# THE STRUCTURE OF RANDOM AUTOMORPHISMS

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Abstract. In order to understand the structure of the "typical" element of an automorphism group, one has to study how large the conjugacy classes of the group are. When typical is meant in the sense of Baire category, a complete description of the size of the conjugacy classes has been given by Kechris and Rosendal. Following Dougherty and Mycielski we investigate the measure theoretic dual of this problem, using Christensen's notion of Haar null sets. When typical means random, that is, almost every with respect to this notion of Haar null sets, the behaviour of the automorphisms is entirely different from the Baire category case. We generalise the theorems of Dougherty and Mycielski about $S_\infty$ to arbitrary automorphism groups of countable structures isolating a new model theoretic property, the *Cofinal Strong Amalgamation Property*. A complete description of the non-Haar null conjugacy classes of the automorphism groups of $(\mathbb{Q}, <)$ and of the random graph is given, in fact, we prove that every non-Haar null class contains a translated copy of a non-empty portion of every compact set. As an application we affirmatively answer the question whether these groups can be written as the union of a meagre and a Haar null set.

## Contents



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## 1. Introduction

The study of typical elements of Polish groups is a flourishing field with a large number of applications. The systematic investigation of typical elements of automorphism groups of countable structures has been initiated by Truss [19]. He conjectured that the existence of a co-meagre conjugacy class can be characterised in model theoretic terms. This question has been answered affirmatively by Kechris and Rosendal [13]. They, extending the work of Hodges, Hodkinson, Lascar and Shelah [11] also investigated the relation between the existence of co-meagre conjugacy classes in every dimension and other group theoretic properties, such as the small index property, uncountable cofinality, automatic continuity and Bergman's property.

The existence and description of typical elements frequently have applications in the theory of dynamical systems as well. For example, it is easy to see that the automorphism group of the countably infinite atomless Boolean algebra is isomorphic to the homeomorphism group of the Cantor set, which is a central object in dynamics. Thus, from their general results Kechris and Rosendal deduced the existence of a co-meagre conjugacy class in the homeomorphism group of the Cantor set. A description of an element with such a class has been given by Glasner and Weiss [7] and from a different perspective by Bernardes and the first author [2].

So, it is natural to ask whether there exist measure theoretic analogues of these results. Unfortunately, on non-locally compact groups there is no natural invariant $\sigma$-finite measure. However, a generalisation of the ideal of measure zero sets can be defined in every Polish group as follows:

**Definition 1.1** (Christensen, [3])**.** Let $G$ be a Polish group and $B \subset G$ be Borel. We say that $B$ is *Haar null* if there exists a Borel probability measure $\mu$ on $G$ such that for every $g, h \in G$ we have $\mu(gBh) = 0$. An arbitrary set $S$ is called Haar null if $S \subset B$ for some Borel Haar null set $B$.

It is known that the collection of Haar null sets forms a $\sigma$-ideal in every Polish group and it coincides with the ideal of measure zero sets in locally compact groups with respect to every left (or equivalently right) Haar measure. Using this definition, it makes sense to talk about the properties of random elements of a Polish group. A property $P$ of elements of a Polish group $G$ is said to *hold almost surely* or *almost every element of $G$ has property $P$* if the set $\{g \in G : g \text{ has property } P\}$ is co-Haar null.

Since we are primarily interested in homeomorphism and automorphism groups, and in such groups conjugate elements can be considered isomorphic, we are only interested in the conjugacy invariant properties of the elements of our Polish groups. Hence, in order to describe the random element, one must give a complete description of the size of the conjugacy classes with respect to the Haar null ideal. The

investigation of this question has been started by Dougherty and Mycielski [5] in the permutation group of a countably infinite set, $S_\infty$. If $f \in S_\infty$ and $a$ is an element of the underlying set then the set $\{f^k(a) : k \in \mathbb{Z}\}$ is called the *orbit of $a$ (under $f$)*, while the cardinality of this set is called *orbit length*. Thus, each $f \in S_\infty$ has a collection of orbits (associated to the elements of the underlying set). It is easy to show that two elements of $S_\infty$ are conjugate if and only if they have the same (possibly infinite) number of orbits for each possible orbit length.

**Theorem 1.2** (Dougherty, Mycielski, [5])**.** *Almost every element of $S_\infty$ has infinitely many infinite orbits and only finitely many finite ones.*

Therefore, almost all permutations belong to the union of a countable set of conjugacy classes.

**Theorem 1.3** (Dougherty, Mycielski, [5])**.** *All of these countably many conjugacy classes are non-Haar null.*

Thus, the above theorems give a complete description of the non-Haar null conjugacy classes and the (conjugacy invariant) properties of a random element. The aim of our paper is to initiate a systematic study of the size of the conjugacy classes of automorphism groups of countable structures. Our work is centred around questions of the following type:

**Question 1.4.** *Let $\mathcal{A}$ be a countable structure.*

*(1) What properties of $\mathcal{A}$ ensure that (an appropriate) generalisation of the theorem of Dougherty and Mycielski holds for* $\mathrm{Aut}(\mathcal{A})$*?*
*(2) Describe the (conjugacy invariant) properties of almost every element of* $\mathrm{Aut}(\mathcal{A})$*: Which conjugacy classes of* $\mathrm{Aut}(\mathcal{A})$ *are non-Haar null? How many non-Haar null conjugacy classes are there? Is almost every element of* $\mathrm{Aut}(\mathcal{A})$ *contained in a non-Haar null class?*

One can prove that in $S_\infty$ the collection of elements that have no infinite orbits is a co-meagre set. This shows that the typical behaviour in the sense of Baire category is quite different from the typical behaviour in the measure theoretic sense. In particular, $S_\infty$ can be decomposed into the union of a Haar null and a meagre set. It is well known that this is possible in every locally compact group, but the situation is not clear in the non-locally compact case. Thus, the following question of the first author arises naturally:

**Question 1.5.** *Suppose that $G$ is an uncountable Polish group. Can it be written as the union of a meagre and a Haar null set?*

We investigate this question for various automorphism groups.

The paper is organised as follows. First, in Section 2 we summarise facts and notations used later, then in Section 3 we give a detailed description of our results. Section 4 contains the general theorems about the automorphism groups of countable structures, while in Sections 5 and 6 we prove the characterization of the non-Haar null conjugacy classes of the automorphism group of $(\mathbb{Q}, <)$ and the random graph, respectively. After this, in Section 8 we investigate the possible cardinality of non-Haar null conjugacy classes of (locally compact and non-locally compact) Polish groups. Finally, we conclude with listing a number of open questions in Section 9.

## 2. Preliminaries and notations

We will follow the notations of [12]. For a detailed introduction to the theory of Polish groups see [1, Chapter 1], while the model theoretic background can be found in [10, Chapter 7]. Nevertheless, we summarise the basic facts which we will use.

As usual, a countable structure $\mathcal{A}$ is a first order structure on a countable set with countably many constants, relations and functions. The underlying set will be denoted by $\mathrm{dom}(\mathcal{A})$. The automorphism group of the structure $\mathcal{A}$ is denoted by $\mathrm{Aut}(\mathcal{A})$ which we consider as a topological (Polish) group with the topology of pointwise convergence. Isomorphisms between topological groups are considered to be group automorphisms that are also homeomorphisms. The structure $\mathcal{A}$ is called *ultrahomogeneous* if every isomorphism between its finitely generated substructures extends to an automorphism of $\mathcal{A}$. The *age* of a structure $\mathcal{A}$ is the collection of the finitely generated substructures of $\mathcal{A}$. An injective homomorphism between structures will be called an *embedding*. A structure is said to be *locally finite* if every finite set of elements generates a finite substructure.

A countable set $\mathcal{K}$ of finitely generated structures of the same language is called a *Fraïssé class* if it satisfies the hereditary (HP), joint embedding (JEP) and amalgamation properties (AP) (see [10, Chapter 7]). We will need the notion of the strong amalgamation property: A Fraïssé class $\mathcal{K}$ satisfies the *strong amalgamation property (SAP)* if for every $\mathcal{B} \in \mathcal{K}$ and every pair of structures $\mathcal{C}, \mathcal{D} \in \mathcal{K}$ and embeddings $\psi : \mathcal{B} \to \mathcal{C}$ and $\chi : \mathcal{B} \to \mathcal{D}$ there exist $\mathcal{E} \in \mathcal{K}$ and embeddings $\psi' : \mathcal{C} \to \mathcal{E}$ and $\chi' : \mathcal{D} \to \mathcal{E}$ such that

$$\psi' \circ \psi = \chi' \circ \chi \text{ and } \psi'(\mathcal{C}) \cap \chi'(\mathcal{D}) = (\psi' \circ \psi)(\mathcal{B}) = (\chi' \circ \chi)(\mathcal{B}).$$

For a Fraïssé class $\mathcal{K}$ the unique countable ultrahomogeneous structure $\mathcal{A}$ with $age(\mathcal{A}) = \mathcal{K}$ is called the *Fraïssé limit of* $\mathcal{K}$. If $G$ is the automorphism group of a structure $\mathcal{A}$, we call a bijection $p$ a *partial automorphism* or a *partial permutation* if it is an automorphism between two finitely generated substructures of $\mathcal{A}$ such that $p \subset g$ for some $g \in G$.

As mentioned before, $S_\infty$ stands for the permutation group of the countably infinite set $\omega$. It is well known that $S_\infty$ is a Polish group with the pointwise convergence topology. This coincides with the topology generated by the sets of the form $[p] = \{f \in S_\infty : p \subset f\}$, where $p$ is a finite partial permutation.

Let $\mathcal{A}$ be a countable structure. By the countability of $\mathcal{A}$, every automorphism $f \in \mathrm{Aut}(\mathcal{A})$ can be regarded as an element of $S_\infty$, and it is not hard to see that in fact $\mathrm{Aut}(\mathcal{A})$ will be a closed subgroup of $S_\infty$. Moreover, the converse is also true, namely every closed subgroup of $S_\infty$ is isomorphic to the automorphism group of a countable structure.

Let $G$ be a closed subgroup of $S_\infty$. The *orbit* of an element $x \in \omega$ (under $G$) is the set $G(x) = \{y \in \omega : \exists g \in G\ (g(x) = y)\}$. For a set $S \subset \omega$ we denote the *pointwise stabiliser* of $S$ by $G_{(S)}$, that is, $G_{(S)} = \{g \in G : \forall s \in S\ (g(s) = s)\}$. In case $S = \{x\}$, we write $G_{(x)}$ instead of $G_{(\{x\})}$.

As in the case of $S_\infty$, for a countable structure $\mathcal{A}$, an element $a \in \mathrm{dom}(\mathcal{A})$ and $f \in \mathrm{Aut}(\mathcal{A})$ the set $\{f^k(a) : k \in \mathbb{Z}\}$ is called the *orbit of* $a$ and denoted by $\mathcal{O}^f(a)$, while the cardinality of this set is called *orbit length*. The *collection of the orbits of* $f$, or *the orbits of* $f$ is the set $\{\mathcal{O}^f(a) : a \in \mathrm{dom}(\mathcal{A})\}$. If $S \subset \mathrm{dom}(\mathcal{A})$ we will also use the notation $\mathcal{O}^f(S)$ for the set $\bigcup_{a \in S} \mathcal{O}^f(a)$.

We will constantly use the following fact.

**Fact 2.1.** *Let $\mathcal{A}$ be a countable structure. A closed subset $\mathcal{K}$ of* $\mathrm{Aut}(\mathcal{A})$ *is compact if and only if for every $a \in \mathrm{dom}(\mathcal{A})$ the set $\{f(a), f^{-1}(a) : f \in \mathcal{K}\}$ is finite.*

We denote by $\mathcal{B}_\infty$ the countable atomless Boolean algebra, by $(\mathbb{Q}, <)$ or $\mathbb{Q}$ the rational numbers as an ordered set. Let us use the notation $\mathcal{R}$ (or $(V, R)$) for the countably infinite random graph, that is, the unique countable graph with the following property: for every pair of finite disjoint sets $A, B \subset V$ there exists $v \in V$ such that $(\forall x \in A)(xRv)$ and $(\forall y \in B)(y \neg Rv)$.

In the investigation of the structure of $\mathrm{Aut}(\mathbb{Q}, <)$ we use the concept of orbitals (defined below, for more details on this topic see [8]). Let $p, q \in \mathbb{Q}$. The interval $(p, q)$ will denote the set $\{r \in \mathbb{Q} : p < r < q\}$. For an automorphism $f \in \mathrm{Aut}(\mathbb{Q}, <)$, we denote the set of fixed points of $f$ by $\mathrm{Fix}(f)$.

**Definition 2.2.** The set of *orbitals* of an automorphism $f \in \mathrm{Aut}(\mathbb{Q}, <)$, $\mathcal{O}_f^*$, consists of the convex hulls (relative to $\mathbb{Q}$) of the orbits of the rational numbers, that is

$$\mathcal{O}_f^* = \{\mathrm{conv}(\{f^n(r) : n \in \mathbb{Z}\}) : r \in \mathbb{Q}\}.$$

It is easy to see that the orbitals of $f$ form a partition of $\mathbb{Q}$, with the fixed points determining one element orbitals, hence "being in the same orbital" is an equivalence relation. Using this fact, we define the relation $<$ on the set of orbitals by letting $O_1 < O_2$ for distinct $O_1, O_2 \in \mathcal{O}_f^*$ if $p_1 < p_2$ for some (and hence for all) $p_1 \in O_1$ and $p_2 \in O_2$. Note that $<$ is a linear order on the set of orbitals.

It is also easy to see that if $p, q \in \mathbb{Q}$ are in the same orbital of $f$ then $f(p) > p \Leftrightarrow f(q) > q$, $f(p) < p \Leftrightarrow f(q) < q$ and $f(p) = p \Leftrightarrow f(q) = q \Rightarrow p = q$. This observation makes it possible to define the *parity function*, $s_f : \mathcal{O}_f^* \to \{-1, 0, 1\}$. Let $s_f(O) = 0$ if $O$ consists of a fixed point of $f$, $s_f(O) = 1$ if $f(p) > p$ for some (and hence, for all) $p \in O$ and $s_f(O) = -1$ if $f(p) < p$ for some (and hence, for all) $p \in O$.

Let us consider the following notion of largeness:

**Definition 2.3.** Let $G$ be a Polish topological group. A set $A \subset G$ is called *compact catcher* if for every compact $K \subset G$ there exist $g, h \in G$ so that $gKh \subset A$. $A$ is *compact biter* if for every compact $K \subset G$ there exist an open set $U$ and $g, h \in G$ so that $U \cap K \neq \emptyset$, and $g(U \cap K)h \subset A$.

If $K$ is a compact set, a set of the form $U \cap K$ will be called a *portion of* $K$ if $U$ is an open set. The following easy observation is one of the most useful tools to prove that a certain set is not Haar null.

**Fact 2.4.** *If $A$ is compact biter then it is not Haar null.*

*Proof.* Suppose that this is not the case and let $B \supset K$ be a Borel Haar null set and $\mu$ be a witness measure for $B$. Then, by the regularity of $\mu$, there exists a compact set $K \subset G$ such that $\mu(K) > 0$. Subtracting the relatively open $\mu$ measure zero subsets of $K$ we can suppose that for every open set $U$ if $U \cap K \neq \emptyset$ then $\mu(U \cap K) > 0$. But, as $A$ is compact biter, so is $B$, thus for some open set $U$ with $\mu(U \cap K) > 0$ there exist $g, h \in G$ so that $g(U \cap K)h \subset B$. This shows that $\mu$ cannot witness that $B$ is Haar null, a contradiction. $\square$

Note that the proof of Theorem 1.3 by Dougherty and Mycielski actually shows that every non-Haar null conjugacy class is compact biter and the unique non-Haar null conjugacy class which contains elements without finite orbits is compact catcher.

It is sometimes useful to consider right and left Haar null sets: a Borel set $B$ is *right (resp. left) Haar null* if there exists a Borel probability measure $\mu$ on $G$ such that for every $g \in G$ we have $\mu(Bg) = 0$ (resp. $\mu(gB) = 0$). An arbitrary set $S$ is called *right (resp. left) Haar null* if $S \subset B$ for some Borel right (resp. left) Haar null set $B$. The following observation will be used several times.

**Lemma 2.5.** *Suppose that $B$ is a Borel set that is invariant under conjugacy. Then $B$ is left Haar null iff it is right Haar null iff it is Haar null.*

*Proof.* Let $\mu$ be a measure witnessing that $B$ is left Haar null. We check that it also witnesses the Haar nullness of $B$. Indeed, let $g, h \in G$ arbitrary, $\mu(gBh) = \mu(ghh^{-1}Bh) = \mu(ghB) = 0$. The proof is analogous when $B$ is right Haar null. □

## 3. Description of the results

3.1. **General results about countable structures.** We start with defining the crucial notion for the description of the orbits of a random element of an automorphism group. Informally, the following definition says that our structure is free enough: if we want to extend a partial automorphism defined on a finite set, there are only finitely many points for which we have only finitely many options.

**Definition 3.1.** Let $G$ be a closed subgroup of $S_\infty$. We say that $G$ *has the finite algebraic closure property (FACP)* if for every finite $S \subset \omega$ the set $\{b : |G_{(S)}(b)| < \infty\}$ is finite.

The following model theoretic property of Fraïssé classes turns out to be essentially a reformulation of the $FACP$ for the automorphism groups of the limits.

**Definition 3.2.** Let $\mathcal{K}$ be a Fraïssé class. We say that $\mathcal{K}$ *has the cofinal strong amalgamation property (CSAP)* if there exists a subclass of $\mathcal{K}$ cofinal under embeddability, which satisfies the strong amalgamation property, or more formally: for every $\mathcal{B}_0 \in \mathcal{K}$ there exists a $\mathcal{B} \in \mathcal{K}$ and an embedding $\phi_0 : \mathcal{B}_0 \to \mathcal{B}$ so that the *strong amalgamation property holds over* $\mathcal{B}$, that is, for every pair of structures $\mathcal{C}, \mathcal{D} \in \mathcal{K}$ and embeddings $\psi : \mathcal{B} \to \mathcal{C}$ and $\chi : \mathcal{B} \to \mathcal{D}$ there exist $\mathcal{E} \in \mathcal{K}$ and embeddings $\psi' : \mathcal{C} \to \mathcal{E}$ and $\chi' : \mathcal{D} \to \mathcal{E}$ such that

$$\psi' \circ \psi = \chi' \circ \chi \text{ and } \psi'(\mathcal{C}) \cap \chi'(\mathcal{D}) = (\psi' \circ \psi)(\mathcal{B}) = (\chi' \circ \chi)(\mathcal{B}).$$

A Fraïssé limit $\mathcal{A}$ is said to *have the cofinal strong amalgamation property* if $age(\mathcal{A})$ has the CSAP.

Generalizing the results of Dougherty and Mycielski we show that the $FACP$ is equivalent to some properties of the orbit structure of a random element of the group.

**Theorem 4.14.** *Let $\mathcal{A}$ be a locally finite Fraïssé limit. Then the following are equivalent:*

*(1) almost every element of* $\mathrm{Aut}(\mathcal{A})$ *has finitely many finite orbits,*
*(2)* $\mathrm{Aut}(\mathcal{A})$ *has the $FACP$,*

*(3) $\mathcal{A}$ has the CSAP.*

*Moreover, any of the above conditions implies that almost every element of $\mathcal{A}$ has infinitely many infinite orbits.*

Note that every relational structure and also $\mathcal{B}_\infty$ is locally finite, moreover, it is well known that the ages of the structures $\mathcal{R}, (\mathbb{Q}, <)$ and $\mathcal{B}_\infty$ have the strong amalgamation property which clearly implies the CSAP (it is also easy to directly check the $FACP$ for these groups). Hence we obtain the following corollary.

**Corollary 3.3.** *In* $\mathrm{Aut}(\mathcal{R}), \mathrm{Aut}(\mathbb{Q}, <)$ *and* $\mathrm{Aut}(\mathcal{B}_\infty)$ *almost every element has finitely many finite and infinitely many infinite orbits.*

As a corollary of our results, in Section 7 we show that a large number of groups can be partitioned in a Haar null and a meagre set.

