x in Lemma 20.5 is that for a nonnegative vector x to decide membership in the particular polyhedron, we only have to check (20.7-20.9) for B_x. This is because if $x \notin \mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}}$, then $x(B_x) = \frac{1}{k}\mathbf{1}^T x^{\vee} < 1$, as $x^{\vee} \notin \mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}}$. If $\mathbf{1}^T x^{\vee} = x(B_x) \geq 1$ for some x with $\mathbf{1}^T x \leq k$, then $x = x^{\vee}$ and $\mathbf{1}^T x = k$, hence $x \in \mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}$. Finally, $x \in \mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$ if and only if $x = x^{\vee}$, that is, $x(B_x) = \mathbf{1}^T x^{\vee} = \frac{1}{k}\mathbf{1}^T x$.

So we can solve the separation (and hence the optimization) problem over $\mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}}$, $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}$ and $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$, if we can efficiently compute transversal B_x for a nonnegative vector x. The proof of Lemma 20.5 provides us with a method to construct B_x, as soon as we can compute $x^{\vee}$ and $x^{\wedge}$. But to construct these vectors according to (20.11), we only have to know how to find $K^{\vee}_x$ and $K^{\wedge}_x$. In what follows we show how to do this.

Let us define monotone function $f : 2^X \to 2^X$ by $f(A) = X \setminus \overline{\mathcal{G}}(X \setminus \mathcal{F}(A))$. Clearly, $K^{\vee}_x = \mathcal{F}(A)$, if A is an inclusionwise maximal fixed set of f such that $\mathcal{F}(A) \subseteq \operatorname{supp} x$; and $K^{\vee}_x = \emptyset$, if such fixed set A does not exist. Our aim is to construct this set A or to decide that it does not exist. As we saw after Theorem 10.1, by the iteration of f starting on X, we can construct the inclusionwise maximal fixed set A_0 of f. A_0 has to contain A, if A exists. Assume that for some $i \in \mathbb{N}$, we have constructed a set A_i in such a way that $A \subseteq A_i$ if A exists. Clearly, $A = f(A) \subseteq f(A_i)$ and by property (11.3) of $\mathcal{F}$, we have $A \cap \mathcal{F}(A_i) \subseteq \mathcal{F}(A) \subseteq \operatorname{supp} x$. Hence

$$A \subseteq A_{i+1} := [A_i \cap f(A_i)] \setminus [\mathcal{F}(A_i) \setminus \operatorname{supp} x].$$

By this definition, we get a decreasing chain $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_i \supseteq \ldots$ which by finiteness stabilizes at some $A_j = A_{j+1}$. It means on one hand that

$$A_j \subseteq f(A_j) = [A_j \setminus \mathcal{F}(A_j)] \cup \mathcal{G}(X \setminus \overline{\mathcal{F}}(A_j)) \tag{20.13}$$

and on the other hand that $\mathcal{F}(A_j) \subseteq \operatorname{supp} x$, as $\mathcal{F}(A_j) \subseteq A_j$ by property (11.2) of $\mathcal{F}$. From the increasing property of $\mathcal{F}$ and $\mathcal{G}$ we see that if fixed element A exists then

$$|\mathcal{F}(A_j)| \geq |\mathcal{F}(A)| = |\mathcal{G}(X \setminus \overline{\mathcal{F}}(A))| \geq |\mathcal{G}(X \setminus \overline{\mathcal{F}}(A_j))|.$$

From here

$$|f(A_j)| = |A_j| - |\mathcal{F}(A_j)| + |\mathcal{G}(X \setminus \overline{\mathcal{F}}(A_j))| \leq |A_j|.$$

By (20.13), this means that $K^{\vee}_x = A_j = f(A_j)$. Otherwise, if fixed element A does not exists, then $A_j \neq f(A_j)$, and $K^{\vee}_x = \emptyset$ follows. So we proved the following theorem.

Theorem 20.6. *Let X be a finite groundset and $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ be increasing comonotone functions described by a value-giving oracle. For any function $c : X \to \mathbb{Z}$ there is an algorithm that constructs an $\mathcal{FG}$-kernel of minimum c-weight and a vector in $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$ with negative c-weight (if it exists) in time polynomial in $|X|$, the size of c and the running time of the oracle.*

Proof. We have seen that there is a polynomial-time separation algorithm over polyhedra $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}, \mathcal{P}^{\uparrow}_{\mathcal{K}_{\mathcal{FG}}}$ and $\mathcal{C}_{\mathcal{K}_{\mathcal{FG}}}$. So by the ellipsoid method (see 5.3), there are polynomial-time optimization algorithms for the above polyhedra. □

Finally, to contrast Theorem 20.4, we prove that it is NP-complete to decide whether a particular element of the groundset can belong to some stable antichain or not. It means that unless P=NP, it is necessary to have some extra assumption (like the increasing property) on the comonotone functions to hope for a good characterization of the corresponding $\mathcal{FG}$-kernel polytope, $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}$. We will use Observation 12.1, the only example of non-increasing comonotone function we have seen so far.

Theorem 20.7. *If undirected graph $G = (V, E)$ and $k \in \mathbb{N}$ are given then it is possible to construct partial orders $\prec$ and $\prec'$ and an element s of their common ground-set X in time polynomial in $|V|$, such that s belongs to a stable antichain of $\prec$ and $\prec'$ if and only if G contains an independent set of size k.*

Proof. We may assume $k \leq |V|$, otherwise the theorem is trivial. Otherwise let

$$X := \{s\} \cup \{a_j, a_j' : 1 \leq j \leq k\} \cup \{v_j, v_j' : v \in V, 1 \leq j \leq k\}.$$

Partial orders $\prec$ and $\prec'$ are determined by

$$a_j \prec s, \qquad u_j \prec v_j', \qquad w_l \prec v_j', \qquad v_j \prec a_j'$$

$$a_j' \prec' s, \qquad u_j' \prec' v_j, \qquad w_l' \prec' v_j, \qquad v_j' \prec' a_j$$

for $1 \leq j \leq k, 1 \leq l \leq k$, $j \neq l$, $u, v, w \in V$, $u \neq v$ and $vw \in E$ or $v = w$.

If G has an independent set $I = \{i^1, i^2, \ldots, i^k\} \subseteq V$ of size k, then $S := \{s\} \cup \{i^j_j, {i^j}'_j : 1 \leq j \leq k\}$ is a stable antichain of $\prec$ and $\prec'$. On the other hand, if s belongs to a stable antichain S then neither a_j, nor a_j' can belong to S. Thus for every j there must exist elements i^j and e^j of V such that $i^j_j, {e^j}'_j \in S$. By stability $i^j = e^j \neq i^l$ and $i^j i^l \notin E$ for $j \neq l$, in other words $I := \{i^1, i^2, \ldots, i^k\}$ is an independent set of G of size k. $\square$

As the decision problem whether there exists an independent set of size k in a graph is NP-complete, it is also NP-complete to solve the kernel-problem in Theorem 20.7. We give another reason why it is NP-complete to optimize kernels. For an undirected graph $G = (V, E)$, function $f : 2^E \to 2^E$, defined by $f(E') = \{e = uv \in E' : D(u) \cup D(v) \subseteq E'\}$ for $E' \subseteq E$ is monotone. Consider comonotone function $\mathcal{F}$, defined by $\mathcal{F}(E') := E' \setminus f(E')$. It is easy to see that $\mathcal{FF}$-stable pairs are $(E(U), E(V \setminus U))$ where $E(U) := \bigcup_{v \in U} D(v)$ (for $U \subseteq V$), and $\mathcal{FF}$-kernels are exactly the edge-sets of the form $D(U)$, for some $U \subseteq V$. If it would be possible to separate over the kernel polytope $P_{\mathcal{FF}}$, then (by the ellipsoid method) it would be possible to find a maximum cut of G in polynomial time. But this latter problem is NP-complete.