**Corollary 7.1.** *Let $G$ be a closed subgroup of $S_\infty$ satisfying the $FACP$ and suppose that the set $F = \{g \in G : \mathrm{Fix}(g)$ is infinite$\}$ is dense in $G$. Then $G$ can be decomposed into the union of an (even conjugacy invariant) Haar null and a meagre set.*

**Corollary 7.2.** $\mathrm{Aut}(\mathcal{R})$, $\mathrm{Aut}(\mathbb{Q}, <)$ *and* $\mathrm{Aut}(\mathcal{B}_\infty)$ *can be decomposed into the union of an (even conjugacy invariant) Haar null and a meagre set.*

However, these results are typically far from the full description of the behaviour of the random elements. We continue with the detailed study of two special cases, $\mathrm{Aut}(\mathbb{Q}, <)$ and $\mathrm{Aut}(\mathcal{R})$.

3.2. $\mathrm{Aut}(\mathbb{Q}, <)$**.** The following is our main result about the automorphism group of the rational numbers.

**Theorem 5.4.** *For almost every element $f$ of* $\mathrm{Aut}(\mathbb{Q}, <)$

*(1) for distinct orbitals $O_1, O_2 \in \mathcal{O}_f^*$ (see Definition 2.2) with $O_1 < O_2$ such that $s_f(O_1) = s_f(O_2) = 1$ or $s_f(O_1) = s_f(O_2) = -1$, there exists an orbital $O_3 \in \mathcal{O}_f^*$ with $O_1 < O_3 < O_2$ and $s_f(O_3) \neq s_f(O_1)$,*

*(2) (follows from Theorem 4.14) $f$ has only finitely many fixed points.*

*These properties characterise the non-Haar null conjugacy classes, i. e., a conjugacy class is non-Haar null if and only if one (or equivalently each) of its elements has properties* (1) *and* (2)*.*

*Moreover, every non-Haar null conjugacy class is compact biter and those non-Haar null classes in which the elements have no rational fixed points are compact catchers.*

This yields the following surprising corollary:

**Corollary 3.4.** *There are continuum many non-Haar null conjugacy classes in* $\mathrm{Aut}(\mathbb{Q}, <)$*, and their union is co-Haar null.*

Note that it was proved by Solecki [17] that in every non-locally compact Polish group that admits a two-sided invariant metric there are continuum many pairwise disjoint non-Haar null Borel sets, thus the above corollary is an extension of his results for $\mathrm{Aut}(\mathbb{Q}, <)$ (see also the case of $\mathrm{Aut}(\mathcal{R})$ below). We would like to point out that in a sharp contrast to this result, in $\mathrm{Homeo}^+([0, 1])$ (that is, in the group of order preserving homeomorphisms of the interval) the random behaviour is quite different (see [4]), more similar to the case of $S_\infty$: there are only countably many non-Haar null conjugacy classes and their union is co-Haar null.

3.3. $\mathrm{Aut}(\mathcal{R})$**.** The characterization of non-Haar null conjugacy classes of the automorphism group of the random graph appears to be similar to the characterization of the non-Haar null classes of $\mathrm{Aut}(\mathbb{Q}, <)$, however their proofs are completely different.

**Theorem 6.29.** *For almost every element $f$ of* $\mathrm{Aut}(\mathcal{R})$

*(1) for every pair of finite disjoint sets, $A, B \subset V$ there exists $v \in V$ such that $(\forall x \in A)(xRv)$ and $(\forall y \in B)(y \neg Rv)$ and $v \notin \mathcal{O}^f(A \cup B)$, i. e., the union of orbits of the elements of $A \cup B$,*

*(2) (from Theorem 4.14) $f$ has only finitely many finite orbits.*

*These properties characterise the non-Haar null conjugacy classes, i. e., a conjugacy class is non-Haar null if and only if one (or equivalently each) of its elements has properties* (1) *and* (2)*.*

*Moreover, every non-Haar null conjugacy class is compact biter and those non-Haar null classes in which the elements have no finite orbits are compact catchers.*

It is not hard to see that this characterisation again yields the following corollary:

**Corollary 3.5.** *There are continuum many non-Haar null classes in* $\mathrm{Aut}(\mathcal{R})$ *and their union is co-Haar null.*

In the proof we use a version (see Lemma 6.3) of the following lemma which is interesting in itself.

**Lemma 3.6.** *(Splitting Lemma, finite version) If $F \subset \mathrm{Aut}(\mathcal{R})$ is a finite set and $A, B \subset V$ are disjoint finite sets, then there exists a vertex $v$ so that for every distinct $f, g \in F$ we have $f(v) \neq g(v)$, $(\forall x \in A)(xRv)$ and $(\forall y \in B)(y \neg Rv)$.*

From the above theorem and the splitting lemma one can give a new proof of well known results of Truss [18] (which was improved by him later) and Rubin, that states that if $f, g$ are non-identity elements in $\mathrm{Aut}(\mathcal{R})$ then $g$ is the product of four conjugates of $f$, see Theorem 7.3.

3.4. **Various behaviours.** Examining any Polish group we can ask the following questions:

1. How many non-Haar null conjugacy classes are there?
2. Is the union of the Haar null conjugacy classes Haar null?

Note that these are interesting even in compact groups. Table 1 summarises our examples and the open questions as well (the left column indicates the number of non-Haar null conjugacy classes, while C, LC $\setminus$ C and NLC stands for compact, locally compact non-compact and non-locally compact groups, respectively). HNN denotes the well known infinite group, constructed by G. Higmann, B. H. Neumann and H. Neumann [9], with two conjugacy classes, while $\mathbb{Q}_d$ stands for the rationals with the discrete topology. The action, $\phi$, of $\mathbb{Z}_2$ on $\mathbb{Z}_3^\omega$ and $\mathbb{Q}_d^\omega$ is the map defined by $a \mapsto -a$.

## 4. General results about countable structures

This section contains our generalisation of the result of Dougherty and Mycielski to automorphism groups of countable structures. For the sake of simplicity we will use the following notation.

| | The union of the Haar null classes is Haar null | | |
|---|---|---|---|
| | C | LC $\setminus$ C | NLC |
| $0$ | – | – | – |
| $n$ | $\mathbb{Z}_n$ | HNN | ??? |
| $\aleph_0$ | ??? | $\mathbb{Z}$ | $S_\infty$ |
| $\mathfrak{c}$ | – | – | $\mathrm{Aut}(\mathbb{Q},<)$; $\mathrm{Aut}(\mathcal{R})$ |
| | The union of the Haar null classes is not Haar null | | |
| | C | LC $\setminus$ C | NLC |
| $0$ | $2^\omega$ | $\mathbb{Z}\times 2^\omega$ | $\mathbb{Z}^\omega$ |
| $n$ | $\mathbb{Z}_n\times(\mathbb{Z}_2\ltimes_\phi\mathbb{Z}_3^\omega)$ | HNN $\times(\mathbb{Z}_2\ltimes_\phi\mathbb{Z}_3^\omega)$ | $\mathbb{Z}_n\times(\mathbb{Z}_2\ltimes_\phi\mathbb{Q}_d^\omega)$ |
| $\aleph_0$ | ??? | $\mathbb{Z}\times(\mathbb{Z}_2\ltimes_\phi\mathbb{Z}_3^\omega)$ | $S_\infty\times(\mathbb{Z}_2\ltimes_\phi\mathbb{Z}_3^\omega)$ |
| $\mathfrak{c}$ | – | – | $\mathrm{Aut}(\mathbb{Q},<)\times(\mathbb{Z}_2\ltimes_\phi\mathbb{Z}_3^\omega)$ |

TABLE 1. Examples of various behaviours

**Definition 4.1.** Let $G$ be a closed subgroup of $S_\infty$ and let $S\subset\omega$ be a finite subset. The *group-theoretic algebraic closure* of $S$ is:

$$\mathrm{ACL}(S)=\{x\in\omega : \text{the orbit of } x \text{ under } G_{(S)} \text{ is finite}\}.$$

Obviously $G$ has the finite algebraic closure property (see Definition 3.1) if and only if for every finite set $S$ the set $\mathrm{ACL}(S)$ is finite. We start with proving a simple observation about the operator ACL.

**Lemma 4.2.** *If a group $G$ has the $FACP$ then the corresponding operator* ACL *is idempotent.*

*Proof.* We have to show that for every finite set $S\subset\omega$ the identity $\mathrm{ACL}(\mathrm{ACL}(S))=\mathrm{ACL}(S)$ holds. Let $S\subset\omega$ be an arbitrary finite set and let $x\in\mathrm{ACL}(\mathrm{ACL}(S))$. We will show that $x$ has a finite orbit under $G_{(S)}$ which implies $x\in\mathrm{ACL}(S)$.

It is enough to show that $G_{(S)}(x)$ is finite. Enumerate the elements of $\mathrm{ACL}(S)$ as $\{x_1,x_2,\dots,x_k\}$. The group $G_{(S)}$ acts on $\mathrm{ACL}(S)^k$ coordinate-wise. Under this group action the stabiliser of the tuple $(x_1,x_2,\dots,x_k)$ is $G_{(\mathrm{ACL}(S))}$. The Orbit-Stabiliser Theorem states that for any group action the index of the stabiliser of an element in the whole group is the same as the cardinality of its orbit. This yields that the index $[G_{(S)}:G_{(\mathrm{ACL}(S))}]$ is the same as the cardinality of the orbit of $(x_1,x_2,\dots,x_k)$. This orbit is finite because the whole space $\mathrm{ACL}(S)^k$ is finite. So $G_{(\mathrm{ACL}(S))}$ has finite index in $G_{(S)}$.

Let $g_1,g_2,\dots,g_n\in G_{(S)}$ be a left transversal for $G_{(\mathrm{ACL}(S))}$ in $G_{(S)}$, then $G_{(S)}=g_1G_{ACL(S)}\cup\dots\cup g_nG_{ACL(S)}$. Since $G_{(S)}(x)=g_1G_{(\mathrm{ACL}(S))}(x)\cup g_2G_{(\mathrm{ACL}(S))}(x)\cup\dots\cup g_nG_{(\mathrm{ACL}(S))}(x)$ is a finite union of finite sets, it must be finite. □

**Lemma 4.3.** *The operator* ACL *is translation invariant in the following sense: if $S\subset\omega$ is a finite set and $g\in G$ is an arbitrary permutation then*

$$\mathrm{ACL}(gS)=g\,\mathrm{ACL}(S).$$

*Proof.* Let $x \in \omega$ be an arbitrary element, then

$$\begin{aligned} &x \text{ and } y \text{ are in the same orbit under } G_{(S)} \Leftrightarrow \\ &\exists h \in G_{(S)} : h(y) = x \Leftrightarrow \exists h \in G_{(S)} : gh(y) = g(x) \Leftrightarrow \\ &\exists h \in G_{(S)} : ghg^{-1}(g(y)) = g(x) \Leftrightarrow \exists f \in G_{(gS)} : f(g(y)) = g(x) \Leftrightarrow \\ &g(x) \text{ and } g(y) \text{ are in the same orbit under } G_{(gS)}. \end{aligned}$$

So an element $x$ has finite orbit under $G_{(S)}$ if and only if $g(x)$ has finite orbit under $G_{(gS)}$. □

Now we describe a process to generate a probability measure on $G$, a closed subgroup of $S_\infty$ that has the $FACP$. This probability measure will witness that certain sets are Haar null (see Theorem 4.13).

Our random process will define a permutation $p \in G$ in stages. It depends on integer sequences $(M_i)_{i\in\omega}$ and $(N_i)_{i\in\omega}$ with $M_i, N_i \geq 1$.

We denote the partial permutation completed in stage $i$ by $p_i$. We start with $p_0 = \emptyset$ and maintain throughout the following Property (i) for every $i \geq 1$, and Properties (ii) and (iii) for $i \in \omega$:

(i) $p_{i-1} \subset p_i$,
(ii) $\mathrm{dom}(p_i)$ and $\mathrm{ran}(p_i)$ are finite sets such that $\mathrm{ACL}(\mathrm{dom}(p_i)) = \mathrm{dom}(p_i)$, $\mathrm{ACL}(\mathrm{ran}(p_i)) = \mathrm{ran}(p_i)$,
(iii) there is a permutation $g \in G$ that extends $p_i$.

Let $O_0, O_1, \ldots \subset \omega$ be a sequence of infinite sets with the property that for every finite set $F \subset \omega$ and every infinite orbit $O$ of $G_{(F)}$, the sequence $(O_i)_{i\in\omega}$ contains $O$ infinitely many times. It is easy to see that such a sequence exists, since there exists only countably many such finite sets $F$, and for each one, there exist only countably many orbits of $G_{(F)}$.

At stage $i \geq 1$, we proceed the following way. First suppose that $i$ is even. We now choose a set $S_i \subset \omega$ with $|S_i| = M_i$ such that $S_i \cap \mathrm{ran}(p_{i-1}) = \emptyset$. If $i \equiv 0 \pmod 4$ we require that $S_i$ contains the least $M_i$ elements of $\omega \setminus \mathrm{ran}(p_{i-1})$, and if $i \equiv 2 \pmod 4$ we require that $S_i$ contains the least $M_i$ elements of $O_{(i-2)/4} \setminus \mathrm{ran}(p_{i-1})$. Now we will extend $p_{i-1}$ to a partial permutation $p_i$ such that

$$\mathrm{ran}(p_i) = \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup S_i). \tag{1}$$

Let us enumerate the elements of $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup S_i) \setminus \mathrm{ran}(p_{i-1})$ as $(x_1, \ldots, x_j)$ such that if $x_1, \ldots, x_{k-1}$ are already chosen then we choose $x_k$ so that

$$\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \ldots, x_k\}) \text{ is minimal with respect to inclusion.} \tag{2}$$

**Claim 4.4.** *For every $1 \leq k \leq \ell \leq m \leq j$, if*

$$x_m \in \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \ldots, x_k\}) \tag{3}$$

*then*

$$x_\ell \in \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \ldots, x_{k-1}\} \cup \{x_m\}) \subset \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \ldots, x_k\}).$$

*Proof.* The last containment holds, using Lemma 4.2 and (3). If $\ell = m$ then there is nothing to prove. Now suppose towards a contradiction that there exists $\ell < m$ violating the statement of the claim, and suppose that $\ell$ is minimal with $k \leq \ell < m$ and

$$x_\ell \notin \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \ldots, x_{k-1}\} \cup \{x_m\}). \tag{4}$$

Using the minimality of $\ell$, $\{x_1, \dots, x_{\ell-1}\} \subset \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{k-1}\} \cup \{x_m\})$, thus an application of Lemma 4.2 and the fact that $k \leq \ell$ shows that $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{k-1}\} \cup \{x_m\}) = \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{\ell-1}\} \cup \{x_m\})$. By (4) it follows that $x_\ell \notin \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{\ell-1}\} \cup \{x_m\})$. Using this, the fact that $k \leq \ell$ and (3), $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{\ell-1}\} \cup \{x_m\}) \subsetneqq \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_\ell\})$ contradicting (2), since $x_\ell$ was chosen after $\{x_1, \dots, x_{\ell-1}\}$ to satisfy that $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_\ell\})$ is minimal. □

We will determine the preimages of $(x_1, x_2, \dots, x_j)$ in this order. Denote the partial permutations defined in these sub-steps by $p_{i,k}$ so that $\mathrm{ran}(p_{i,k}) = \mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_k\}$ for $k = 0, \dots, j$. If the first $k$ preimages are determined then there are two possibilities for $x_{k+1}$:

(a) The set of possible preimages of $x_{k+1}$ under $p_{i,k}$ is finite, that is, the set $\{g^{-1}(x_{k+1}) : g \in G, g \supset p_{i,k}\}$ is finite. Then choose one from them randomly with uniform distribution.
(b) The set of possible preimages of $x_{k+1}$ under $p_{i,k}$ is infinite. Then choose one from the smallest $N_i$ many possible values uniformly.

We note that the orbit of $x_k$ under the stabiliser $G_{(\mathrm{ran}(p_{i-1}))}$ is infinite because $x_k \notin \mathrm{ran}(p_{i-1}) = \mathrm{ACL}(\mathrm{ran}(p_{i-1}))$ so

$$\text{possibility (b) must occur for at least } x_1. \tag{5}$$

Let $p_i = p_{i,j}$. Properties (i) and (iii) obviously hold for $i$. Let $g \in G$ be a permutation with $g \supset p_i$. Now $\mathrm{ran}(p_i) = \mathrm{ACL}(\mathrm{ran}(p_i))$ using (1) and Lemma 4.2. Then $\mathrm{dom}(p_i) = g^{-1}\,\mathrm{ran}(p_i)$, hence using Lemma 4.3, $\mathrm{ACL}(\mathrm{dom}(p_i)) = \mathrm{ACL}(g^{-1}\,\mathrm{ran}(p_i)) = g^{-1}\,\mathrm{ACL}(\mathrm{ran}(p_i)) = g^{-1}\,\mathrm{ran}(p_i) = \mathrm{dom}(p_i)$, showing Property (ii). This concludes the case where $i$ is even.

If $i$ is odd we let $S_i \subset \omega$ be the set of the least $M_i$ elements of $\omega \setminus \mathrm{dom}(p_{i-1})$, if $i \equiv 1 \pmod 4$ and the least $M_i$ elements of $O_{(i-3)/4} \setminus \mathrm{dom}(p_{i-1})$, if $i \equiv 3 \pmod 4$. We extend $p_{i-1}$ to a partial permutation $p_i$ such that

$$\mathrm{dom}(p_i) = \mathrm{ACL}(\mathrm{dom}(p_{i-1}) \cup S_i). \tag{6}$$

Again, we enumerate the elements of $\mathrm{ACL}(\mathrm{dom}(p_{i-1}) \cup S_i) \setminus \mathrm{dom}(p_{i-1})$ as $(x_1, \dots, x_j)$ such that if $x_1, \dots, x_{k-1}$ are already chosen then we choose $x_k$ from the rest so that $\mathrm{ACL}(\mathrm{dom}(p_{i-1}) \cup \{x_1, \dots, x_k\})$ is minimal with respect to inclusion. The proof of the following claim is analogous to the proof of Claim 4.4.

**Claim 4.5.** *For every* $1 \leq k \leq \ell \leq m \leq j$, $x_m \in \mathrm{ACL}(\mathrm{dom}(p_{i-1}) \cup \{x_1, \dots, x_k\})$ *implies* $x_\ell \in \mathrm{ACL}(\mathrm{dom}(p_{i-1}) \cup \{x_1, \dots, x_{k-1}\} \cup \{x_m\}) \subset \mathrm{ACL}(\mathrm{dom}(p_{i-1}) \cup \{x_1, \dots, x_k\})$.

We determine the images of $(x_1, x_2, \dots, x_j)$ in this order. Denote the partial permutations defined in these sub-steps by $p_{i,k}$ so that $\mathrm{dom}(p_{i,k}) = \mathrm{dom}(p_{i-1}) \cup \{x_1, \dots, x_k\}$ for $k = 0, \dots, j$. If the first $k$ images are determined then there are two possibilities for $x_{k+1}$:

(a) The set of possible images of $x_{k+1}$ under $p_{i,k}$ is finite, that is, the set $\{g(x_{k+1}) : g \in G, g \supset p_{i,k}\}$ is finite. Then choose one from them randomly with uniform distribution.
(b) The set of possible images of $x_{k+1}$ under $p_{i,k}$ is infinite. Then choose one from the smallest $N_i$ many possible values uniformly.

Again, the orbit of $x_k$ under the stabiliser $G_{(\mathrm{dom}(p_{i-1}))}$ is infinite because $x_k \notin \mathrm{dom}(p_{i-1}) = \mathrm{ACL}(\mathrm{dom}(p_{i-1}))$ for every $k$, so possibility (b) must occur for at least $x_1$.

Let $p_i = p_{i,j}$. Again, Properties (i) and (iii) hold for $i$. Let $g \in G$ be a permutation with $g \supset p_i$. Now $\mathrm{dom}(p_i) = \mathrm{ACL}(\mathrm{dom}(p_i))$ using (6) and Lemma 4.2. Then using Lemma 4.3, $\mathrm{ACL}(\mathrm{ran}(p_i)) = \mathrm{ACL}(g\,\mathrm{dom}(p_i)) = g\,\mathrm{ACL}(\mathrm{dom}(p_i)) = g\,\mathrm{dom}(p_i) = \mathrm{ran}(p_i)$, showing Property (ii). This concludes the construction for odd $i$.

Now let $p = \bigcup_i p_i$. This makes sense using (i).

**Claim 4.6.** *$p \in G$.*

*Proof.* First we show that $p \in S_\infty$. Using (iii), each $p_i$ is a partial permutation, hence injective. Using (i), $p$ is the union of compatible injective functions, hence $p$ is an injective function. It is clear from the construction that $\{0, 1, \dots, i-1\} \subset \mathrm{dom}(p_{4i}) \cap \mathrm{ran}(p_{4i})$ for every $i$, hence $p \in S_\infty$.

Using (iii), we can find an element $g_i \in G$ such that $g_i \supset p_i$. It is clear that $g_i \to p$, and since $G$ is a closed subgroup of $S_\infty$, we conclude that $p \in G$. □

The following lemma is crucial in proving that almost every element of $G$ has finitely many finite and infinitely many infinite orbits.

**Lemma 4.7.** *Suppose that the parameters of the random process $M_1, \dots, M_i$ and $N_1, \dots, N_{i-1}$ are given along with the numbers $K \in \omega$ and $\varepsilon > 0$. Then we can choose $N_i$ so that for every set $S \subset \omega$ with $|S| = K$, the probability that $S \cap (\mathrm{dom}(p_i) \setminus \mathrm{dom}(p_{i-1})) \neq \emptyset$ if $i$ is even, or that $S \cap (\mathrm{ran}(p_i) \setminus \mathrm{ran}(p_{i-1})) \neq \emptyset$ if $i$ is odd, is at most $\varepsilon$.*

*Proof.* We suppose that $i$ is even and prove the lemma only in this case. The proof for the case when $i$ is odd is analogous.

One can easily see using induction on $i$ that if $M_1, \dots, M_{i-1}$ and $N_1, \dots, N_{i-1}$ are given then the random process can yield only finitely many different $p_{i-1}$ as a result.

Let $p_{i-1}$ be one of the possible outcomes, and let $(x_1, x_2, \dots, x_j)$ denote the elements of $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup S_i) \setminus \mathrm{ran}(p_{i-1})$ enumerated in the same order as they appear during the construction. Note that this only depends on $p_{i-1}$ and $M_i$. Let $a_1$ be the index for which $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1\}) = \mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{a_1}\}$, such an index exists using Claim 4.4. Hence, for every $m \leq a_1$, $x_m \in \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1\})$, thus using Claim 4.4 again, it follows that

$$x_1 \in \mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_m\}) \text{ for every } 1 \leq m \leq a_1. \tag{7}$$

**Claim 4.8.** *For every such $m$, there is a unique positive integer $k_m$ such that if $q$ is an extension of $p_{i-1}$ with $\mathrm{ran}(q) = \mathrm{ran}(p_{i-1}) \cup \{x_m\}$ (such that $q \subset g$ for some $g \in G$) then $|\{g^{-1}(x_1) : g \in G, g \supset q\}| = k_m$.*

*Proof.* Let $H = G_{(\mathrm{ran}(p_{i-1}) \cup x_m)}$, then

$$k = |\{g(x_1) : g \in H\}| = |\{g^{-1}(x_1) : g \in H\}| \tag{8}$$

is finite using (7) and the fact that $H$ is a subgroup. It is enough to show that if $q$ is an extension of $p_{i-1}$ with $\mathrm{ran}(q) = \mathrm{ran}(p_{i-1}) \cup \{x_m\}$ then $|\{g^{-1}(x_1) : g \in G, g \supset q\}| = k$.

Let $g_1, \dots, g_k \in H$ with $g_\ell^{-1}(x_1) \neq g_n^{-1}(x_1)$ if $\ell \neq n$. If $h \in G$ is a permutation with $h \supset q$ then $g_n h \supset q$ for every $1 \leq n \leq k$. Then using the identity $(g_n h)^{-1}(x_1) = h^{-1}(g_n^{-1}(x_1))$, $(g_\ell h)^{-1}(x_1) \neq (g_n h)^{-1}(x_1)$ if $\ell \neq n$. This shows that $|\{g^{-1}(x_1) : g \in G, g \supset q\}| \geq k$.

To prove the other inequality, suppose towards a contradiction that there exist $g_1, \dots, g_{k+1}$ with $g_n \supset q$ for every $n \leq k+1$ and $g_\ell^{-1}(x_1) \neq g_n^{-1}(x_1)$ for every $\ell \neq n$. It is easy to see that $g_n g_1^{-1} \in H$ for every $n$, but the values $(g_n g_1^{-1})^{-1}(x_1) = g_1(g_n^{-1}(x_1))$ are pairwise distinct, contradicting (8). Thus the proof of the claim is complete. □

Now let $k = \max\{k_2, k_3, \dots, k_{a_1}\}$, if $a_1 \geq 2$, otherwise let $k = 1$.

**Claim 4.9.** *If $N_i > \frac{kKj}{\varepsilon}$ then for every fixed set $S \subset \omega$ with $|S| = K$ we have $\mathbb{P}(p_i^{-1}(x_m) \in S) < \frac{\varepsilon}{j}$ for every $1 \leq m \leq a_1$.*

*Proof.* This is immediate for $m = 1$, since $k \geq 1$, and the preimage of $x_1$ is chosen from $N_i$ many elements using (5). Now let $m > 1$, using Claim 4.8 and the fact that $k \geq k_m$, it follows that for every $y \in \omega$, $|\{g^{-1}(x_1) : g \in G, g \supset p_{i-1}, g(y) = x_m\}| \leq k$, hence for the set $R = \{g^{-1}(x_1) : g \in G, g \supset p_{i-1}, g^{-1}(x_m) \in S\}$, $|R| \leq kK$. In order to be able to extend $p_{i-1}$ to $p_i$ with $p_i^{-1}(x_m) \in S$, we need to choose $p_i^{-1}(x_1)$ from $R$. Since during the construction of the random automorphism, $p_i^{-1}(x_1)$ is chosen uniformly from a set of size $N_i > \frac{kKj}{\varepsilon}$, we conclude that $\mathbb{P}(p_i^{-1}(x_m) \in S) \leq \mathbb{P}(p_i^{-1}(x_1) \in R) \leq \frac{|R|}{N_i} < \frac{\varepsilon}{j}$. □

For the rest of the proof, we need to repeat the above argument until we reach $j$. If $a_1 < j$, let $a_2$ be the index satisfying $\mathrm{ACL}(\mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{a_1+1}\}) = \mathrm{ran}(p_{i-1}) \cup \{x_1, \dots, x_{a_2}\}$, such an index exists using Claim 4.4 as before. Again, we can set a lower bound for $N_i$ so that the for every $a_1 < m \leq a_2$, $\mathbb{P}(p_i^{-1}(x_m) \in S) < \frac{\varepsilon}{j}$. Repeating the argument, we can choose $N_i$ so that $\mathbb{P}(p_i^{-1}(x_m) \in S) < \frac{\varepsilon}{j}$ for every $1 \leq m \leq j$, thus $\mathbb{P}(p_i^{-1}(\{x_1, \dots, x_j\}) \cap S \neq \emptyset) < \varepsilon$. Completing the proof of the lemma. □

**Proposition 4.10.** *Let $G \leq S_\infty$ be a closed subgroup. If $G$ has the $FACP$ then the sets*

$$\mathcal{F} = \{g \in G : g \text{ has finitely many finite orbits}\},$$
$$\mathcal{C} = \{g \in G : \forall F \subset \omega \text{ finite } \forall x \in \omega \text{ (if } G_{(F)}(x) \text{ is infinite then it is not covered by finitely many orbits of } g)\}$$

*are co-Haar null.*

*Proof.* We first show the following lemma.

**Lemma 4.11.** *The sets $\mathcal{F}$ and $\mathcal{C}$ are conjugacy invariant Borel sets.*

*Proof.* The fact that $\mathcal{F}$ is conjugacy invariant follows form the fact that conjugation does not change the orbit structure of a permutation.

To show that $\mathcal{C}$ is conjugacy invariant, let $c \in \mathcal{C}$, $h \in G$, we need to show that $h^{-1}ch \in \mathcal{C}$. Suppose towards a contradiction that $h^{-1}ch \notin \mathcal{C}$, that is, for some finite $F \subset \omega$ and $x_0 \in \omega$ the orbit $G_{(F)}(x_0)$ is covered by finitely many orbits of $h^{-1}ch$. This means that there exist finitely many elements $y_0, \dots, y_{n-1} \in \omega$ such that

$$G_{(F)}(x_0) \subset \{(h^{-1}ch)^k(y_i) : k \in \mathbb{Z}, i < n\}. \tag{9}$$

Let $F' = \{h(x) : x \in F\}$, it is easy to see that

$$g \in G_{(F)} \Leftrightarrow hgh^{-1} \in G_{(F')}, \tag{10}$$

moreover, since the conjugating map $g \mapsto hgh^{-1}$ is bijective,

$$\text{every element of } G_{(F')} \text{ is of the form } hgh^{-1} \text{ for some } g \in G_{(F)}. \tag{11}$$

To get a contradiction with the fact that $c \in \mathcal{C}$ we show that the orbit

$$G_{(F')}(h(x_0)) \text{ is infinite} \tag{12}$$

but is covered by finitely many orbits of $c$, in fact,

$$G_{(F')}(h(x_0)) \subset \{c^k(h(y_i)) : k \in \mathbb{Z}, i < n\}. \tag{13}$$

Showing this would be enough to prove that $\mathcal{C}$ is conjugacy invariant.

Using that the orbit $G_{(F)}(x_0)$ is infinite, we can choose permutations $(g_k)_{k\in\omega} \subset G_{(F)}$ such that

$$g_i(x_0) \neq g_j(x_0) \text{ if } i \neq j. \tag{14}$$

Then, using (10), $hg_ih^{-1} \in G_{(F')}$ for every $i \in \omega$ and if $hg_ih^{-1}(h(x_0)) = hg_jh^{-1}(h(x_0))$ then $hg_i(x_0) = hg_j(x_0)$, hence, using that $h$ is a permutation, $g_i(x_0) = g_j(x_0)$, thus $i = j$, using (14). This shows (12).

To prove (13), let us choose a permutation from $G_{(F')}$. Using (11), we can suppose that this permutation is of the form $hgh^{-1}$ for some $g \in G_{(F)}$. We need to show that

$$hgh^{-1}(h(x_0)) \in \{c^k(h(y_i)) : k \in \mathbb{Z}, i < n\}. \tag{15}$$

Using (9), $g(x_0) = (h^{-1}ch)^k(y_i)$ for some $k \in \mathbb{Z}$ and $i < n$. Then $g(x_0) = h^{-1}c^kh(y_i)$, hence $hg(x_0) = c^kh(y_i)$, proving (15) and also (13). Thus the proof of the conjugacy invariance of $\mathcal{C}$ is complete.

To show that $\mathcal{F}$ is Borel, notice that the set of permutations containing a given finite orbit is open for every finite orbit. Thus for any finite set of finite orbits the set of permutations containing those finite orbits in their orbit decompositions is open: it can be obtained as the intersection of finitely many open sets. Thus for every $n \in \omega$ the set of permutations containing at least $n$ finite orbits is open: it can be obtained as the union of open sets (one open set for each possible set of $n$ orbits). Thus $S_\infty \setminus \mathcal{F}$ is $G_\delta$: it is the intersection of the above open sets. Hence $\mathcal{F}$ is Borel.

Now we show that $\mathcal{C}$ is also Borel. It is enough to show that if $H \subset \omega$ is arbitrary then the set $H^* = \{g \in G : \text{finitely many orbits of } g \text{ cannot cover } H\}$ is Borel, since $\mathcal{C}$ can be written as the countable intersection of such sets. And $H^*$ can be easily seen to be Borel for any $H$, since its complement, $\{g \in G : \exists n\ \forall m \in H\ \exists k\ \exists i < n\ (g^k(i) = m)\}$ is $G_{\delta\sigma}$, hence $H^*$ is $F_{\sigma\delta}$. □

To prove the proposition, we use the above construction to generate a random permutation $p$. We set $M_i = 2^i$ for every $i \geq 1$ and we define $(N_i)_{i\geq 1}$ recursively. If $N_1, \dots, N_{i-1}$ are already defined, then, as before, the random process can yield only finitely many distinct $p_{i-1}$. Hence, there is a bound $m_i$ depending only on $N_1, \dots, N_{i-1}$ such that $|\operatorname{dom}(p_i)| = |\operatorname{ran}(p_i)| \leq m_i$, since $|\operatorname{ran}(p_i)| = |\operatorname{ACL}(\operatorname{ran}(p_{i-1}) \cup S_i)|$ if $i$ is even and $|\operatorname{dom}(p_i)| = |\operatorname{ACL}(\operatorname{dom}(p_{i-1}) \cup S_i)|$ if $i$ is odd, which is independent of $N_i$. Now we use Lemma 4.7 to choose $N_i$ so that the conclusion of the lemma is true with $K = m_i$ and $\varepsilon = \frac{1}{2^i}$.

Using Lemma 2.5 and the fact that the sets $\mathcal{F}$ and $\mathcal{C}$ are conjugacy invariant, it is enough to show that

$$\mathbb{P}(ph \text{ has finitely many finite orbits}) = 1 \tag{16}$$

and

$$\mathbb{P}(\text{finitely many orbits of } ph \text{ do not cover } O) = 1 \tag{17}$$

for every $h \in G$, every finite $F \subset \omega$ and every infinite orbit $O$ of $G_{(F)}$, since there exist only countably many such orbits. So let us fix $h \in G$ and an infinite orbit $O \subset \omega$ of $G_{(F)}$ for some finite $F \subset \omega$ for the rest of the proof.

For a partial permutation $q$, a *partial path* in $q$, is a sequence $(y, q(y), \dots, q^n(y))$ with $n \geq 1$, $q^n(y) \notin \operatorname{dom}(q)$ and $y \notin \operatorname{ran}(q)$. Note that $p_i h$ is considered a partial permutation with $\operatorname{dom}(p_i h) = h^{-1}(\operatorname{dom}(p_i))$ and $\operatorname{ran}(p_i h) = \operatorname{ran}(p_i)$.

During the construction of the random permutation, an *event* occurs when the partial permutation is extended to a new element at some stage regardless of whether it happens for possibility (a) or (b). Suppose that during an event, the partial permutation $p'$ is extended to $p'' = p' \cup (x, y)$. We call this event *bad* if the number of partial paths decreases or $h^{-1}(x) = y$. Note that an event is bad if the extension connects two partial paths of $p'h$ or it completes an orbit (possibly a fixed point).

**Claim 4.12.** *Almost surely, only finitely many bad events happen.*

*Proof.* Let $i$ be fixed and suppose first that it is even. It is easy to see that a bad event can only happen at stage $i$ if a preimage is chosen from $h(\operatorname{ran}(p_{i-1}))$, that includes the case when a fixed point is constructed. Note that $|\operatorname{ran}(p_i)| \leq m_i$, thus the probability of choosing a preimage from this set is at most $\frac{1}{2^i}$, using Lemma 4.7.

We proceed similarly if $i$ is odd. Then to connect partial paths or complete orbits, an image has to be chosen from the set $h^{-1}(\operatorname{dom}(p_{i-1}))$. Since $|\operatorname{dom}(p_i)| \leq m_i$, the probability of choosing from this set is at most $\frac{1}{2^i}$.

Using the Borel–Cantelli lemma, the number of $i$ such that a bad event happens at stage $i$ is finite almost surely. The fact that only a finite number of bad events can happen at a particular stage completes the proof of the claim. □

Since a finite orbit can only be created during a bad event, (16) follows immediately from the claim. Thus $\mathcal{F}$ is co-Haar null.

Now we prove that $\mathcal{C}$ is also co-Haar null by showing (17). Let $n_0, n_1, \ldots \in \omega$ be a sequence with $n_0 < n_1 < \ldots$ and $O_{n_i} = O$ for every $i \in \omega$. Let $c_i$ be the number of partial paths of $p_{4n_i+2}h$ intersecting $O$. It is enough to show that the sequence $(c_i)_{i\in\omega}$ is unbounded almost surely, since using Claim 4.12, only finitely many of such partial paths can be connected in later stages, hence infinitely many orbits of $ph$ will intersect $O$, almost surely.

At stage $4n_i + 2$, $p_{4n_i+1}$ is extended to $p_{4n_i+2}$ with $\operatorname{ran}(p_{4n_i+2}) \setminus \operatorname{ran}(p_{4n_i+1}) \supset S_{4n_i+2}$, $|S_{4n_i+2}| = M_{4n_i+2} = 2^{4n_i+2}$ and $S_{4n_i+2} \subset O_{(4n_i+2-2)/4} = O_{n_i} = O$. Hence, it is enough to prove that apart from a finite number of exceptions, the elements of $\operatorname{ran}(p_{4n_i+2}) \setminus \operatorname{ran}(p_{4n_i+1})$ are in different partial paths in $p_{4n_i+2}h$, almost surely.

The proof of this fact is similar to the proof of Claim 4.12. An element $y \in O \cap (\operatorname{ran}(p_{4n_i+2}) \setminus \operatorname{ran}(p_{4n_i+1}))$ can only be contained in a completed orbit (of $p_{4n_i+2}h$), if $h^{-1}p_{4n_i+2}^{-1}(y) \in \operatorname{ran}(p_{4n_i+2})$, hence $p_{4n_i+2}^{-1}(y) \in h(\operatorname{ran}(p_{4n_i+2}))$. Similarly, if $y, y' \in O \cap (\operatorname{ran}(p_{4n_i+2}) \setminus \operatorname{ran}(p_{4n_i+1}))$ are in the same partial path (in $p_{4n_i+2}h$)

such that $y$ is the not the first element of this path, then $p^{-1}_{4n_i+2}(y) \in h(\mathrm{ran}(p_{4n_i+2}))$. Again using Lemma 4.7, the probability of this happening at stage $4n_i + 2$ is at most $\frac{1}{2^{4n_i+2}}$, since $|\,\mathrm{ran}(p_{4n_i+2})| \leq m_{4n_i+2}$. As before, the application of the Borel–Cantelli lemma completes the proof of (17). And thus the proof of the proposition is also complete. □

**Theorem 4.13.** *Let $G \leq S_\infty$ be a closed subgroup. If $G$ has the $FACP$ then the sets*

$$\mathcal{F} = \{g \in G : g \text{ has finitely many finite orbits}\},$$
$$\mathcal{I} = \{g \in G : g \text{ has infinitely many infinite orbits}\}$$

*are both co-Haar null. Moreover, if $\mathcal{F}$ is co-Haar null then $G$ has the $FACP$.*

*Proof.* The fact that $\mathcal{F}$ is co-Haar null follows immediately from Proposition 4.10. Let $\mathcal{C}$ denote the set as in Proposition 4.10. If $g \in \mathcal{C}$ then $g$ contains infinitely many orbits, since otherwise finitely many orbits of $g$ could cover $\omega$, hence every infinite orbit of $G_{(F)}$ for some finite $F \subset \omega$. It follows that the co-Haar null set $\mathcal{C} \cap \mathcal{F}$ is contained in $\mathcal{I}$, hence $\mathcal{I}$ is also co-Haar null. And thus the proof of the first part of the theorem is complete.

Now we prove the second assertion. We have to show that if $G$ does not have the $FACP$ then $\mathcal{F}$ is not co-Haar null. If $G$ does not have the $FACP$ then there is a finite set $S \subset \omega$ such that $\mathrm{ACL}(S)$ is infinite. This means that all of the permutations in $G_{(S)}$ have infinitely many finite orbits, hence $G_{(S)} \cap \mathcal{F} = \emptyset$. The stabiliser $G_{(S)}$ is a non-empty open set, thus it cannot be Haar null. Therefore the proof of the theorem is complete. □

Now we are ready to prove the main result of this section.