21 Lattice polyhedra

In this section, we point out a connection between results in Section 20 to the theory of lattice polyhedra. We shall see that Corollary 20.4 is a straightforward consequence of this theory and the theory of blocking polyhedra. Also, we are going to describe antiblocking type polytope $\mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}}$ for increasing comonotone functions $\mathcal{F}$ and $\mathcal{G}$. In fact, our proof for Theorem 20.3 is simply the specialization to the comonotone setting of the proof of the Hoffman-Schwartz theorem (Theorem 21.1), a basic result in the theory of lattice polyhedra. I hope that our application of that theory to kernel polyhedra helps to recognize the importance of lattice polyhedra in Combinatorial Optimization and Game Theory.

To state the Hoffman-Schwartz theorem, a basic result on lattice polyhedra, we need to formulate some assumptions. Fix a groundset X and a family $\mathcal{L}$ of subsets of X. A partial order $\prec$ on $\mathcal{L}$ is called *consistent* if $A \cap C \subseteq B$ holds for any members A, B, C of $\mathcal{L}$ with $A \prec B \prec C$. Quadruple $(\mathcal{L}, \prec, \wedge, \vee)$ is a *(consistent) quasilattice* if $\prec$ is a (consistent) partial order on $\mathcal{L}$ and $\wedge$ and $\vee$ are binary operations such that $A \wedge B = B \wedge A \prec A$, $A \prec A \vee B = B \vee A$ holds for any members A, B of $\mathcal{L}$ and $A \prec B$ implies $A = A \wedge B$ and $B = A \vee B$. (So if $\prec$ is a lattice order with lattice operations $\wedge, \vee$ then $(\mathcal{L}, \prec, \wedge, \vee)$ is a quasilattice.) Family $\mathcal{L}$ is an *upper clutter* if there is a consistent quasilattice $(\mathcal{L}, \prec, \wedge, \vee)$ such that $(A \wedge B) \cup (A \vee B) \subseteq A \cup B$ holds for any members A, B of $\mathcal{L}$. Family $\mathcal{L}$ is a *lower clutter* if there is a consistent quasilattice $(\mathcal{L}, \prec, \wedge, \vee)$ such that $A \cup B \subseteq (A \wedge B) \cup (A \vee B)$ and $A \cap B \subseteq (A \wedge B) \cap (A \vee B)$ holds for any members A, B of $\mathcal{L}$. (Note that for a quasilattice $(\mathcal{L}, \prec, \wedge, \vee)$ the consistency of $\prec$ implies $(A \wedge B) \cap (A \vee B) \subseteq A \cap B$ for any $A, B \in \mathcal{L}$. Hence it is redundant to require this for upper clutters, and for lower clutters we have $(A \wedge B) \cap (A \vee B) = A \cap B$. This means that $\mathcal{L}$ is an upper clutter if $\chi^A + \chi^B \geq \chi^{A \wedge B} + \chi^{A \vee B}$ holds for any members A, B of $\mathcal{L}$, and $\mathcal{L}$ is a lower clutter if the opposite inequality is true.)

Theorem 21.1 (Hoffman-Schwartz [53]). *Let $(\mathcal{L}, \prec, \wedge, \vee)$ be a consistent quasilattice on groundset X and $d : X \to \mathbb{N} \cup \{\infty\}$ be and arbitrary function. If $\mathcal{L}$ is a lower clutter for this quasilattice and $r : X \to \mathbb{N}$ is submodular then system*

$$\{x \in \mathbb{R}^X : 0 \leq x \leq d,\ x(A) \leq r(A) \text{ for any } A \in \mathcal{L}\}$$

is TDI.

If $\mathcal{L}$ is an upper clutter for the above quasilattice and $r : X \to \mathbb{N}$ is supermodular then system

$$\{x \in \mathbb{R}^X : 0 \leq x \leq d,\ x(A) \geq r(A) \text{ for any } A \in \mathcal{L}\}$$

is TDI. □

(Here, $r : X \to \mathbb{N}$ is *submodular* if $r(A) + r(B) \geq r(A \wedge B) + r(A \vee B)$ holds for any $A, B \in \mathcal{L}$; r is *supermodular* if the reverse inequality is true.)