**Theorem 4.14.** *Let $\mathcal{A}$ be a locally finite Fraïssé limit. Then the following are equivalent:*

*(1) almost every element of* $\mathrm{Aut}(\mathcal{A})$ *has finitely many finite orbits,*
*(2)* $\mathrm{Aut}(\mathcal{A})$ *has the FACP,*
*(3) $\mathcal{A}$ has the CSAP.*

*Moreover, any of the above conditions implies that almost every element of $\mathcal{A}$ has infinitely many infinite orbits.*

*Proof.* The equivalence (1) $\iff$ (2), and the last statement of the theorem is just the application of Theorem 4.13 to $G = \mathrm{Aut}(\mathcal{A})$. Thus, it is enough to show that (2) $\iff$ (3).

Let $\mathcal{K} = age(\mathcal{A})$. Since $\mathcal{A}$ is the limit of $\mathcal{K}$, using that $\mathcal{A}$ is ultrahomogeneous it follows that $\mathcal{K}$ has the extension property, that is, for every $\mathcal{B}, \mathcal{C} \in \mathcal{K}$ and embeddings $\phi : \mathcal{B} \to \mathcal{C}$ and $\psi : \mathcal{B} \to \mathcal{A}$ there exists an embedding $\psi' : \mathcal{C} \to \mathcal{A}$ with $\psi' \circ \phi = \psi$. Thus, the embeddings between the structures in $\mathcal{K}$ can be considered as partial automorphisms of $\mathcal{A}$.

((2) $\Rightarrow$ (3)) Take an arbitrary $\mathcal{B}_0 \in \mathcal{K}$ and fix an isomorphic copy of it inside $\mathcal{A}$. Let $\mathcal{B} = \mathrm{ACL}(\mathrm{dom}(\mathcal{B}_0))$ and note that by the fact that $\mathrm{Aut}(\mathcal{A})$ has the $FACP$ $\mathcal{B}$ is a finite substructure of $\mathcal{A}$. We will show that over $\mathcal{B}$ the strong amalgamation property holds (see Definition 3.2). In order to see this, let $\mathcal{C}, \mathcal{D} \in \mathcal{K}$ and let $\psi : \mathcal{B} \to \mathcal{C}$ and $\phi : \mathcal{B} \to \mathcal{D}$ be embeddings. By the extension property we can suppose that $\mathcal{B} < \mathcal{C} < \mathcal{A}$, $\mathcal{B} < \mathcal{D} < \mathcal{A}$ and $\psi = \phi = id_{\mathcal{B}}$. By Lemma 4.2 $\mathrm{ACL}(\mathrm{dom}(\mathcal{B})) = \mathcal{B}$, hence the $\mathrm{Aut}(\mathcal{A})_{(\mathrm{dom}(\mathcal{B}))}$ orbit of every point in $\mathrm{dom}(\mathcal{C}) \setminus \mathrm{dom}(\mathcal{B})$ is infinite. By

M. Neumann's Lemma [10, Corollary 4.2.2.] $\mathrm{dom}(\mathcal{C}) \setminus \mathrm{dom}(\mathcal{B})$ has infinitely many pairwise disjoint copies under the action of $\mathrm{Aut}(\mathcal{A})_{(\mathrm{dom}(\mathcal{B}))}$. In particular, by the pigeonhole principle, there exists an $f \in \mathrm{Aut}(\mathcal{A})_{(\mathrm{dom}(\mathcal{B}))}$ such that $f(\mathcal{C}) \cap \mathcal{D} = \mathcal{B}$. Letting $\mathcal{E}$ to be the substructure of $\mathcal{A}$ generated by $\mathrm{dom}(f(\mathcal{C})) \cup \mathrm{dom}(\mathcal{D})$, $\psi' = f|_{\mathcal{C}}$ and $\phi' = \mathrm{id}_{\mathcal{D}}$ shows that SAP holds over $\mathcal{B}$ and hence CSAP holds as well.

((2) $\Leftarrow$ (3)) Let $S \subset \mathrm{dom}(\mathcal{A})$ be finite. Let $\mathcal{B}_0$ be the substructure generated by $S$. Clearly, $\mathcal{B}_0 \in \mathcal{K}$, hence there exists a $\mathcal{B} \in \mathcal{K}$ over which the strong amalgamation property holds and which contains an isomorphic copy of $\mathcal{B}_0$. By the extension property of $\mathcal{A}$ we can suppose that $\mathcal{B}$ and all the structures constructed later on in this part of the proof are substructures of $\mathcal{A}$ containing $\mathcal{B}_0$.

We claim that for every $b \in \mathrm{dom}(\mathcal{A}) \setminus \mathrm{dom}(\mathcal{B})$ the orbit $\mathrm{Aut}(\mathcal{A})_{(\mathrm{dom}(\mathcal{B}))}(b)$ is infinite. Indeed, let $\mathcal{C}$ be the substructure generated by $\mathrm{dom}(\mathcal{B}) \cup \{b\}$. Using the strong amalgamation property repeatedly, first for $\mathcal{B}, \mathcal{C}$ and $\mathcal{D} = \mathcal{C}$ obtaining an $\mathcal{E}_1$, then for $\mathcal{B}, \mathcal{C}$ and $\mathcal{D} = \mathcal{E}_1$ obtaining an $\mathcal{E}_2$ etc. for every $n$ we can find a substructure $\mathcal{E}_n$ of $\mathcal{A}$ which contains $n+1$ isomorphic copies of $\mathcal{C}$ which intersect only in $\mathcal{B}$, and the isomorphisms between these copies fix $\mathcal{B}$. Extending the isomorphisms to automorphisms of $\mathrm{Aut}(\mathcal{A})$ shows that the orbit $\mathrm{Aut}(\mathcal{A})_{(\mathrm{dom}(\mathcal{B}))}(b)$ is infinite. □

**Remark 4.15.** It is not hard to construct countable Fraïssé classes to show that CSAP is neither equivalent to SAP, nor to AP. An example showing that CSAP $\not\Rightarrow$ SAP is $age(\mathcal{B}_\infty)$. Indeed, using a result of Schmerl [15] that states that a Fraïssé class has the SAP if and only if its automorphism group has no algebraicity (that is, $\mathrm{ACL}(F) = F$ for every finite $F$), $age(\mathcal{B}_\infty)$ cannot have the SAP.

To see that AP $\not\Rightarrow$ CSAP, let $\mathcal{Z}$ be the structure on the set $\mathbb{Z}$ of integers with a relations $R_n$ for each $n \geq 1$, $n \in \mathbb{N}$ satisfying that $aR_nb \Leftrightarrow |a-b| = n$ for each $a, b \in \mathbb{Z}$ and $n \geq 1$. It can be easily checked that $age(\mathcal{Z})$ satisfies AP, but $\mathrm{Aut}(\mathcal{Z})$ does not satisfy FACP, since the algebraic closure of any two points is $\mathbb{Z}$. Thus Theorem 4.14 implies that $\mathcal{Z}$ cannot satisfy CSAP.

## 5. $\mathrm{Aut}(\mathbb{Q}, <)$

5.1. **Christensen's theorem revisited.** We will need a straightforward generalisation of a theorem proved by Christensen [3], here we reiterate Rosendal's proof (see [14]).

**Theorem 5.1.** *(Christensen) Let $A \subset G$ be a conjugacy invariant set and suppose that there exists a cover of $A$ by Borel sets $A = \bigcup_{n\in\omega} A_n$ (in particular, $A$ is also a Borel set) and a conjugacy invariant set $B$ so that $1 \in \overline{B}$ and $B \cap \bigcup_{n\in\omega} A_n^{-1}A_n = \emptyset$. Then $A$ is Haar null.*

**Remark 5.2.** Some authors use a definition for Haar null sets which slightly differs from Definition 1.1. Namely, according to that version, a set $S$ is Haar null, if there exists a Borel probability measure $\mu$ on $G$ and a *universally measurable set* $U$ such that $S \subset U$ and for every $g, h \in G$ we have $\mu(gUh) = 0$. These two notions differ in general (see [6]), although they coincide for analytic sets (see [16]). We would like to point out that the above theorem and Corollary 5.3 remain true (and can be proved in the same way) if we change Borel to universally measurable everywhere and use the mentioned alternative definition of Haar null sets.

*Proof of Theorem 5.1.* We claim that there exists a sequence $\{g_i : i \in \omega\} \subset B$ with $g_i \to 1$ and the following properties:

- for every $(\varepsilon_i)_{i\in\omega} \in 2^\omega$ we have that the sequence $(g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n})_{n\in\omega}$ converges,
- the map $\phi : 2^\omega \to G$ defined by $(\varepsilon_n)_{n\in\omega} \mapsto g_0^{\varepsilon_0} g_1^{\varepsilon_1} g_2^{\varepsilon_2} \dots$ is continuous (the right hand side expression makes sense because of the convergence).

We can choose such a sequence by induction: fix a compatible complete metric and suppose that we have already selected $g_0, g_1, \dots, g_n$. Now notice that for every $(\varepsilon_0, \dots, \varepsilon_n) \in 2^{n+1}$ the set $\{x \in G : d(g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n} x, g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n}) < 2^{-n-1}\}$ contains a neighbourhood of the identity. Therefore we can choose a

$$g_{n+1} \in B \cap \bigcap_{(\varepsilon_0,\dots,\varepsilon_n)\in 2^{n+1}} \{x \in G : d(g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n} x, g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n}) < 2^{-n-1}\}.$$

One can easily show that for every $(\varepsilon_n)_{n\in\omega} \in 2^\omega$ the sequence $(g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n})_{n\in\omega}$ is Cauchy and the function $\phi$ is continuous.

Let $\lambda$ be the usual measure on $2^\omega$ and let $\lambda_* = \phi_*\lambda$, its push forward. We claim that $\lambda_*$ witnesses that $A$ is left-Haar null which is equivalent to its Haar nullness, by the fact that $A$ is conjugacy invariant and by using Lemma 2.5.

Suppose not, then there exists an $f \in G$ so that $\lambda_*(fA) > 0$, therefore $\lambda_*(fA_k) > 0$ for some $k \in \omega$. This is equivalent to $\lambda(\phi^{-1}(fA_k)) > 0$ and if we regard $2^\omega$ as $\mathbb{Z}_2^\omega$, by Weil's theorem (see e.g. [14]) we have that $\phi^{-1}(fA_k) - \phi^{-1}(fA_k)$ contains a neighbourhood of $(0, 0, \dots)$, the identity in $\mathbb{Z}_2^\omega$. Then there exists an element in $\phi^{-1}(fA_k) - \phi^{-1}(fA_k)$ which is zero at every coordinate except for one. Thus, $\phi^{-1}(fA_k)$ contains two elements of the form $(\varepsilon_0, \dots, \varepsilon_{n-1}, 0, \varepsilon_{n+1}, \dots)$ and $(\varepsilon_0, \dots, \varepsilon_{n-1}, 1, \varepsilon_{n+1}, \dots)$, i. e., differing at exactly one place. Then taking the $\phi$ images of these elements we obtain that there exist $h_1, h_2 \in G$ so that $h_1 h_2 \in fA_k$ and $h_1 g_n h_2 \in fA_k$. This implies

$$h_2^{-1} h_1^{-1} h_1 g_n h_2 \in A_k^{-1} A_k$$

thus

$$h_2^{-1} g_n h_2 \in A_k^{-1} A_k$$

but by the conjugacy invariance of $B$ we get

$$h_2^{-1} g_n h_2 \in B \cap A_k^{-1} A_k,$$

contradicting the initial assumptions of the theorem.

□

Letting $A = A_n$ for every $n \in \omega$ and using that if $A$ is conjugacy invariant then so is $G \setminus A^{-1}A$ we can deduce the following corollary:

**Corollary 5.3.** *If $A$ is a conjugacy invariant Borel set that is not Haar null then $A^{-1}A$ contains a neighbourhood of the identity.*

5.2. **Characterisation of the non-Haar null conjugacy classes of** $\mathrm{Aut}(\mathbb{Q}, <)$**.** In this section we prove the following result:

**Theorem 5.4.** *For almost every element $f$ of* $\mathrm{Aut}(\mathbb{Q}, <)$

*(1) for distinct orbitals $O_1, O_2 \in \mathcal{O}_f^*$ (see Definition 2.2) with $O_1 < O_2$ such that $s_f(O_1) = s_f(O_2) = 1$ or $s_f(O_1) = s_f(O_2) = -1$, there exists an orbital $O_3 \in \mathcal{O}_f^*$ with $O_1 < O_3 < O_2$ and $s_f(O_3) \neq s_f(O_1)$,*

*(2) (from Theorem 4.14) $f$ has only finitely many fixed points.*

*These properties characterise the non-Haar null conjugacy classes, i. e., a conjugacy class is non-Haar null if and only if one (or equivalently each) of its elements has properties* (1) *and* (2).

*Moreover, every non-Haar null conjugacy class is compact biter and those non-Haar null classes in which the elements have no rational fixed points are compact catchers.*

We split the proof into two parts, proving the following two theorems separately.

**Theorem 5.5.** *The conjugacy class of $f \in \mathrm{Aut}(\mathbb{Q}, <)$ is non-Haar null if and only if $\mathrm{Fix}(f)$ is finite, and for distinct orbitals $O_1, O_2 \in \mathcal{O}_f^*$ with $O_1 < O_2$ such that $s_f(O_1) = s_f(O_2) = 1$ or $s_f(O_1) = s_f(O_2) = -1$, there exists an orbital $O_3 \in \mathcal{O}_f^*$ with $O_1 < O_3 < O_2$ and $s_f(O_3) \neq s_f(O_1)$.*

*In fact, every non-Haar null conjugacy class is compact biter and those non-Haar null classes in which the elements have no fixed points are compact catchers.*

**Theorem 5.6.** *The union of the Haar null conjugacy classes of $\mathrm{Aut}(\mathbb{Q}, <)$ is Haar null.*

We say that an automorphism is *good* if it satisfies the conditions of Theorem 5.4. In the proof of Theorems 5.5 and 5.6, we use the following lemma to check conjugacy between good automorphisms.

**Lemma 5.7.** *Let $f$ and $g$ be good automorphisms. Suppose that there exists a function $\phi : \mathbb{Q} \to \mathcal{O}_f^*$ with the following properties: it is monotonically increasing (not necessarily strictly), surjective, $|\phi^{-1}(p)| = 1$ for every $p \in \mathrm{Fix}(f)$, and for each $q \in \mathbb{Q}$,*

*(1) $g(q) = q \Leftrightarrow s_f(\phi(q)) = 0$;*
*(2) $g(q) > q \Leftrightarrow s_f(\phi(q)) = 1$;*
*(3) $g(q) < q \Leftrightarrow s_f(\phi(q)) = -1$.*

*Then $f$ and $g$ are conjugate automorphisms.*

*Proof.* We use the characterization in [8, Theorem 2.2.5] to check the conjugacy of automorphisms: $f$ and $g$ are conjugate if and only if there exists an order preserving bijection $\psi : \mathcal{O}_g^* \to \mathcal{O}_f^*$ such that $s_g(O) = s_f(\psi(O))$ for every $O \in \mathcal{O}_g^*$.

We now show that it is legal to define the appropriate bijection $\psi$ as $\psi(O) = O'$ where $O' = \phi(p)$ for some $p \in O$. To show that it is a well-defined map, we need to prove that given $O \in \mathcal{O}_g^*$ and $p, q \in O$, $\phi(p) = \phi(q)$. Suppose the contrary, then $p \neq q$, hence $s_g(O) = 1$ or $s_g(O) = -1$. We now suppose that $s_g(O) = 1$, the case where $s_g(O) = -1$ is analogous. Then $g(p) > p$ and $g(q) > q$, hence $s_f(\phi(p)) = s_f(\phi(q)) = 1$. Since $f$ is good, and $\phi(p) \neq \phi(q)$ by our assumption, there is an orbital $O' \in \mathcal{O}_f^*$ such that $\phi(p) < O' < \phi(q)$ and $s_f(O') \neq 1$. Using that $\phi$ is surjective and monotone increasing, there exists an $r \in (p, q)$ such that $\phi(r) = O'$. Then $s_f(\phi(r)) \neq 1$, but $r$ is in the same orbital as $p$ and $q$, since orbitals are convex, hence $g(r) > r$. This contradicts (2).

The map $\psi$ is increasing and surjective, since $\phi$ is increasing and surjective. One can easily check that conditions (1), (2) and (3) imply that for every $O \in \mathcal{O}_g^*$, $s_g(O) = s_f(\psi(O))$. Hence it remains to show that $\psi$ is injective.

Let $O, O' \in \mathcal{O}_g^*$ be distinct orbitals with $\psi(O) = \psi(O')$. Then using conditions (1), (2) and (3), we have $s_g(O) = s_g(O')$. If $s_g(O) = s_g(O') = 0$ then $s_f(\psi(O)) = 0$, hence $\psi(O)$ is a set consisting of a fixed point, let $\{q\} = \psi(O)$. Then $|\phi^{-1}(q)| = 1$

using the assumption of the lemma, contradicting the fact that $O, O' \subset \phi^{-1}(q)$. If $s_g(O) = s_g(O') = 1$ then using that $g$ is good, there exists an orbital $O'' \in \mathcal{O}_g^*$ between $O$ and $O'$ such that $s_g(O'') \neq 1$. Then using the monotonicity of $\psi$ one obtains $\psi(O'') = \psi(O')$, hence $1 \neq s_f(\psi(O'')) = s_f(\psi(O)) = 1$, a contradiction. An analogous argument shows that $s_g(O) = s_g(O') = -1$ also leads to a contradiction, hence the proof of the lemma is complete. □

Now we turn to the proof of the theorems.

*Proof of Theorem 5.5.* First we show the "only if" part. Using Corollary 3.3, for almost every $f \in \mathrm{Aut}(\mathbb{Q}, <)$, $\mathrm{Fix}(f)$ is finite. Since the cardinality of fixed points is the same for conjugate automorphisms, it is clear that the conjugacy class of $f$ can only be non-Haar null if $\mathrm{Fix}(f)$ is finite.

The property that between any two distinct orbitals $O_1, O_2 \in \mathcal{O}_f^*$ with either $s_f(O_1) = s_f(O_2) = 1$ or $s_f(O_1) = s_f(O_2) = -1$, there exists an orbital $O_3 \in \mathcal{O}_f^*$ with $s_f(O_3) \neq s_f(O_1)$, is also conjugacy invariant. Hence it is enough to prove the following lemma to finish the "only if" part of the theorem.

**Lemma 5.8.** *For almost every $f \in \mathrm{Aut}(\mathbb{Q}, <)$, for distinct orbitals $O_1, O_2 \in \mathcal{O}_f^*$ with $O_1 < O_2$ such that either $s_f(O_1) = s_f(O_2) = 1$ or $s_f(O_1) = s_f(O_2) = -1$, there exists an orbital $O_3 \in \mathcal{O}_f^*$ with $O_1 < O_3 < O_2$ and $s_f(O_3) \neq s_f(O_1)$.*

*Proof.* Let $\mathcal{H}$ be the set of those automorphisms that satisfy the property in the lemma, and let $\mathcal{A}$ denote the set of functions $f \in \mathcal{H}^c$ such that we can choose distinct orbitals $O_1^f, O_2^f \in \mathcal{O}_f^*$ such that $O_1^f < O_2^f$, $s_f(O_1^f) = s_f(O_2^f) = 1$, and between $O_1^f$ and $O_2^f$, there is no orbital $O_3 \in \mathcal{O}_f^*$ with $s_f(O_3) \neq 1$. Also, let us denote by $\mathcal{A}'$ the set of functions $f \in \mathcal{H}^c$, such that we can choose distinct orbitals $O_1^f, O_2^f \in \mathcal{O}_f^*$ such that $O_1^f < O_2^f$, $s_f(O_1^f) = s_f(O_2^f) = -1$, and between $O_1^f$ and $O_2^f$, there is no orbital $O_3 \in \mathcal{O}_f^*$ with $s_f(O_3) \neq -1$. We show that $\mathcal{A}$ is Haar null and the same can be proved similarly for $\mathcal{A}'$. Since it is easy to see that $\mathcal{H}^c = \mathcal{A} \cup \mathcal{A}'$, proving this will finish the proof of the lemma.

We use Theorem 5.1 to show that the conjugacy invariant set $\mathcal{A}$ is Haar null. Let $(p_0, q_0), (p_1, q_1), \dots$ be an enumeration of all pairs $(p, q)$ with $p < q$, and for all $n \in \omega$, let

$$\begin{aligned}\mathcal{A}_n =& \{f \in \mathrm{Aut}(\mathbb{Q}, <) : p_n \text{ and } q_n \text{ are in distinct orbitals with respect to } f \\ & \text{and } f(r) > r \text{ for every } r \in [p_n, q_n]\} \\ =& \bigcap_{k \in \mathbb{Z}} \bigcap_{r \in [p_n, q_n]} \{f \in \mathrm{Aut}(\mathbb{Q}, <) : f^k(p_n) < q_n \text{ and } f(r) > r\}.\end{aligned}$$

Note that $\mathcal{A} = \bigcup_{n \in \omega} \mathcal{A}_n$ and $\mathcal{A}_n$ is Borel for every $n \in \omega$. Using Theorem 5.1, it is enough to show that there is a conjugacy invariant set $\mathcal{B}$ with $1 = \mathrm{id}_{\mathbb{Q}} \in \overline{\mathcal{B}}$ and $\mathcal{B} \cap \bigcup_{n \in \omega} \mathcal{A}_n^{-1} \mathcal{A}_n = \emptyset$. Let

$$\begin{aligned}\mathcal{B} = \{f \in \mathrm{Aut}(\mathbb{Q}, <) :& \\ & \mathrm{Fix}(f) \text{ is finite, } |\mathcal{O}_f^*| = 2|\mathrm{Fix}(f)| + 1 \text{ and } f(r) \geq r \text{ for every } r \in \mathbb{Q}\}.\end{aligned}$$

Note that the condition $|\mathcal{O}_f^*| = 2|\mathrm{Fix}(f)| + 1$ essentially states that between neighbouring fixed points, every point is in the same orbital (roughly speaking, this means that there are no "irrational fixed points").

It is easy to see that $\mathcal{B}$ is a conjugacy invariant set, and also that $\mathrm{id}_{\mathbb{Q}} \in \overline{\mathcal{B}}$. Let $n \in \omega$ be arbitrary, it remains to show that if $f,\ g \in \mathcal{A}_n$ then $f^{-1}g \notin \mathcal{B}$. Let $O$ be the orbit of $p_n$ with respect to $g$ and let $I = \{r \in \mathbb{Q} : r \leq r' \text{ for some } r' \in O\}$. Note that $I$ is convex, and since $g(r) > r$ for every $r \in (p_n, q_n)$ but $p_n$ and $q_n$ are in different orbitals (with respect to both $f$ and $g$), $q_n \notin I$.

There are two cases with respect to the relationship of $I$ and the orbitals of $f$. Suppose first that $I$ does not split orbitals of $f$, that is, there is no $r \in I$ and $k \in \mathbb{Z}$ such that $f^k(r) \notin I$. Then the sets $I$ and $\mathbb{Q} \setminus I$ are invariant under both $f$ and $g$ (and $f^{-1}$ and $g^{-1}$), thus $I$ does not split any orbitals of $f^{-1}g$. Moreover, $I$ has no greatest element, nor $\mathbb{Q} \setminus I$ has a least element, since any such element would need to be a fixed point of $g$, but $g$ does not have a fixed point in the interval $(p_n, q_n)$. Now suppose that $f^{-1}g \in \mathcal{B}$. Then it has a greatest fixed point (if any) that belongs to $I$ and a least fixed point (if any) that belongs to $\mathbb{Q} \setminus I$, hence between the two, every point is in the same orbital. This contradicts the fact that $I$ does not split the orbitals of $f^{-1}g$.

Now suppose that $I$ splits an orbital of $f$, thus there exist $r \in I$ and $k \in \mathbb{Z}$ such that $f^k(r) \notin I$. Since $f(r) > r$ for every $r \in (p_n, q_n)$, it follows that there is an $r \in (p_n, \infty) \cap I$ such that $f(r) \notin I$. Then $g^{-1}(f(r)) \notin I$, since $I$ does not split orbitals of $g$. By setting $r' = g^{-1}(f(r))$, we see that $f^{-1}g(r') = r \in I$, thus $f^{-1}g(r') < r'$, hence $f^{-1}g \notin \mathcal{B}$ also in this case, finishing the proof of the lemma. $\square$

Now we prove the "if" part of the theorem. Let $f$ be a good automorphism, we prove that $\mathcal{C}$, the conjugacy class of $f$ is non-Haar null, in fact, we show that it is compact biter and that if $f$ has no fixed points then its conjugacy class is compact catcher, so Fact 2.4 will yield the theorem. We use the notation $\mathcal{O} = \mathcal{O}_f^*$ and $s = s_f$.

To show this, take a compact set $\mathcal{F} \subset \mathrm{Aut}(\mathbb{Q}, <)$. In our proof, we partition $\mathbb{Q}$ into finitely many intervals bounded by the fixed points of $f$, and on each interval, we define a suitable part of $g$.

Let $\{p_1, p_2, \dots, p_{k-1}\}$ be the set of fixed points of $f$ (which is necessarily finite) with $p_1 < p_2 < \dots < p_{k-1}$. We now choose $q_1, q_2, \dots, q_{k-1} \in \mathbb{Q}$ such that if we set $\mathcal{U} = \{h \in \mathrm{Aut}(\mathbb{Q}, <) : \forall i (h(p_i) = q_i)\}$ then $\mathcal{U} \cap \mathcal{F} \neq \emptyset$. Let $\mathcal{K} = \mathcal{U} \cap \mathcal{F}$. Note that $\mathcal{U}$ is clopen, hence $\mathcal{K}$ is compact, and also that if $f$ has no fixed points then $\mathcal{F} = \mathcal{K}$. Hence, it is enough to construct an automorphism $g$ with $\mathcal{K} \subset g\mathcal{C}$ to finish the proof of the theorem. During the construction, we use the fact that the sets $\{h(p) : h \in \mathcal{K}\}$ and $\{h^{-1}(p) : h \in \mathcal{K}\}$ are finite, since $\mathcal{K}$ is compact.

Let us use the notation $p_0 = q_0 = -\infty$ and $p_k = q_k = +\infty$. We construct $g$ and a function $\phi : \mathbb{Q} \times \mathcal{K} \to \mathcal{O}$ separately on each interval $(p_i, p_{i+1})$, recursively. So let $i < k$ be fixed for now and let $r_1, r_2, \dots$ be an enumeration of $(p_i, p_{i+1})$ and $t_1, t_2, \dots$ be an enumeration of $(q_i, q_{i+1})$. Let $O_1, O_2, \dots$ be an infinite sequence of elements of $\mathcal{O}$ that are subsets of $(p_i, p_{i+1})$, containing every such element at least once. Note that there may be only finitely many such intervals, hence the sequence may contain the same element more than once. We let $\mathcal{O}' = \{O_1, O_2, \dots\}$. At the $n$th step of the recursive construction, we have a finite set $H_n \subset (p_i, p_{i+1})$ and functions $g_n$ and $\phi_n$. We preserve the following properties of these sets and functions:

For every $n \in \omega$, $h, h_1, h_2 \in \mathcal{K}$ and $p, p', p'' \in H_n$, where $p' < p''$ and $(p', p'') \cap H_n = \emptyset$,

(i) $H_0 \subset H_1 \subset \dots$, $g_0 \subset g_1 \subset \dots$ and $\phi_0 \subset \phi_1 \subset \dots$;
(ii) $H_n \subset (p_i, p_{i+1})$ is finite;
(iii) $g_n : H_n \to (q_i, q_{i+1})$ is strictly increasing;
(iv) $\phi_n : H_n \times \mathcal{K} \to \mathcal{O}'$, and $\phi_n(., h)$ is increasing;
(v) $r_1, \dots, r_{n+1} \in H_{3n+1}$, $t_1, \dots, t_{n+1} \in g_{3n+2}(H_{3n+2})$ and $O_1, \dots, O_{n+1} \in \phi_{3n+3}(H_{3n+3}, h)$;
(vi) it cannot happen that $h_1(p') < g_n(p') < h_2(p')$, $h_1(p'') > g_n(p'') > h_2(p'')$;
(vii) if $h(p') > g_n(p')$ and $h(p'') > g_n(p'')$ then $h(r) \geq g_n(p'')$ for every $r \in [p', p'']$; similarly, if $h(p') < g_n(p')$ and $h(p'') < g_n(p'')$ then $h(r) \leq g_n(p')$ for every $r \in [p', p'']$ (thus extending $g_n$ in any way to a strictly increasing function on $[p', p'']$, there is no $r \in [p', p'']$ where the value of the extension can be equal to $h(r)$);
(viii) $s(\phi_n(p, h)) = 1 \Leftrightarrow g_n(p) < h(p)$ and $s(\phi_n(p, h)) = -1 \Leftrightarrow g_n(p) > h(p)$;
(ix) $\phi_n(H_n, h_1) = \phi_n(H_n, h_2)$;
(x) the value of $s$ is alternating on the image $\phi_n(H_n, h)$, that is, either $\phi_n(p', h) = \phi_n(p'', h)$ or $s(\phi_n(p', h)) \neq s(\phi_n(p'', h))$;
(xi) $h_i(p') > g_n(p')$ and $h_i(p'') < g_n(p'')$ $(i = 1, 2)$ (or similarly, $h_i(p') < g_n(p')$ and $h_i(p'') > g_n(p'')$ $(i = 1, 2)$) implies that $\phi_n(p', h_1) = \phi_n(p', h_2)$ and $\phi_n(p'', h_1) = \phi_n(p'', h_2)$.

**Remark 5.9.** Conditions (vi) and (vii) are equivalent to the following fact: the rectangle $\mathrm{conv}((p', g_n(p')), (p'', g_n(p')), (p'', g_n(p'')), (p', g_n(p'')))$ has two sides that are opposite such that no $h \in \mathcal{K}$ intersects the interior of any of those sides.

First we prove the following.

**Claim 5.10.** *On each interval $(p_i, p_{i+1})$, the sets and functions $H_n$, $g_n$ and $\phi_n$ can be constructed with the above properties.*

*Proof.* We prove the claim by induction on $n$. For $n = 0$, let $H_0 = g_0 = \phi_0 = \emptyset$. Now suppose that $H_n$, $g_n$ and $\phi_n$ are given with the above properties, using them, we construct the suitable $H_{n+1}$, $g_{n+1}$ and $\phi_{n+1}$. There are three cases according to the remainder of $n$ mod 3.

Case 1: $n = 3m$. At this step, we make sure that $r_{m+1} \in H_{n+1}$. If already $r_{m+1} \in H_n$ then let $H_{n+1} = H_n$, $g_{n+1} = g_n$ and $\phi_{n+1} = \phi_n$. Otherwise, there are multiple cases according to the existence of $p' \in H_n$ with $p' < r_{m+1}$, $p'' \in H_n$ with $r_{m+1} < p''$, and whether $g_n(p') < h(p')$ or $g_n(p') > h(p')$, $g_n(p'') < h(p'')$ or $g_n(p'') > h(p'')$.

Case 1a: there are neither $p' \in H_n$ with $p' < r_{m+1}$ nor $p'' \in H_n$ with $r_{m+1} < p''$ (that is, $H_n = \emptyset$, $n = 0$). If $s(O_1) = 1$ then we find $q \in (q_i, q_{i+1})$ with $q < h(r_{m+1})$ for every $h \in \mathcal{K}$, otherwise, we find $q \in (q_i, q_{i+1})$ with $q > h(r_{m+1})$ for every $h \in \mathcal{K}$. Such a $q$ exists, since $\mathcal{K}$ is compact, thus $\{h(r_{m+1}) : h \in \mathcal{K}\}$ is finite. Now we set $H_{n+1} = \{r_{m+1}\}$, $g_{n+1}(r_{m+1}) = q$, $\phi_{n+1}(r_{m+1}, h) = O_1$ for every $h \in \mathcal{K}$.

Case 1b: there is a $p' \in H_n$ with $p' < r_{m+1}$ but there is no $p'' \in H_n$ with $r_{m+1} < p''$. Let $p'$ be the largest element in $H_n$, clearly $p' < r_{m+1}$. Let $q' = g_n(p')$. Using (ix), $\phi_n(p', h)$ is the same for every $h \in \mathcal{K}$, since it is the largest element in the common image $\phi_n(H_n, h)$. Let $O = \phi_n(p', h)$ for some $h \in \mathcal{K}$. Depending on $s(O)$, $g_n(p') < h(p')$ for every $h \in \mathcal{K}$ or $g_n(p') > h(p')$ for every $h \in \mathcal{K}$ using (viii). In the first case, choose $t \in (q_i, q_{i+1})$ such that $q' < t < h(p')$ for every $h \in \mathcal{K}$. Then set $H_{n+1} = H_n \cup \{r_{m+1}\}$ and let $g_{n+1}$ extend $g_n$ with $g_{n+1}(r_{m+1}) = t$, and let $\phi_{n+1}$ extend $\phi_n$ with $\phi_{n+1}(r_{m+1}, h) = O$ for every $h \in \mathcal{K}$.

In the second case, let $h(r_{m+1}) < t$ for every $h \in \mathcal{K}$, also satisfying $t \in (q', q_{i+1})$. Choose $q \in (q', t)$ such that $q > h(r_{m+1})$ for every $h \in \mathcal{K}$. As $h^{-1}(q') > p'$ for every $h \in \mathcal{K}$, there exists $p \in (p', r_{m+1})$ such that $p < h^{-1}(q')$ for every $h \in \mathcal{K}$. Now set $H_{n+1} = H_n \cup \{r_{m+1}, p\}$, and let $g_{n+1}$ and $\phi_{n+1}$ extend the appropriate functions with $g_{n+1}(r_{m+1}) = t$, $g_{n+1}(p) = q$ and $\phi_{n+1}(r_{m+1}, h) = \phi_{n+1}(p, h) = O$ for every $h \in \mathcal{K}$.

Case 1c: there is no $p' \in H_n$ with $p' < r_{m+1}$ but there is a $p'' \in H_n$ with $r_{m+1} < p''$. This case can be handled similarly as Case 1b.

Case 1d: there is a $p' \in H_n$ with $p' < r_{m+1}$, there is a $p'' \in H_n$ with $r_{m+1} < p''$, and for the largest such $p'$ and the smallest such $p''$, there is no $h \in \mathcal{K}$ with $g_n(p') < h(p')$ and $g_n(p'') > h(p'')$. In this case, let $\mathcal{K}' = \{h \in \mathcal{K} : g_n(p') > h(p')$ and $g_n(p'') < h(p'')\}$, where $p' \in H_n$ is the largest with $p' < r_{m+1}$ and $p'' \in H_n$ is the smallest with $p'' > r_{m+1}$. Note that $\mathcal{K}'$ may be the empty set. Let $q' = g_n(p')$ and $q'' = g_n(p'')$. Choose $t \in (q', q'')$ such that $t > h(r_{m+1})$ for each $h \in \mathcal{K}'$ with $h(r_{m+1}) < q''$. Such a $t$ exists, since the compactness of $\mathcal{K}'$ implies that $\{h(r_{m+1}) : h \in \mathcal{K}', h(r_{m+1}) < q''\}$ is finite. We will set $g_{n+1}(r_{m+1}) = t$, but we need to define the value of $g_{n+1}$ at one more place. Choose $q \in (q', t)$ with $q > h(r_{m+1})$ for each $h \in \mathcal{K}'$ with $h(r_{m+1}) < q''$. For every $h \in \mathcal{K}'$ we have $h(p') < q'$, hence also $p' < h^{-1}(q')$. Therefore there is a $p \in (p', r_{m+1})$ for which $p < h^{-1}(q')$ for every $h \in \mathcal{K}'$.

Now let $H_{n+1} = H_n \cup \{p, r_{m+1}\}$, $g_{n+1}$ extend $g_n$ with $g_{n+1}(p) = q$, $g_{n+1}(r_{m+1}) = t$. For $h \in \mathcal{K}'$, either $h(r_{m+1}) < t$ or $h(r_{m+1}) > t$. If $h(r_{m+1}) < t$ then let $\phi_{n+1}(r_{m+1}, h) = \phi_n(p', h)$, if $h(r_{m+1}) > t$ then let $\phi_{n+1}(r_{m+1}, h) = \phi_n(p'', h)$. In both cases, let $\phi_{n+1}(p, h) = \phi_n(p', h)$. If $h \in \mathcal{K} \setminus \mathcal{K}'$ then let $\phi_{n+1}(p, h) = \phi_{n+1}(r_{m+1}, h) = \phi_n(p', h)$. Note that using (viii), $s(\phi_n(p', h)) = s(\phi_n(p'', h))$, thus (x) implies that $\phi_n(p', h) = \phi_n(p'', h)$. All of the properties can be checked easily.

Case 1e: there is a $p' \in H_n$ with $p' < r_{m+1}$, there is a $p'' \in H_n$ with $r_{m+1} < p''$, and for the largest such $p'$ and the smallest such $p''$, there is no $h \in \mathcal{K}$ with $g_n(p') > h(p')$ and $g_n(p'') < h(p'')$. Now let $\mathcal{K}' = \{h \in \mathcal{K} : g_n(p') < h(p')$ and $g_n(p'') > h(p'')\}$, where again, $p' \in H_n$ is the largest with $p' < r_{m+1}$ and $p'' \in H_n$ is the smallest with $p'' > r_{m+1}$. Let $q' = g_n(p')$ and $q'' = g_n(p'')$. The set $\{h(p'') : h \in \mathcal{K}'\}$ is finite, hence there is a $t \in (q', q'')$ with $t > h(p'')$ for every $h \in \mathcal{K}'$. Let $H_{n+1} = H_n \cup \{r_{m+1}\}$, $g_{n+1}(r_{m+1}) = t$ and $\phi_{n+1}(r_{m+1}, h) = \phi_n(p'', h)$ for every $h \in \mathcal{K}$. Using the fact that for no $h \in \mathcal{K}$ can $h$ and any strictly increasing extension of $g_{n+1}$ have the same values on $[r_{m+1}, p'']$, one can easily check that every property is satisfied.

Using (vi), these cover all sub-cases of Case 1. Now we turn to the second case.

Case 2: $n = 3m + 1$. At this step, we make sure that $t_{m+1} \in g_{n+1}(H_{n+1})$. If already $t_{m+1} \in g_n(H_n)$ then let $H_{n+1} = H_n$, $g_{n+1} = g_n$ and $\phi_{n+1} = \phi_n$. Otherwise, similarly as in Case 1, there are multiple sub-cases according to the existence of $q' \in g_n(H_n)$ with $q' < t_{m+1}$, $q'' \in g_n(H_n)$ with $t_{m+1} < q''$, and whether there exists an $h \in \mathcal{K}$ such that $g_n(p') < h(p')$ or $g_n(p') > h(p')$, $g_n(p'') < h(p'')$ or $g_n(p'') > h(p'')$, where $p' = g_n^{-1}(q')$ and $p'' = g_n^{-1}(q'')$. These sub-cases can be handled similarly as in Case 1, but we quickly go though them. Since $r_{m+1} \in H_n$ we do not have to deal with the case $H_n = \emptyset$.

Case 2a: there is a $q' \in g_n(H_n)$ with $q' < t_{m+1}$ but there is no $q'' \in g_n(H_n)$ with $t_{m+1} < q''$. Let $q'$ be the largest element in $g_n(H_n)$, clearly $q' < t_{m+1}$. As before, $g_n(p') < h(p')$ for every $h \in \mathcal{K}$ or $g_n(p') > h(p')$ for every $h \in \mathcal{K}$, where $p' = g_n^{-1}(q')$.