Next we observe that Theorem 21.1 is relevant in our setting.

Observation 21.2. *If $\mathcal{F}, \mathcal{G} : 2^X \to 2^X$ are increasing comonotone functions then family $\mathcal{K}_{\mathcal{FG}}$ of $\mathcal{FG}$-kernels is an upper and lower clutter.*

Proof. If $A \prec B \prec C$ then there are $\mathcal{FG}$-stable pairs $(A_\mathcal{F}, A_\mathcal{G}), (B_\mathcal{F}, B_\mathcal{G})$ and $(C_\mathcal{F}, C_\mathcal{G})$ with $\mathcal{F}(A_\mathcal{F}) = A = \mathcal{G}(A_\mathcal{G})$, $\mathcal{F}(B_\mathcal{F}) = B = \mathcal{G}(B_\mathcal{G})$, $\mathcal{F}(C_\mathcal{F}) = C = \mathcal{G}(C_\mathcal{G})$ and $A_\mathcal{F} \subset B_\mathcal{F} \subset C_\mathcal{F}$. By (11.2) and (11.3), $A \cap C = \mathcal{F}(A_\mathcal{F}) \cap \mathcal{F}(C_\mathcal{F}) \subseteq A_\mathcal{F} \cap \mathcal{F}(C_\mathcal{F}) \subseteq B_\mathcal{F} \cap \mathcal{F}(C_\mathcal{F}) \subseteq \mathcal{F}(B_\mathcal{F}) = B$, hence lattice order $\prec_{\mathcal{FG}}$ is consistent. By Lemma 20.1, $\mathcal{K}_{\mathcal{FG}}$ is a lower and upper clutter. □

For comonotone functions $\mathcal{F}$ and $\mathcal{G}$ on the same groundset X, define the *antiblocker* of $\mathcal{FG}$ kernels by

$$\mathcal{A}_{\mathcal{FG}} := \{A \subseteq X : |A \cap K| \leq 1 \text{ for any memeber } K \text{ of } X\}$$

and the submissive of the kernel-antiblocker polytope by

$$\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}} := \{\chi^A : A \in \mathcal{A}_{\mathcal{FG}}\}^{\downarrow}.$$

Applying the Hoffman-Schwartz theorem on $\mathcal{K}_{\mathcal{FG}}$, we get the following generalization of Theorem 20.3.

Theorem 21.3. *If $\mathcal{FG}$ are increasing comonotone functions on groundset X then*

$$\mathcal{P}^{\uparrow}_{\mathcal{B}_{\mathcal{FG}}} = \{x \in \mathbb{R}^X : x \geq \mathbf{0} \textit{ and } x(K) \geq 1 \textit{ for any } K \in \mathcal{K}_{\mathcal{FG}}\} \textit{ and} \quad (21.1)$$

$$\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}} = \{x \in \mathbb{R}^X : x \geq \mathbf{0} \textit{ and } x(K) \leq 1 \textit{ for any } K \in \mathcal{K}_{\mathcal{FG}}\}. \quad (21.2)$$

Proof. Obviously, the polyhedra on the left hand side of (21.1,21.2) are the integer hulls of the polyhedra described by right hand sides.

By Observation 21.2, $\mathcal{K}_{\mathcal{FG}}$ is an upper and lower clutter. Let $d(v) := \infty$ and $r(K) := 1$ for all $v \in X$ and $K \in \mathcal{K}_{\mathcal{FG}}$. Clearly, r is sub- and supermodular. By Theorem 21.1, linear systems in (21.1,21.2) are TDI, hence the polyhedra on the right hand sides are integer. □

Using the theory of antiblocking polyhedra, we can describe the submissive of the kernel polytope.