In the first case, choose $r \in (p', p_{i+1})$ and $r > h^{-1}(t_{m+1})$ for every $h \in \mathcal{K}$. Such an $r$ exists, since $h(p_{i+1}) = q_{i+1}$ for every $h \in \mathcal{K}$, and $\{h^{-1}(t_{m+1}) : h \in \mathcal{K}\}$ is finite. Let $q \in (q', t_{m+1})$ with $q < h(p')$ for every $h \in \mathcal{K}$, and choose $p \in (p', r)$ with $p > h^{-1}(t_{m+1})$ for every $h \in \mathcal{K}$. Then let $H_{n+1} = H_n \cup \{r, p\}$, and let $g_{n+1}$ and $\phi_{n+1}$ extend $g_n$ and $\phi_n$, respectively, with $g_{n+1}(r) = t_{m+1}$, $g_{n+1}(p) = q$ and $\phi_{n+1}(r, h) = \phi_{n+1}(p, h) = \phi_n(p', h)$ for every $h \in \mathcal{K}$.

In the second case, choose $r \in (p', p_{i+1})$ with $r < h^{-1}(q')$ for every $h \in \mathcal{K}$. Such an $r$ exists, since for every $h \in \mathcal{K}$, $h(p') < q'$ implies $p' < h^{-1}(q')$ and $\{h^{-1}(q') : h \in \mathcal{K}\}$ is finite. Then set $H_{n+1} = H_n \cup \{r\}$, and let $g_{n+1}(r) = t_{m+1}$ and $\phi_{n+1}(p, h) = \phi_n(p', h)$ for every $h \in \mathcal{K}$.

Case 2b: there is no $q' \in g_n(H_n)$ with $q' < t_{m+1}$ but there is a $q'' \in g_n(H_n)$ with $t_{m+1} < q''$. This case can be handled similarly to Case 2a.

Case 2c: there is a $q' \in g_n(H_n)$ with $q' < t_{m+1}$, there is a $q'' \in H_n$ with $t_{m+1} < q''$, and for the largest such $q'$ and the smallest such $q''$, there is no $h \in \mathcal{K}$ with $g_n(p') < h(p')$ and $g_n(p'') > h(p'')$, where $p' = g_n^{-1}(q')$ and $p'' = g_n^{-1}(q'')$. This is analogous to Case 1e. There exists $r \in (p', p'')$ with $h^{-1}(q') > r$ for every $h \in \mathcal{K}$ such that $g_n(p') > h(p')$ and $g_n(p') < h(p')$. As before, set $H_{n+1} = H_n \cup \{r\}$ and let $g_{n+1}$ extend $g_n$ with $g_{n+1}(r) = t_{m+1}$, and $\phi_{n+1}$ extend $\phi_n$ with $\phi_{n+1}(r, h) = \phi_n(p', h)$ for every $h \in \mathcal{K}$.

Case 2d: there is a $q' \in g_n(H_n)$ with $q' < t_{m+1}$, there is a $q'' \in H_n$ with $t_{m+1} < q''$, and for the largest such $q'$ and the smallest such $q''$, there is no $h \in \mathcal{K}$ with $g_n(p') > h(p')$ and $g_n(p'') < h(p'')$, where $p' = g_n^{-1}(q')$ and $p'' = g_n^{-1}(q'')$. This is analogous to Case 1d. Let $\mathcal{K}' = \{h \in \mathcal{K} : g_n(p') < h(p') \text{ and } g_n(p'') > h(p'')\}$, this may again be the empty set. Choose $r \in (p', p'')$ such that $r < h^{-1}(t_{m+1})$ for each $h \in \mathcal{K}'$ with $h^{-1}(t_{m+1}) > p'$. There is a $q \in (t_{m+1}, q'')$ with $q > h(p'')$ for every $h \in \mathcal{K}'$. Choose $p \in (r, p'')$ with $p < h^{-1}(t_{m+1})$ for each $h \in \mathcal{K}'$ with $h^{-1}(t_{m+1}) > p'$.

Now let $H_{n+1} = H_n \cup \{p, r\}$, $g_{n+1}$ extend $g_n$ with $g_{n+1}(r) = t_{m+1}$, $g_{n+1}(p) = q$. For $h \in \mathcal{K}'$, either $h^{-1}(t_{m+1}) \leq p'$ or $h^{-1}(t_{m+1}) > p$. If $h^{-1}(t_{m+1}) \leq p'$ then let $\phi_{n+1}(r, h) = \phi_n(p', h)$, if $h^{-1}(t_{m+1}) > p$ then let $\phi_{n+1}(r, h) = \phi_n(p'', h)$. In both cases, let $\phi_{n+1}(p, h) = \phi_n(p'', h)$. If $h \in \mathcal{K} \setminus \mathcal{K}'$ then let $\phi_{n+1}(p, h) = \phi_{n+1}(r, h) = \phi_n(p', h)$.

Again using (vi), these cover all sub-cases of Case 2. Now we turn to the third case.

Case 3: $n = 3m + 2$. At this step, we make sure that $O_{m+1} \in \phi_{n+1}(H_{n+1}, h)$ for every $h \in \mathcal{K}$. Note throughout that there is no $O \in \mathcal{O}'$ with $s(O) = 0$. If $O_{m+1} \in \phi_n(H_n, h)$ for any (hence, by (ix) for every) $h \in \mathcal{K}$ then let $H_{n+1} = H_n$, $g_{n+1} = g_n$ and $\phi_{n+1} = \phi_n$. If this is not the case, we consider the sub-cases according to $\phi_n(H_n, h_0)$ for a fixed $h_0 \in \mathcal{K}$. We suppose throughout that $s(O_{m+1}) = 1$. The case $s(O_{m+1}) = -1$ is similar. Also, note that $H_n \neq \emptyset$, as, for example, $r_1 \in H_n$.

Case 3a: $O_{m+1} > O$ for every $O \in \phi_n(H_n, h_0)$, and for the largest $O \in \phi_n(H_n, h_0)$ (with respect to $<$), $s(O) = -1$. Let $p$ be the largest element in $H_n$, $q = g_n(p)$, then $O = \phi_n(p, h_0)$. This means, using (ix) and (viii) that $\phi_n(p, h) = O$ and $g_n(p) > h(p)$ for every $h \in \mathcal{K}$. As $h(p_{i+1}) = q_{i+1}$ for every $h \in \mathcal{K}$ and $g_n : H_n \to (q_i, q_{i+1})$, we can choose $t \in (q, q_{i+1})$ and as $\{h^{-1}(t) : h \in \mathcal{K}\}$ is finite, there exists $r \in (p, p_{i+1})$ with $r > h^{-1}(t)$ for every $h \in \mathcal{K}$. Now let $H_{n+1} = H_n \cup \{r\}$, let $g_{n+1}$ extend $g_n$ with $g_{n+1}(r) = t$ and let $\phi_{n+1}$ extend $\phi_n$ with $\phi_{n+1}(r, h) = O_{m+1}$ for every $h \in \mathcal{K}$. One can easily check that the necessary conditions still hold.

Case 3b: $O_{m+1} > O$ for every $O \in \phi_n(H_n, h_0)$, and for the largest $O \in \phi_n(H_n, h_0)$ (with respect to $<$), $s(O) = 1$. Let $p$ be the largest element in $H_n$, $q = g_n(p)$, then $O = \phi_n(p, h_0)$. Using that $f$ is good, there exists $O' \in \mathcal{O}'$ with $O < O' < O_{m+1}$ and $s(O') = -1$. Now choose $r' \in (p, p_{i+1}$ and choose $t' \in (q, q_{i+1})$ with $t' > h(r')$ for every $h \in \mathcal{K}$. Then choose $t'' \in (t', q_{i+1})$ and choose $r'' \in (r', p_{i+1})$ with $r'' > h^{-1}(t'')$ for every $h \in \mathcal{K}$. Now let $H_{n+1} = H_n \cup \{r', r''\}$, and let $g_{n+1}$ extend $g_n$ with $g_{n+1}(r') = t'$ and $g_{n+1}(r'') = t''$, and let $\phi_{n+1}$ extend $\phi_n$ with $\phi_{n+1}(r', h) = O'$ and $\phi_{n+1}(r'', h) = O_{m+1}$ for every $h \in \mathcal{K}$.

The cases where $O_{m+1} < O$ for every $O \in \phi_n(H_n, h_0)$ are similar to the ones above.

Case 3c: $O_{m+1}$ is between elements of $\phi_n(H_n, h_0)$, and if $O'$ is the largest element of $\phi_n(H_n, h_0)$ with $O' < O_{m+1}$ and $O''$ is the smallest element of $\phi_n(H_n, h_0)$ with $O_{m+1} < O''$ then $s(O') = -1$ and $s(O'') = 1$. In this case, choose $O \in \mathcal{O}'$ with $O_{m+1} < O < O''$ and $s(O) = -1$, again, such an $O$ exists because $f$ is good. The orbitals $O'$ and $O''$ are neighbouring ones in $\phi_n(H_n, h)$ for every $h \in \mathcal{K}$.

Notice that for every $h \in \mathcal{K}$ there exists a unique pair of neighbouring points $p', p'' \in H_n$ with $\phi_n(p', h) = O'$ and $\phi_n(p'', h) = O''$. Therefore, we can partition $\mathcal{K}$ into finitely many compact sets according to this pair. We define $g_{n+1}$ separately on each such interval $(p', p'')$, that is, where $p'$ and $p''$ are neighbouring points in $H_n$ and $\phi_n(p', h) = O'$, $\phi_n(p'', h) = O''$ for some $h \in \mathcal{K}$.

So let $p', p''$ be such elements of $H_n$ and let $\mathcal{K}' = \{h \in \mathcal{K} : \phi_n(p', h) = O' \text{ and } \phi_n(p'', h) = O''\}$. Using the facts that $s(O') = -1$, $s(O'') = 1$ and (viii), we have $g_n(p') > h(p')$ and $g_n(p'') < h(p'')$ for every $h \in \mathcal{K}'$. Let $q' = g_n(p')$ and $q'' = g_n(p'')$, and choose $q \in (q', q'')$. Let $\{r^1, r^2, \dots, r^c\} = \{h^{-1}(q) : h \in \mathcal{K}'\}$, where $r^1 < r^2 < \dots < r^c$. Note that $h(p') < g_n(p') = q' < q < q'' = g_n(p'') < h(p'')$ for every $h \in \mathcal{K}'$, hence $p' < r^1$ and $r^c < p''$. For $1 \le j \le c$, let $\mathcal{K}^j = \{h \in \mathcal{K}' : h^{-1}(q) = r^j\}$.

Choose $t \in (q', q)$ with $t > h(r^j)$ for every $1 \le j \le c$ and every $h \in \mathcal{K}'$ such that $h(r^j) < q$. From now on, the values of $g_{n+1}|_{(p', p'')}$ on newly defined points will always be at least $t$. This will achieve that if we add new points to take care of the functions in $\mathcal{K}^j$ for some $j$, then our choices will not interfere with the functions in $\mathcal{K}' \setminus \mathcal{K}^j$.

Choose $r \in (p', p'')$ with $r < h^{-1}(q')$ for every $h \in \mathcal{K}'$. By setting $g_{n+1}(r) = t$ and extending it to a strictly increasing function, it can be easily seen that the extension cannot have a common value with any $h \in \mathcal{K}'$ on the interval $(p', r)$. Let $t_1^1 \in (t, q)$ be arbitrary and choose $r_1^1 \in (r, r^1)$ with $t_1^1 < h(r_1^1)$ for every $h \in \mathcal{K}^1$. Then choose $r_2^1 \in (r_1^1, r^1)$ and choose $t_2^1 \in (t_1^1, q)$ such that $t_2^1 > h(r_2^1)$ for every $h \in \mathcal{K}^1$. Then let $r_3^1 = r^1$ and choose $t_3^1 \in (t_2^1, q)$.

We handle the families $\mathcal{K}^j$ for $j \ge 2$ similarly. Choose $t_1^j \in (t_3^{j-1}, q)$ and then choose $r_1^j \in (r_3^{j-1}, r^j)$ such that $h(r_1^j) > t_1^j$ for every $h \in \mathcal{K}^j$. Then let $r_2^j \in (r_1^j, r^j)$ and choose $t_2^j \in (t_1^j, q)$ with $t_2^j > h(r_2^j)$ for every $h \in \mathcal{K}^j$. Then let $r_3^j = r^j$ and choose $t_3^j \in (t_2^j, q)$.

After recursively choosing the rational numbers above for every $j \le c$, we choose $p \in (r_3^c, p'')$ such that $p > h^{-1}(q'')$ for every $h \in \mathcal{K}'$. Now we will set $H_{n+1} \cap (p', p'') = (H_n \cap (p', p'')) \cup \{r, p, r_\ell^j : 1 \le j \le c, 1 \le \ell \le 3\}$. Let $g_{n+1}$ extend $g_n$ with $g_{n+1}(r) = t$, $g_{n+1}(p) = q$ and $g_{n+1}(r_\ell^j) = t_\ell^j$ for every $1 \le j \le c$ and $1 \le \ell \le 3$. Let $\phi_{n+1}$ extend $\phi_n$ with $\phi_{n+1}(r, h) = O'$, $\phi_{n+1}(p, h) = O''$ for every $h \in \mathcal{K}'$. Also, let $\phi_{n+1}(r_\ell^j, h) = O'$ for every $\ell$ if $h \in \mathcal{K}^{j'}$ with $j' > j$, and

$\phi_{n+1}(r_\ell^j, h) = O''$ if $j' < j$. If $j' = j$ then let $\phi_{n+1}(r_1^j, h) = O_{m+1}$, $\phi_{n+1}(r_2^j, h) = O$ and $\phi_{n+1}(r_3^j, h) = O''$. For every $h \in \mathcal{K} \setminus \mathcal{K}'$, let $\phi_{n+1}(x, h) = \phi_n(p', h) = \phi_n(p'', h)$, for every $x \in \{r, p, r_\ell^j : 1 \le j \le c, 1 \le \ell \le 3\}$, where we used (x) for the last equality.

We do the same in every interval of the form $(p', p'')$, where $p'$ and $p''$ are neighbours in $H_n$, and $\phi_n(h, p') = O'$ and $\phi_n(h, p'') = O''$ for some $h \in \mathcal{K}$. Extending $g_n$ and $\phi_n$ appropriately, one obtains $H_{n+1}$, $g_{n+1}$ and $\phi_{n+1}$ with the necessary conditions. We note that the choice of $t$ ensures that condition (vii) is satisfied.

Case 3d: $O_{m+1}$ is between elements of $\phi_n(H_n, h_0)$, and if $O'$ is the largest element of $\phi_n(H_n, h_0)$ with $O' < O_{m+1}$ and $O''$ is the smallest element of $\phi_n(H_n, h_0)$ with $O_{m+1} < O''$ then $s(O') = 1$ and $s(O'') = -1$. This case can be handled quite similarly as Case 3c. Choose $O \in \mathcal{O}'$ with $O' < O < O_{m+1}$ and $s(O) = -1$. Again the unique pairs of neighbouring points $p', p'' \in H_n$ with $\phi_n(p', h) = O'$ and $\phi_n(p'', h) = O''$ define a partition of $\mathcal{K}'$. So let $p', p'' \in H_n$ be such a pair, we set $q' = g_n(p')$ and $q'' = g_n(p'')$.

Let $p \in (p', p'')$ be arbitrary and let $\{t^1, \dots, t^c\} = \{h(p) : h \in \mathcal{K}'\}$, where $\mathcal{K}' = \{h \in \mathcal{K} : h(p') > g_n(p') \text{ and } h(p'') < g_n(p'')\}$, such that $t^1 < \dots < t^c$. We set $\mathcal{K}^j = \{h \in \mathcal{K}' : h(p) = t^j\}$. Now one can choose $r \in (p', p)$ with $h^{-1}(t^j) < r$ for every $h \in \mathcal{K}'$ and $1 \le j \le c$ if $h^{-1}(t^j) < p$. Let $t \in (q', t^1)$ be such that $t < h(p')$ for every $h \in \mathcal{K}'$. Now suppose that for $j' < j$ and $1 \le \ell \le 3$ the points $r_\ell^{j'}$ and $t_\ell^{j'}$ are given. Then choose $r_1^j$ arbitrarily for the set $(r, p)$ if $j = 1$ and from $(r_3^{j-1}, p)$ if $j > 1$. Then choose $t_1^j$ from $(t, t^j)$ if $j = 1$ and from $(t_3^{j-1}, t^j)$ if $j > 1$ such that $h(r_1^j) < t_1^j$ for every $h \in \mathcal{K}^j$. Then choose $t_2^j \in (t_1^j, t^j)$ and choose $r_2^j \in (r_1^j, p)$ such that $h(r_2^j) > t_2^j$ for every $h \in \mathcal{K}^j$. Finally, choose $r_3^j \in (r_2^j, p)$ and set $t_3^j = t^j$.

After recursively choosing the points $r_\ell^j$ and $t_\ell^j$, choose $q \in (t^c, q'')$ such that $q > h(p'')$ for every $h \in \mathcal{K}'$. As before, let $H_{n+1} \cap (p', p'') = (H_n \cap (p', p'')) \cup \{r, p, r_\ell^j : 1 \le j \le c, 1 \le \ell \le 3\}$, and define $g_{n+1}(r) = t$, $g_{n+1}(p) = q$ and $g_{n+1}(r_\ell^j) = t_\ell^j$ for every $1 \le j \le c$ and $1 \le \ell \le 3$. For $h \in \mathcal{K}^j$ let $\phi_{n+1}(r, h) = O'$, $\phi_{n+1}(r_\ell^{j'}, h) = O'$ for every $j' < j$ and $1 \le \ell \le 3$, $\phi_{n+1}(r_1^j, h) = O$, $\phi_{n+1}(r_2^j, h) = O_{m+1}$, $\phi_{n+1}(r_3^j, h) = O''$, and $\phi_{n+1}(r_\ell^{j''}, h) = \phi_{n+1}(p, h) = O''$ for every $j'' > j$, $1 \le \ell \le 3$. For every $h \in \mathcal{K} \setminus \mathcal{K}'$ we set $\phi_{n+1}(x, h) = \phi_n(p', h)$ for every $x \in (H_{n+1} \cap (p', p'')) \setminus H_n$.

It is straightforward to check that $H_{n+1}$, $g_{n+1}$ and $\phi_{n+1}$ obtained in this way satisfy the conditions. □

Now we show the following to complete the proof of the theorem.

**Claim 5.11.** *There is an automorphism $g \in \mathrm{Aut}(\mathbb{Q}, <)$ such that $g^{-1}\mathcal{K} \subset \mathcal{C}$.*

*Proof.* Suppose that $H_n^i$, $g_n^i$ and $\phi_n^i$ are the corresponding sets and functions on the interval $(p_i, p_{i+1})$ for $i < k$. Let $g(p_i) = q_i$ for every $1 \le i < k$, and let $g|_{(p_i, p_{i+1})} = \bigcup_n g_n^i$ for every $i < k$. This makes sense, since $\bigcup_n g_n^i$ is an increasing bijection between $(p_i, p_{i+1})$ and $(q_i, q_{i+1})$ using (i), (iii) and (v). Also, let $\phi(p_i, h) = \{p_i\}$ for every $1 \le i < k$ and $h \in \mathcal{K}$, and let $\phi(., h)|_{(p_i, p_{i+1})} = \bigcup_n \phi_n^i(., h)$ for every $0 \le i < k$ and $h \in \mathcal{K}$. This also makes sense, since using (i), (iv) and (v), $\bigcup_n \phi_n^i(., h)$ is an increasing surjective function from $(p_i, p_{i+1})$ to those elements of $\mathcal{O}_f^*$ that are subsets of $(p_i, p_{i+1})$.

We now show that $f$, $g^{-1}h$ and $\phi(., h)$ satisfy the conditions of Lemma 5.7 to prove that $f$ and $g^{-1}h$ are conjugate automorphisms for every $h \in \mathcal{K}$. We start

by showing that $g^{-1}h$ is good for every $h \in \mathcal{K}$. First of all, it only has the finitely many fixed points that $f$ has, since if $p \in \mathbb{Q}$ is not among the fixed points of $f$, then $p$ is in some interval of the form $(p_i, p_{i+1})$, and as (viii) covers all cases, $g_n(p) \neq h(p)$, hence $g^{-1}h(p) \neq p$. Now suppose towards a contradiction that there are distinct orbitals $O_1, O_2 \in \mathcal{O}^*_{g^{-1}h}$ such that either $s_{g^{-1}h}(O_1) = s_{g^{-1}h}(O_2) = 1$ or $s_{g^{-1}h}(O_1) = s_{g^{-1}h}(O_2) = -1$ and there is no orbital $O_3 \in \mathcal{O}^*_{g^{-1}h}$ with $s_{g^{-1}h}(O_3) \neq s_{g^{-1}h}(O_1)$ between them. We suppose for the rest of the proof that $s_{g^{-1}h}(O_1) = s_{g^{-1}h}(O_2) = 1$, the case when they equal $-1$ is analogous. Note that in this case,

$$g(p) < h(p) \text{ for every } p \in (O_1 \cup O_2). \tag{18}$$

There is no fixed point of $g^{-1}h$, or equivalently, there is no fixed point of $f$ between $O_1$ and $O_2$, thus $O_1, O_2 \subset (p_i, p_{i+1})$ for some $i < k$. Let $p' \in O_1$ and $p'' \in O_2$ be arbitrary. Then $\phi(p', h), \phi(p'', h) \in \mathcal{O}^*_f$. We consider the following two cases separately.

Case 1: $\phi(p', h) = \phi(p'', h)$. Let $O = \phi(p', h)$. Then, using the fact that $\phi(., h)$ is increasing provided by (iv), $\phi(p, h) = O$ for every $p \in (p', p'')$. Using (18) and (viii), $s_f(\phi(p', h)) = s_f(O) = 1$. Hence if $p \in (p', p'')$ then $g(p) < h(p)$ using (viii) and the fact that $\phi(p, h) = O$. Let $n$ be large enough such that $p', p'' \in H^i_n$ and let $\{r^1, \dots, r^m\} = H^i_n \cap [p', p'']$ where $p' = r^1 < \dots < r^m = p''$. Then applying (vii) to each of the intervals $[r^j, r^{j+1}]$, the facts that $h(r^j) > g^i_n(r^j)$ and $h(r^{j+1}) > g^i_n(r^{j+1})$ imply $h(r) \geq g^i_n(r^{j+1})$ for every $r \in [r^j, r^{j+1}]$. It follows (since $g$ is an increasing extension of $g^i_n$) that $g^{-1}h(r^1) \geq g^{-1}(g(r^2)) = r^2$. Using induction, one can show with the same argument that $(g^{-1}h)^{m-1}(r^1) \geq r^m$, hence $(g^{-1}h)^{m-1}(p') \geq p''$. This fact implies that $p'$ and $p''$ are in the same orbital with respect to $g^{-1}h$, contradicting our assumption.

Case 2: $\phi(p', h) \neq \phi(p'', h)$. Again using (18) and (viii) twice, $g(p') < h(p')$ and $g(p'') < h(p'')$, hence $s_f(\phi(p', h)) = s_f(\phi(p'', h)) = 1$. Using the fact that $f$ is good, there is $O \in \mathcal{O}^*_f$ between $\phi(p', h)$ and $\phi(p'', h)$ with $s_f(O) = -1$, since there is no fixed point between $O_1$ and $O_2$. Using that $\phi(., h)$ is increasing and surjective provided by (iv) and (v), there is $p \in (p', p'')$ with $\phi(p, h) = O$. Then (viii) ensures that $g(p) > h(p)$, hence $g^{-1}h(p) < p$, therefore there exists $O' \in \mathcal{O}^*_{g^{-1}h}$ with $O_1 < O' < O_2$ and $s_{g^{-1}h}(O') = -1$, contradicting our assumptions. This completes the proof of the fact that $g^{-1}h$ is good.

The function $\phi(., h) : \mathbb{Q} \to \mathcal{O}^*_f$ is increasing and surjective using its construction and (iv), (v). The fact that $|\phi(., h)^{-1}(p)| = 1$ for every $p \in \mathrm{Fix}(f)$ readily follows from the construction of $\phi$. Condition (1) of Lemma 5.7 follows from the fact that (viii) covers all cases, hence there is no fixed points of $g^{-1}h$ on any interval of the form $(p_i, p_{i+1})$. Now we check condition (2). Let $q \in \mathbb{Q}$ be fixed. For both direction, both the facts that $g^{-1}h(q) > q$ and $s_f(\phi(q, h)) = 1$ imply separately that $q \neq p_i$ for any $i$, hence $q \in (p_i, p_{i+1})$ for some $i$. If $n$ is large enough such that $q \in H^i_n$ then (viii) implies both direction in (2). The proof is analogous for (3).

Therefore the conditions of Lemma 5.7 are satisfied for $f$, $g^{-1}h$ and $\phi$, hence $f$ and $g^{-1}h$ are conjugate automorphisms for every $h \in \mathcal{K}$. This completes the proof of the lemma. □

And thus the proof of the theorem is also complete. □

*Proof of Theorem 5.6.* Using Theorem 5.5, the union of the Haar null conjugacy classes is exactly the union of the automorphisms with infinitely many fixed points

and those that violate the condition of Lemma 5.8. The former set is Haar null using Theorem 4.14, and the latter is Haar null by Lemma 5.8. Hence the union of the two is also Haar null. □

## 6. $\mathrm{Aut}(\mathcal{R})$

6.1. **Notations.** In this section we investigate the automorphism group of the random graph. We first introduce the notations and conventions we will use. We fix an enumeration of $\{v_0, v_1, \dots\}$ of $V$, the vertex set of the graph. If $\mathcal{K} \subset \mathrm{Aut}(\mathcal{R})$ and $M \subset V$ then $\mathcal{K}(M) = \{f(v) : v \in M, f \in \mathcal{K}\}$, similarly $\mathcal{K}^{-1}(M) = \{f^{-1}(v) : v \in M, f \in \mathcal{K}\}$ and $\mathcal{K}|_M = \{f|_M : f \in \mathcal{K}\}$. For a set $M \subset V$ we will denote by $M^*$ the set $M \cup \mathcal{K}^{-1}(M)$. We shall also abuse this notation, for $v \in V$ letting $\mathcal{K}(v) = \mathcal{K}(\{v\})$. Moreover, we will also use the notation $\mathcal{K}^2 = \{ff' : f, f' \in \mathcal{K}\}$ and $\mathcal{K}^{-1} = \{f^{-1} : f \in \mathcal{K}\}$. Note that by Fact 2.1 if $\mathcal{K}$ is compact and $M$ is finite then $\mathcal{K}(M)$ and $\mathcal{K}|_M$ are also finite sets. If $f$ is a function let us use the notation $\mathrm{rd}(f)$ for the set $\mathrm{ran}(f) \cup \mathrm{dom}(f)$. Finally, as before, for the adjacency (resp. non-adjacency) of vertices $x$ and $y$ we will use the notation $xRy$ (resp. $x \neg Ry$).

6.2. **The Splitting Lemma.** In this subsection we prove a theorem, which is interesting on its own.

**Definition 6.1.** Suppose that $M \subset V$ is a finite set and $\tau : M \to 2$ a function. We say that a vertex $v \in V$ *realises* $\tau$ if for every $w \in M$ we have

$$wRv \iff \tau(v) = 1.$$

**Definition 6.2.** Let $M \subset V$ be a finite set and $\mathcal{K} \subset \mathrm{Aut}(\mathcal{R})$ be compact. We call a vertex $v$ a *splitting point for $M$ and $\mathcal{K}$* if for every $h, h' \in \mathcal{K}$ so that $h|_M \neq h'|_M$ we have $h(v) \neq h'(v)$ and $h^{-1}(v) \neq h'^{-1}(v)$.

**Lemma 6.3.** *(Splitting Lemma) Let $\mathcal{K} \subset \mathrm{Aut}(\mathcal{R})$ be a compact set, $M \subset V$ finite, $\tau : M \to 2$ a function and $n \in \omega$. There exists a splitting point for $M$ and $\mathcal{K}$, $v \in V \setminus \{v_i : i \leq n\}$ that realises $\tau$.*

We start the proof of the lemma with a slightly modified special case, namely when we would like to find a splitting point for a pair of automorphisms.

**Lemma 6.4.** *Let $p, p'$ be finite partial automorphisms, $w_0$ a vertex with $p(w_0) \neq p'(w_0)$ and $N \in \omega$. There exist two disjoint finite sets of vertices $A, A' \subset V \setminus \{v_i : i \leq N\}$ with the following property: for a vertex $v$ if for every $w \in A$ we have $wRv$ and for every $w' \in A'$ we have $w' \neg Rv$ then $h(v) \neq h'(v)$ for each $h \in [p] \cap \mathcal{K}$ and $h' \in [p'] \cap \mathcal{K}$.*

*Proof.* Let us use the notation $\mathcal{L} = [p] \cap \mathcal{K}$ and $\mathcal{L}' = [p'] \cap \mathcal{K}$. Take a vertex $w_1 \notin \mathcal{L}(\{v_i : i \leq N\}) \cup \mathcal{L}'(\{v_i : i \leq N\})$ with $w_1 Rp(w_0)$ and $w_1 \neg Rp'(w_0)$, this can be done by the compactness of $\mathcal{L}$ and $\mathcal{L}'$. Now let $A = \mathcal{L}^{-1}(w_1)$ and $A' = \mathcal{L}'^{-1}(w_1)$, again these sets are finite by compactness. Moreover, if $x \in A$ then $x = h^{-1}(w_1)$ for some $h \in \mathcal{L}$. Since $p(w_0) = h(w_0)$ and $w_1 Rp(w_0)$, we have that $w_1 Rh(w_0)$, hence $h^{-1}(w_1) Rw_0$, that is, $xRw_0$. Analogously, $w_0 \neg Rx$ for every $x \in A'$, in particular $A \cap A' = \emptyset$. Notice that $w_1 \notin \mathcal{L}(\{v_i : i \leq N\}) \cup \mathcal{L}'(\{v_i : i \leq N\})$ is equivalent to $\emptyset = (\mathcal{L}^{-1}(w_1) \cup \mathcal{L}'^{-1}(w_1)) \cap \{v_i : i \leq N\}$, thus $(A \cup A') \cap \{v_i : i \leq N\} = \emptyset$.

Finally, we have to check that $A$ and $A'$ have the required property, so take a vertex $v$ with $wRv$ and $w' \neg Rv$ for every $w \in A$ and $w' \in A'$ and two automorphisms

$h \in \mathcal{L}$ and $h' \in \mathcal{L}'$. Clearly, $h^{-1}(w_1) \in \mathcal{L}^{-1}(w_1) = A$ and $h'^{-1}(w_1) \in \mathcal{L}'^{-1}(w_1) = A'$ so $h^{-1}(w_1)Rv$ and $h'^{-1}(w_1)\neg Rv$, consequently, $w_1 Rh(v)$ and $w_1 \neg Rh'(v)$, in particular $h(v) \neq h'(v)$. □

*Proof of the Splitting Lemma.* Let $m_0 > n$ so that $M \cup \mathcal{K}(M) \subset \{v_i : i \leq m_0\}$ and $\mathcal{K}_1 = \mathcal{K} \cup \mathcal{K}^{-1}$. By the compactness of $\mathcal{K}$ the set $M \cup \mathcal{K}(M)$ is finite and $\mathcal{K}_1$ is compact. List the pairs of distinct finite partial automorphisms in $\mathcal{K}_1|_{\{v_i : i \leq m_0\}}$ as $\{(p_j, p'_j) : j < k\}$. Again, from the compactness of $\mathcal{K}_1$ it follows that there are only finitely many such pairs. Using Lemma 6.4 we can inductively define a sequence $m_0 < m_1 < \cdots < m_k$ of natural numbers and a sequence of disjoint finite sets $A_j, A'_j \subset \{v_{m_j}, v_{m_j+1}, \ldots, v_{m_{j+1}}\}$ with the property given by the lemma, that is, for every $j < k$ and $h \in [p_j] \cap \mathcal{K}_1$ and $h' \in [p'_j] \cap \mathcal{K}_1$ if a vertex $v$ is connected to every vertex in $A_j$ and not connected to every vertex in $A'_j$ then $h(v) \neq h'(v)$.

Now take a vertex $v \in V \setminus \{v_i : i \leq m_k\}$ that realises $\tau$ and $v$ is connected to each vertex in $\cup_{j<k} A_j$ and not connected to every vertex in $\cup_{j<k} A'_j$. Clearly, the selection of the sequence $(m_j)_{j<k}$ and $(A_j, A'_j)_{j<k}$ shows that such a vertex exists.

Let $h, h' \in \mathcal{K}$ be arbitrary with $h|_M \neq h'|_M$ and choose a $w_0 \in M$ so that $h(w_0) \neq h'(w_0)$. Then by the definition of $m_0$ clearly $h(w_0) \in \{v_i : i \leq m_0\}$, moreover $h'^{-1}(h(w_0)) \neq h^{-1}(h(w_0)) = w_0$. Consequently, $h^{-1}|_{\{v_i : i \leq m_0\}} \neq h'^{-1}|_{\{v_i : i \leq m_0\}}$. By $M \subset \{v_0, \ldots, v_{m_0}\}$ and using $h, h', h^{-1}, h'^{-1} \in \mathcal{K}_1$ we have that there exist indices $i, j < k$ so that $h|_{\{v_i : i \leq m_0\}} = p_i$ and $h'|_{\{v_i : i \leq m_0\}} = p'_i$ and $h^{-1}|_{\{v_i : i \leq m_0\}} = p_j$ and $h'^{-1}|_{\{v_i : i \leq m_0\}} = p'_j$, consequently, by the choice of $(A_i, A'_i)$ and $(A_j, A'_j)$ we obtain $h(v) \neq h'(v)$ and $h^{-1}(v) \neq h'^{-1}(v)$, which finishes the proof of the theorem. □

6.3. **Translation of compact sets, special case.** In this subsection we will prove that certain types of conjugacy classes are compact biters.

**Definition 6.5.** Let $f \in \mathrm{Aut}(\mathcal{R})$. We say that $f$ has property $(*)_0$ (resp. $(*)_1$) if

- $f$ has only finitely many finite orbits and infinitely many infinite orbits,
- for every finite set $M \subset V$ and $\tau : M \to 2$ there exists a $v$ that realises $\tau$, $v \notin \mathcal{O}^f(M)$ and $v \neg Rf(v)$ (resp. $vRf(v)$).

**Theorem 6.6.** *Suppose that $f$ has property $(*)_0$ or $(*)_1$ and denote by $N$ the union of finite orbits of $f$. Suppose that $\mathcal{K} \subset \mathrm{Aut}(\mathcal{R})$ is a compact set so that for every $h \in \mathcal{K}$ we have $h|_N = f|_N$. Then $\mathcal{K}$ can be translated into the conjugacy class of $f$.*

*In fact, there exist $g, (\phi_h)_{h \in \mathcal{K}} \in \mathrm{Aut}(\mathcal{R})$ so that $g|_N = \phi_h|_N = id_N$ and for every $h \in \mathcal{K}$ we have $\phi_h \circ h \circ g = f \circ \phi_h$.*

Clearly, by the symmetry it is enough to show this theorem for automorphisms having property $(*)_0$.

The idea of the proof is rather simple: we construct $g$ and $(\phi_h)_{h \in \mathcal{K}}$ inductively from finite approximations, every time extending the approximations of $g$ by splitting points for $\mathcal{K}$ and certain finite sets, we also select the new points from far enough (see below the definition of $d_{\mathcal{K}}$). Using this, we will be able to ensure that the requirements on the extensions of the approximations of $\phi_h$ will not interfere.

In order to prove the theorem we need a couple of definitions.

**Definition 6.7.** Let us define a new graph with the same vertex set as $\mathcal{R}$ as follows. Let

$$xEy \iff (\exists h \in \mathcal{K})(h(x) = y \text{ or } h^{-1}(x) = y).$$

We will denote by $d_{\mathcal{K}}(x,y)$ the length of the shortest path between $x$ and $y$ and let it be equal to $\infty$ if there is no such path. For sets of vertices $M, M'$ let

$$d_{\mathcal{K}}(M,M') = \min\{d_{\mathcal{K}}(x,y) : x \in M, y \in M'\}.$$

We will denote by $d_{\mathcal{K}}(M,x)$ the number $d_{\mathcal{K}}(M,\{x\})$.

Note that the function $d_{\mathcal{K}} : V \times V \to \omega \cup \{\infty\}$ is an extended metric.

**Corollary 6.8.** *Suppose that $M$ is a finite set and $\tau : M \to 2$ is a function. There exists a vertex $v$ that is a splitting point for $M$ and $\mathcal{K}$, realises $\tau$ and $d_{\mathcal{K}}(v,M) > 3$.*

*Proof.* By the compactness of $\mathcal{K} \cup \mathcal{K}^{-1}$ the set

$$M \cup \bigcup_{h_1,h_2,h_3 \in \mathcal{K} \cup \mathcal{K}^{-1}} h_1 h_2 h_3(M)$$

is finite, so we can take an $n \in \omega$ so that it is contained in $\{v_i : i \leq n\}$. By the Splitting Lemma (Lemma 6.3) there exists a $v$ so that $v \notin \{v_i : i \leq n\}$ and $v$ realises $\tau$. Clearly, $d_{\mathcal{K}}(v,M) > 3$ holds as well. □

**Definition 6.9.** Let $g$ be a finite partial automorphism and $w \in V$. Suppose that for every $i \in \mathbb{Z} \setminus \{0\}$ we have $g^i(w) \neq w$. Then we will denote by $e(w,g)$ the vertex $g^i(w)$ so that

$$i = \max\{j \in \omega : g^j(w) \text{ is defined,}$$
$$\text{i. e., for every } k \text{ with } 0 \leq k < j \text{ we have } g^k(w) \in \mathrm{dom}(g)\},$$

and similarly we denote by $b(w,g)$ the vertex $g^{-i}(w)$ so that

$$i = \max\{j \in \omega : g^{-j}(w) \text{ is defined,}$$
$$\text{i. e., for every } k \text{ with } 0 \leq k < j \text{ we have } g^{-k}(w) \in \mathrm{ran}(g)\},$$

or equivalently, the vertex $e(w,g^{-1})$.

Note that if $w \notin \mathrm{dom}(g)$ then $e(w,g) = w$ and also if $w \notin \mathrm{ran}(g)$ then $b(w,g) = w$.

In the next two definitions we will describe possible set-ups that could be obstacles to carry out the inductive procedure.

**Definition 6.10.** Let $h, h' \in \mathcal{K}$ and $g$, $\phi_h$ and $\phi_{h'}$ be partial automorphisms. We call the following set-up an $(h, h', \phi_h, \phi_{h'}, g)$ *bad situation*: there exist vertices $x, x', y \in V$ so that

(B1) $x \in N$ or $x = b(x, h \circ g)$,
$x' \in N$ or $x' = b(x', h' \circ g)$,
$y \in N$ or $y = e(y, h \circ g) = e(y, h' \circ g)$,
(B2) $h^{-1}(x) = h'^{-1}(x')$,
(B3) (a) $x, y \in \mathrm{dom}(\phi_h)$, $x', y \in \mathrm{dom}(\phi_{h'})$
(b) it is not true that

$$\phi_h(x) R f(\phi_h(y)) \iff \phi_{h'}(x') R f(\phi_{h'}(y)).$$

In case we would like to specify the roles of vertices, we will also call such a set-up an $(h, h', \phi_h, \phi_{h'}, g, x, x', y)$ *bad situation*, or when clear from the context, an $(h, h', x, x', y)$ *bad situation.*

**Definition 6.11.** Let $h, h' \in \mathcal{K}$ and $g$, $\phi_h$ and $\phi_{h'}$ be partial automorphisms. We call the following set-up an $(h, h', \phi_h, \phi_{h'}, g)$ *ugly situation*: there exist vertices $x, y \in V$ so that $(h, h', \phi_h, \phi_{h'}, x, y, y)$ has Properties (B1), (B2) of bad situations,

(U1) $y \notin \mathrm{dom}(\phi_{h'})$,
(U2) $x, y \in \mathrm{dom}(\phi_h)$ and $\phi_h(x) R f(\phi_h(y))$.

We will use the conventions used at bad situations in the naming of ugly situations as well.

Now we are ready to formulate our inductive assumptions. Recall that $M^*$ stands for $M \cup \mathcal{K}^{-1}(M)$.