Theorem 21.4. *If $\mathcal{FG}$ are increasing comonotone functions on groundset X then*

$$\mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}} = \{x \in \mathbb{R}^X : x \geq \mathbf{0} \textit{ and } x(A) \leq 1 \textit{ for any } A \in \mathcal{K}_{\mathcal{FG}}\}. \quad (21.3)$$

Proof. By (21.2) and (5.6), $\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}} = A(\mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}})$. Using Theorem 5.8 we get that $\mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}} = A(\mathcal{P}^{\downarrow}_{\mathcal{A}_{\mathcal{FG}}})$, and Theorem 21.4 follows from (5.6). □

22 Open questions

The first interesting problem is to find similar characterizations to (20.7-20.9) and (21.3) for the corresponding matroid-kernel polyhedra. The difficulty here is that when we proved the existence of a matroid-kernel, we used a greedy algorithm after fixing an order of the elements of the matroid by increasing costs. However, this linear extension is not unique, and if we pick different ones, then the $\mathcal{M}_1\mathcal{M}_2$-kernels we find can be completely different. One extreme case, when all elements have different costs, that is when the cost function is injective. Then the linear description of the corresponding kernel polyhedra is given by (20.7-20.9). The other extreme is the case of constant cost function. Then $\mathcal{P}_{\mathcal{K}_{\mathcal{FG}}}$ is the convex hull of the inclusionwise maximal common independent sets. In this case the antiblocking polytope $\mathcal{P}^{\downarrow}_{\mathcal{K}_{\mathcal{FG}}}$ has been characterized by Edmonds (see Theorem 5.14). Thus it would be interesting to find a good characterization for the antiblocking polytope of matroid kernels.

Another possible direction of further research would be a generalization of Theorem 9.10 to the kernel polytope $\mathcal{P}_{\mathcal{K}_{\mathcal{F}\mathcal{G}}}$ for increasing comonotone functions. We have indeed given a linear description for an extended class of comonotone kernel polytopes in Theorem 20.4, but this theorem is not a generalization of the former one. Namely, we talk about the blocker of a certain family, whereas in the description of Vande Vate and Rothblum the constraints are given concretely. In [52], Hoffman provides a certain characterization of the blocker of upper clutters and the antiblocker of lower clutters. Still, I see no direct application of this result that generalizes Theorem 9.10.

According to the suggestion of András Frank, another interesting application of matroid-kernels would be to find a similar theorem to Galvin's about list-colourings of matroids.

Conjecture 22.1. *Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be matroids on the same groundset. Assume that E can be covered by k common independent sets of $\mathcal{M}_1$ and $\mathcal{M}_2$ and $k \leq \sum_{i\in[m]} \chi^{C_i}$ for sets $C_1, C_2, \dots, C_m$. Then there exist $C_i \supseteq I_i \in \mathcal{I}_1 \cap \mathcal{I}_2$ such that $E = \bigcup_{i\in[m]} I_i$.*

If Conjecture 22.1 is true, then apart from Galvin's theorem, it would extend the following result of Seymour.

Theorem 22.2 (Seymour [95]). *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and assume that there are k independent sets $I_1, I_2, \dots, I_k \in \mathcal{I}$ such that $E = \bigcup_{i\in[k]} I_i$. If $k \cdot \chi^E \leq \sum_{i\in[m]} \chi^{L_i}$ for subsets $L_1, L_2, \dots, L_m$ of E, then there are subsets J_i of L_i for $i \in [m]$ such that $J_i \in \mathcal{I}$ and $E = \bigcup_{i\in[m]} J_i$.*

In Section 18, we indicated a connection between Brouwer's topological fixed point theorem and some fractional kernel results. The main topic of this chapter was the relation of Tarski's fixed point theorem to integral kernel results. I think that the relation of these theorems is not yet fully understood. So without formulating a specific question, I ask whether there are more interconnections between these approaches.

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Curriculum Vitæ

Tamás Fleiner was born on the 8th of April, 1971, in Budapest. In 1989, he finished high school at the Fazekas Mihály Fővárosi Gyakorló Általános Iskola és Gimnázium, and started to study Mathematics at the Eötvös Loránd University, in Budapest. For the academic year 1992/93 he was a TEMPUS scholar at the University of Sussex in Brighton (supervisors: J.W.P. Hirschfeld and R.P. Lewis). In 1995, under the supervision of András Frank, he received a diploma with distinction at the Eötvös university, and the same year he started PhD studies at the Mathematical Institute of the Hungarian Academy of Sciences (today called Alfréd Rényi Institute of Mathematics), with supervisor Ervin Győri. In 1996, he moved to Amsterdam, where he started to do PhD studies at the Centrum voor Wiskunde en Informatica, under the supervision of Bert Gerards. From the 1st of September, 2000, he works as a young research fellow at the Alfréd Rényi Institute of Mathematics and has a half teaching position at the Budapest University of Technology and Economics.

In 1996, Tamás Fleiner got married to Rita Fleiner and in 1998 his daughter, Zsófia Fleiner was born.

During his high school studies, in 1989, Tamás Fleiner was a member of the Hungarian team for the International Mathematical Olympics, where he received a silver medal. On the Kürschák József Matematikai Tanulóverseny, which is one of the most prestigious national Mathematical competitions in Hungary, he won a second prize in 1987, a first prize in 1988 and finished on the first place in 1989.

Tamás Fleiner's interest in Mathematics is mainly oriented to Combinatorics, especially to Combinatorial Optimization, Polyhedral Combinatorics, Graph and Matroid Theory and Extremal Combinatorics.

Samenvatting

Dit proefschrift bestaat - behoudens inleidende paragrafen - uit twee delen. Het eerste deel, Hoofdstuk II, behandelt zogenaamde kruisende structuren. De motivatie hiervoor komt voornamelijk uit de Besliskunde, maar de behaalde resultaten hebben ook consequenties voor onder andere de Speltheorie en de theorie van eindige metrische ruimten. Het tweede deel, Hoofdstuk III, behandelt de relatie tussen onderwerpen uit uiteenlopende gebieden als Besliskunde, Speltheorie, Grafentheorie, Wiskundige Economie en Topologie.

In de eerste paragraaf van Hoofdstuk II beschouwen we een zogenaamd ontkruisingsspel. Twee deelverzamelingen van een verzameling X *kruisen (in X)* als zij niet-lege doorsnede hebben, niet in elkaar bevat zijn en samen X niet overdekken. In het ontkruisingsspel spelen de *ontkruiser* en de *spelbreker* met een collectie verzamelingen. In elke stap van het spel neemt de ontkruiser twee kruisende verzamelingen uit de collectie en vervangt ze door hun doorsnede en vereniging. Vervolgens stopt de spelbreker een van de twee door de ontkruiser verwijderde verzamelingen in de collectie terug. De ontkruiser wint als het systeem geen kruisende verzamelingen meer bevat; de spelbreker wint als de collectie in een eerdere toestand terugkomt. Hurkens, Lovász, Schrijver en Tardos [54] hebben een strategie voor de ontkruiser ontwikkeld om dit spel in polynomiale tijd te winnen. In dit proefschrift generaliseren we deze strategie naar een soortgelijk ontkruisingsspel op een collectie van paren verzamelingen. Deze gegeneraliseerde strategie levert een polynomiaal algoritme om aan te tonen dat een zekere verbetering in de samenhang van een gericht netwerk zo goedkoop mogelijk is.

Het tweede resultaat in Hoofdstuk II is het bewijs van een vermoeden van András Frank over symmetrische ordeningen. Dit resultaat, een minmax relatie, generaliseert de stelling van Dilworth [18] met betrekking tot de opdeling van een partiële ordening in zo min mogelijk lineaire ordeningen. Ons bewijs van dit vermoeden gebruikt de Tutte-Berge formule voor maximale koppelingen in algemene grafen. Het is in dit verband interessant op te merken dat de stelling van Dilworth afgeleid kan worden uit de stelling van Kőnig over maximale koppelingen in bipartiete grafen. De door ons bewezen minmax relatie heeft gevolgen voor het inbedden van eindige metrische ruimten in een l_1-ruimte met minimale dimensie.

Het derde resultaat van Hoofdstuk II is dat elke collectie deelverzamelingen van

$\{1, \ldots, n\}$ zonder paarsgewijs kruisende drietallen hooguit $10n$ leden heeft. Pevzner [76] geeft een lang en moeilijk leesbaar argument voor ongeveer dezelfde bewering; ons bewijs is kort en eenvoudig. De motivatie voor het onderzoek naar collecties verzamelingen zonder paarsgewijs kruisende drietallen is de Locking Stelling van Karzanov en Lomonosov [57, 58] over stromen met meerdere goederen. Een bekend vermoeden van Karzanov en Lomonosov zegt dat een collectie deelverzamelingen van $\{1, \ldots, n\}$ zonder paarsgewijs kruisende k-tallen hooguit $O(kn)$ leden heeft.

Uitgangspunt van Hoofdstuk III is de Stabiele-huwelijksstelling van Gale and Shapley. Als in een dorp een man en vrouw elkaar verkiezen boven hun eventuele eigen huwelijkspartner, dan is de collectie huwelijken binnen dat dorp instabiel. Ontbreekt zo'n paar potentiële echtbrekers dan heet de collectie huwelijken *stabiel*. Gale en Shapley [42] hebben bewezen dat uitgaande van gegeven vaste prioriteiten van elke man en vrouw binnen een populatie een stabiele collectie huwelijken bestaat. Zij geven ook een onderhandelingsprocedure van aanzoeken en afwijzingen die tot een stabiele collectie huwelijken leidt. Deze stabiele-huwelijksproblematiek laat zich natuurlijk makkelijk vertalen naar andere toewijzingen, zoals studenten aan universiteiten en medewerkers aan bedrijven. Wat niet zo voor de hand ligt is dat, zoals in dit proefschrift blijkt, de wiskundige ideeën verder reiken, naar onderwerpen waarbij de analogie niet zo duidelijk is. Voorbeelden zijn "graafkernen" (van belang in de theorie van de perfecte grafen), "matroïdenkernen" en collecties disjuncte paden.

Een van de belangrijkste bouwstenen van de in Hoofdstuk III ontwikkelde unificerende theorie is de observatie dat de Stabiele-huwelijksstelling en veel andere bestaande resultaten, binnen verschillende vakgebieden, afgeleid kunnen worden uit de bekende dekpuntstelling van Tarski. Deze observatie leidt niet alleen tot eenvoudige bewijzen van reeds bekende resultaten maar ook tot nieuwe generalisaties in de context van matroïden en ordeningen. Door matroïdenkernpolytopen met stelsels lineaire ongelijkheden te beschrijven, laten we zien dat het mogelijk is om met behulp van de ellipsoïdenmethode over matroïdenkernen te optimaliseren. Het bewijs van deze lineaire beschrijvingen is een nieuwe interessante toepassing van Hoffman's theorie van roosterpolyeders.

Ook de topologische dekpuntstelling van Brouwer heeft interessante gevolgen voor matroïdenkernen. We bewijzen dat in situaties algemener dan behandeld met behulp van Tarski's dekpuntstelling fractionele kernen bestaan. Een voorbeeld is het probleem van stabiele koppeling van kamergenoten. Omdat in dat probleem de onderliggende graaf niet bipartiet hoeft te zijn, is het mogelijk dat er geen stabiel schema is. Maar een zeker stabiel fractioneel schema bestaat altijd. De lineare beschrijving van het bipartiete-koppelingspolytoop impliceert dat als de graaf bipartiet is, elk stabiel fractioneel schema een convexe combinatie van (geheeltallige) stabiele schema's is. Dit levert een interessant nieuw bewijs voor de stelling van Gale en Shapley.

Een gelijksoortige situatie doet zich voor bij matroïdenkernen: uit de dekpuntstelling van Brouwer volgt het bestaan van een fractionele kern voor elke collectie matroïden. Gecombineerd met Edmonds' beschrijving van het matroïdenintersectiepolytoop, volgt daaruit de existentie van een (geheeltallige) kern voor elke collectie van twee matroïden.

We besluiten Hoofdstuk III met enkele interessante open vragen.