**Definition 6.12.** We say that the triple $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ is *good* if the following conditions hold for every $h, h' \in \mathcal{K}$:

(i) $M$ is a finite set of vertices, $g$ and $\phi_h$ are partial automorphisms,
(ii) $\mathrm{dom}(\phi_h) \supset \mathrm{rd}(h \circ g)$ and $N \cup \mathrm{rd}(g) \cup \mathrm{dom}(\phi_h) \subset M$,
(iii) $N \subset \mathrm{rd}(g)$, $N \subset \mathrm{dom}(\phi_h)$, $g|_N = \phi_h|_N = id|_N$,
(iv) $\phi_h \circ h \circ g = f \circ \phi_h$, i. e., whenever both of the sides of the equation are defined then they are equal,
(v) for vertices $w, w' \in \mathrm{dom}(\phi_h) \setminus N$ we have that $\mathcal{O}^{h\circ g}(w) \neq \mathcal{O}^{h\circ g}(w')$ implies $\mathcal{O}^f(\phi_h(w)) \neq \mathcal{O}^f(\phi_h(w'))$,
(vi) if $h|_{M^*} = h'|_{M^*}$ then $\phi_h = \phi_{h'}$,
(vii) if $w \in V \setminus N$ then for every $i \in \mathbb{Z}\setminus\{0\}$ we have $(h\circ g)^i(w) \neq w$, in particular, the functions $b(w, h \circ g), e(w, h \circ g)$ are defined for every $w \in V \setminus N$,
(viii) for every $w \in \mathrm{dom}(\phi_h)$ if $f(\phi_h(w)) R \phi_h(w)$ and $h'^{-1}(w) = h^{-1}(w)$ hold then $w \in \mathrm{dom}(\phi_{h'})$,
(ix) there are no $(h, h', \phi_h, \phi_{h'}, g)$ ugly situations,
(x) there are no $(h, h', \phi_h, \phi_{h'}, g)$ bad situations.

We start the proof with a couple of trivial observations.

**Remark 6.13.** It is easy to see that if $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ is a good triple and $\overline{M} \supset M$ is finite then $(g, (\phi_h)_{h\in\mathcal{K}}, \overline{M})$ is also a good triple.

**Lemma 6.14.** *$(id_N, (id_N)_{h\in\mathcal{K}}, N)$ is a good triple.*

*Proof.* Properties (i)-(viii) are obvious. We check the remaining two properties:

(ix) note that if $\mathrm{dom}(\phi_h) = \mathrm{dom}(\phi_{h'})$ then there are no $(h, h', \phi_h, \phi_{h'}, g)$ ugly situations as the conjunction of (U1) and (U2) cannot be true,
(x) if we had an $(h, h', id_N, id_N, id_N, x, x', y)$ bad situation, then by property (B3.a) we would have $x, x' \in N$ so by (B2), $h|_N = h'|_N = f|_N$ and the fact that $N$ is the union of orbits of $f$ clearly $x = x'$, but then (B3.b) could not be true.

□

**Lemma 6.15.** *Let $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ be a good triple and $h, h' \in \mathcal{K}$. Suppose that $g \subset \overline{g}$, $\phi_h \subset \overline{\phi}_h$ and $\phi_{h'} \subset \overline{\phi}_{h'}$ are partial automorphisms. Suppose moreover that if $\phi_h \subsetneqq \overline{\phi}_h$ then we have $\{v\} = \mathrm{dom}(\overline{\phi}_h) \setminus \mathrm{dom}(\phi_h)$ and similarly if $\phi_{h'} \subsetneqq \overline{\phi}_{h'}$ then we have $\{v'\} = \mathrm{dom}(\overline{\phi}_{h'}) \setminus \mathrm{dom}(\phi_{h'})$ so that*

*(a) $h^{-1}(v) \neq h'^{-1}(v)$,*
*(b) $d_{\mathcal{K}}(v, M) > 2$ and $d_{\mathcal{K}}(v', M) > 2$*

*then*

*(1) there are no $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, \overline{g})$ ugly situations.*
*(2) if for some $x, x', y$ there exists an $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, \overline{g}, x, x', y)$ bad situation then either*

- *$y \in M$, $x = v$ and $x' = v'$ OR*
- *$y = v = v'$ and $x, x' \in M$,*

*(in particular, $v$ and $v'$ must exist).*

*Proof.* First notice that in the definition of both ugly and bad situations the automorphism $g$ is only used in property (B1), and this property does not use $\phi_h$ or $\phi_{h'}$. Moreover, by the definition of functions $b$ and $e$ clearly if $\overline{g} \supset g$ and $x = b(x, h \circ \overline{g})$ then $x = b(x, h \circ g)$ and similarly for $e$. Therefore, if (B1) holds for $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, \overline{g}, x, x', y)$ then it also holds for $(h, h', \phi_h, \phi_{h'}, g, x, x', y)$.

Notice that using this observation about (B1) we can conclude that if $x, y \in \mathrm{dom}(\phi_h)$ and $x', y \in \mathrm{dom}(\phi_{h'})$ and there is an $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, \overline{g}, x, x', y)$ then it is also an $(h, h', \phi_h, \phi_{h'}, g, x, x', y)$ bad situation (and similarly with $x, y \in \mathrm{dom}(\phi_h)$ for ugly situations). So in order to prove the impossibility of an $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, \overline{g}, x, x', y)$ bad situation, since $(g, (\phi_h)_{h \in \mathcal{K}}, M)$ is a good triple it is enough to show that $x, y \in \mathrm{dom}(\phi_h)$ and $x', y \in \mathrm{dom}(\phi_{h'})$ (and analogously for ugly situations).

Now we prove the statements of the lemma.

(1) If there exists a $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, \overline{g}, x, y)$ ugly situation then by the above argument and the fact that we have (possibly) extended $\phi_h$ only to $v$ and $\phi_{h'}$ only to $v'$, we get $\{x, y\} \cap \{v, v'\} \neq \emptyset$. Moreover, from the definition of an ugly situation (B2) holds for $x$ and $x' = y$ so clearly $d_{\mathcal{K}}(x, y) \leq 2$. This implies by assumption (b) that $x, y \notin M$. Using (U2) we obtain $\{x, y\} \subset \mathrm{dom}(\overline{\phi}_h) \setminus M$ and by Property (ii) of good triples $\mathrm{dom}(\phi_h) \subset M$, so $\{x, y\} \subset \mathrm{dom}(\overline{\phi}_h) \setminus \mathrm{dom}(\phi_h) = \{v\}$. But (B2) gives that $h^{-1}(x) = h'^{-1}(y)$, so $h^{-1}(v) = h'^{-1}(v)$ contradicting the assumption (a) of the lemma.

Now we prove (2).

Suppose $y \notin M$. Then by (B3.a) we have $y \in \mathrm{dom}(\overline{\phi}_h) \cap \mathrm{dom}(\overline{\phi}_{h'}) \setminus M$, which is only possible using Property (ii) of good triples if $y = v = v'$. Since $x \in \mathrm{dom}(\overline{\phi}_h)$ clearly, $x \notin M$ can happen only if $x = v = y$. Then, by (B2) we have $d_{\mathcal{K}}(x, x') \leq 2$, so $x' \in \mathrm{dom}(\overline{\phi}_h) \setminus M$, therefore $x' = v' = y$. But then, using again (B2) we get $h^{-1}(v) = h^{-1}(x) = h'^{-1}(x') = h'^{-1}(v)$, contradicting (a). So $x \in M$ and a similar argument shows $x' \in M$.

So assume $y \in M$, in particular by (B3.a) and the assumptions of the lemma $y \in \mathrm{dom}(\phi_h) \cap \mathrm{dom}(\phi_{h'})$. Suppose now that $x \neq v$ (with the possibility that $v$ does not exists). Since by (B3.a) we have $x \in \mathrm{dom}(\overline{\phi}_h)$ and $\mathrm{dom}(\overline{\phi}_h) \subset \{v\} \cup M$ clearly $x \in M$. Using property (B2) and (B3.a) we get $d_{\mathcal{K}}(x, x') \leq 2$ and $x' \in \mathrm{dom}(\overline{\phi}_{h'})$ but by assumption (b) this can happen only if $x' \in M$, so $x' \in \mathrm{dom}(\phi_{h'})$. Therefore $x, y \in \mathrm{dom}(\phi_h)$ and $x', y \in \mathrm{dom}(\phi_{h'})$ which is impossible. Thus, $x = v$ and similarly $x' = v'$. □

Now we prove a lemma which ensures that a good triple can be extended.

**Lemma 6.16.** *Suppose that $(g, (\phi_h)_{h \in \mathcal{K}}, M)$ is a good triple and $v \in \bigcap_{h \in \mathcal{K}} \mathrm{dom}(\phi_h)$. Then there exist extensions $\overline{g} \supset g$, $\overline{\phi}_h \supset \phi_h$ and $\overline{M} \supset M$ so that $(\overline{g}, (\overline{\phi}_h)_{h \in \mathcal{K}}, \overline{M})$ is a good triple and $v \in \mathrm{dom}(\overline{g})$.*

*Proof.* We will find a suitable vertex $\overline{v}$ and let $\overline{g} = g \cup \langle v, \overline{v} \rangle$.

Define a map $\tau_g : \mathrm{ran}(g) \to 2$ as follows:

$$\tau_g(w) = 1 \iff g^{-1}(w) R v, \tag{19}$$

and maps $\tau_h : h^{-1}(\mathrm{dom}(\phi_h)) \to 2$ for each $h \in \mathcal{K}$ as

$$\tau_h(w) = 1 \iff \phi_h(h(w)) R f(\phi_h(v)). \tag{20}$$

**Claim 6.17.** *The maps $\tau_g, (\tau_h)_{h\in\mathcal{K}}$ are compatible, i. e., $\tau = \tau_g \cup \bigcup_{h\in\mathcal{K}} \tau_h$ is a function.*

*Proof of the Claim. $\tau_g$ and $\tau_h$ are compatible.* Let $w \in \mathrm{ran}(g) = \mathrm{dom}(\tau_g)$ and let $h \in \mathcal{K}$ be arbitrary. Clearly, $g^{-1}(w) \in \mathrm{dom}(h \circ g) \subset \mathrm{dom}(\phi_h)$ and $(h \circ g)(g^{-1}(w)) \in \mathrm{ran}(h \circ g) \subset \mathrm{dom}(\phi_h)$ by Property (ii) of good triples. Therefore, we can use Property (iv) for $g^{-1}(w)$ (that is, in the following equation both of the sides are defined):

$$(\phi_h \circ h \circ g)(g^{-1}(w)) = (f \circ \phi_h)(g^{-1}(w))$$

so we get

$$f^{-1}(\phi_h(h(w))) = \phi_h(g^{-1}(w)). \tag{21}$$

As $f$ is an automorphism

$$\tau_h(w) = 1 \iff \phi_h(h(w)) R f(\phi_h(v)) \iff f^{-1}(\phi_h(h(w))) R \phi_h(v). \tag{22}$$

So by (21), (22) and the fact that $\phi_h$ is a partial automorphism we have

$$\tau_h(w) = 1 \iff \phi_h(g^{-1}(w)) R \phi_h(v) \iff g^{-1}(w) R v.$$

Comparing this equation to the definition of $\tau_g$ we obtain that $\tau_g$ and $\tau_h$ are indeed compatible.

*$\tau_h$ and $\tau_{h'}$ are compatible.* Now, using the first case it is enough to check compatibility for $w \notin \mathrm{ran}(g)$. We will use Property (x), that there are no bad situations. Let us consider the sequence $(h, h', \phi_h, \phi_{h'}, g, h(w), h'(w), v)$. Clearly, since $w \notin ran(g)$, we have $h(w) \notin \mathrm{ran}(h \circ g)$ thus $b(h(w), h \circ g) = h(w)$ and similarly $b(h'(w), h' \circ g) = h'(w)$. Moreover, as $v \notin \mathrm{dom}(g)$ we have $e(v, h \circ g) = e(v, h' \circ g) = v$, so Property (B1) of bad situations hold. Moreover, $h^{-1}(h(w)) = w = h'^{-1}(h'(w))$, therefore Property (B2) is also true. Clearly, by the assumptions of Lemma 6.16 we have $v, h(w) \in \mathrm{dom}(\phi_h)$ and $v, h'(w) \in \mathrm{dom}(\phi_{h'})$. Hence, as there are no bad situations Property (B3.b) must fail, consequently

$$\phi_h(h(w)) R f(\phi_h(v)) \iff \phi_{h'}(h'(w)) R f(\phi_{h'}(v)),$$

so, using this and the definition of $\tau_h$ and $\tau_{h'}$ we get

$$\tau_h(w) = 1 \iff \phi_h(h(w)) R f(\phi_h(v)) \iff$$
$$\phi_{h'}(h'(w)) R f(\phi_{h'}(v)) \iff \tau_{h'}(w) = 1.$$

This finishes the proof of the claim. □

Now we return to the proof of Lemma 6.16. By Corollary 6.8 there exists a splitting point $\overline{v}$ for $M^*$ and $\mathcal{K}$ that realises $\tau$ and $d_{\mathcal{K}}(\overline{v}, M^*) > 3$ (in particular, by $M \subset M^*$ we have $d_{\mathcal{K}}(\overline{v}, M) > 3$) extending $\tau$ to the whole $M^*$ arbitrarily if necessary. Let $\overline{g} = g \cup \langle v, \overline{v} \rangle$, $\overline{M} = M \cup \{\overline{v}, h(\overline{v}) : h \in \mathcal{K}\}$ and for every $h \in \mathcal{K}$ let $\overline{\phi}_h = \phi_h \cup \langle h(\overline{v}), f(\phi_h(v)) \rangle$.

We claim that $(\overline{g}, (\overline{\phi}_h)_{h\in\mathcal{K}}, \overline{M})$ is a good triple.

(i) By compactness $\overline{M}$ is finite. We check that $\overline{g}$ and $\overline{\phi}_h$ are partial automorphisms. Since $d_{\mathcal{K}}(\overline{v}, M) > 3$ and Property (ii) of good triples $\mathrm{ran}(g) \subset M$ so the function $\overline{g}$ is injective.

We check the injectivity of the functions $\overline{\phi}_h$. If for some $w$ we have

$$\overline{\phi}_h(h(\overline{v})) = f(\phi_h(v)) = \phi_h(w) \tag{23}$$

then using the facts that $\phi_h|_N = id|_N$ and that $N$ is the union of the finite orbits of $f$ we can conclude that $w \in N$ would imply $\phi_h(v) \in N$, so $v \in N \subset \mathrm{dom}(g)$ which is impossible. So $w \notin N$ and also $\phi_h(w) \notin N$. By (23) we have that $\mathcal{O}^f(\phi_h(v)) = \mathcal{O}^f(\phi_h(w))$ and clearly, $v \notin N$, so using Property (v) of good triples we obtain $\mathcal{O}^{h\circ g}(v) = \mathcal{O}^{h\circ g}(w)$. Then as $v \notin \mathrm{dom}(g)$ clearly $v = (h \circ g)^k(w)$ for some $k \geq 0$. Suppose $k > 0$. Applying $\phi_h$ to both sides and using (iv) of good triples we get

$$\phi_h(v) = \phi_h((h \circ g)^k(w)) = f(\phi_h(h \circ g)^{k-1}(w)) = \cdots = f^k(\phi_h(w)),$$

but then $f(\phi_h(v)) = f^{k+1}(\phi_h(w))$ therefore $f^{k+1}(\phi_h(w)) = \phi_h(w)$, contradicting the fact that $f$ has only infinite orbits outside of $N$. Thus $k = 0$ and $v = w$, so $\overline{\phi}_h$ is indeed injective.

So we only have to check $\overline{g}$ and $\overline{\phi}_h$ preserve the relation, that is, for every $w, w' \in \mathrm{dom}(\overline{g})$ distinct we have

$$wRw' \iff \overline{g}(w)R\overline{g}(w'),$$

and it is enough to check this condition if $\{w, w'\} \not\subset \mathrm{dom}(g)$ (and similarly for $\phi_h$). So suppose that $w \in \mathrm{dom}(g)$ and $w' \in \mathrm{dom}(\overline{g}) \setminus \mathrm{dom}(g)$, that is, $w' = v$. Then by the fact that $g(w) \in \mathrm{ran}(g) = \mathrm{dom}(\tau_g)$, (19) and the definition of $\tau$ we have

$$\overline{g}(w')R\overline{g}(w) \iff \overline{g}(v)R\overline{g}(w) \iff \overline{v}Rg(w) \iff \tau(g(w)) = 1 \iff$$

$$\tau_g(g(w)) = 1 \iff g^{-1}(g(w))Rv \iff wRv \iff wRw',$$

so indeed, $\overline{g}$ preserves the relation.

Now if $w \in \mathrm{dom}(\phi_h)$ and $w' \in \mathrm{dom}(\overline{\phi}_h) \setminus \mathrm{dom}(\phi_h)$, that is, $w' = h(\overline{v})$ then $h^{-1}(w) \in h^{-1}(\mathrm{dom}(\phi_h)) = \mathrm{dom}(\tau_h)$. Then we have

$$\overline{\phi}_h(w)R\overline{\phi}_h(w') \iff \phi_h(w)R\overline{\phi}_h(h(\overline{v}))$$

which is by the definition of $\overline{\phi}_h$

$$\iff \phi_h(w)Rf(\phi_h(v)) \iff \phi_h(h(h^{-1}(w)))Rf(\phi_h(v))$$

using the definition of $\tau$ and (20) we get

$$\iff \tau_h(h^{-1}(w)) = 1 \iff h^{-1}(w)R\overline{v} \iff wRh(\overline{v}) \iff wRw',$$

so we are done.

(ii) By the definition of $\overline{\phi}_h$ we have $\mathrm{dom}(\overline{\phi}_h) = \mathrm{dom}(\phi_h) \cup \{h(\overline{v})\} \supset \mathrm{rd}(h \circ g) \cup \{v, h(\overline{v})\}$ and using the fact that $d_{\mathcal{K}}(\overline{v}, \mathrm{dom}(\overline{g})) > 3$ we obtain that $h(\overline{v}) \notin \mathrm{dom}(g)$, thus $\mathrm{rd}(h \circ g) \cup \{v, h(\overline{v})\} = \mathrm{rd}(h \circ \overline{g})$. Moreover, $\mathrm{rd}(\overline{g}) \cup \mathrm{dom}(\overline{\phi}_h) \subset \overline{M}$.

(iii) Obvious.

(iv) It is enough to check equality $(\overline{\phi}_h \circ h \circ \overline{g})(v_0) = (f \circ \overline{\phi}_h)(v_0)$ for $v_0 = v$, as for $v_0 \in \mathrm{dom}(g)$ we have $v_0 \in \mathrm{dom}(g) \subset \mathrm{dom}(\phi_h)$ and $h(g(v_0)) \in \mathrm{rd}(h \circ g) \subset \mathrm{dom}(\phi_h)$ so the equality holds because we started with a good triple. But using the definition of $\overline{\phi}_h$ we have $\overline{\phi}_h(h(\overline{g}(v))) = \overline{\phi}_h(h(\overline{v})) = f(\overline{\phi}_h(v))$.

(v) Suppose that for vertices $w, w' \in \mathrm{dom}(\overline{\phi}_h) \setminus N$ we have that $\mathcal{O}^{h\circ\overline{g}}(w) \neq \mathcal{O}^{h\circ\overline{g}}(w')$. Then of course $w \neq w'$. We have extended $\phi_h$ only to $h(\overline{v})$ so it is enough to check the property with $w = h(\overline{v})$ and $w' \in \mathrm{dom}(\phi_h)$. But $\mathcal{O}^{h\circ\overline{g}}(h(\overline{v})) = \mathcal{O}^{h\circ\overline{g}}(h(\overline{g}(v))) = \mathcal{O}^{h\circ\overline{g}}(v)$, thus, $\mathcal{O}^{h\circ g}(v) \cap \mathcal{O}^{h\circ g}(w') \subset \mathcal{O}^{h\circ\overline{g}}(h(\overline{v})) \cap \mathcal{O}^{h\circ\overline{g}}(w') = \emptyset$. Therefore, by the fact that $h(\overline{v}) \notin N$ implies $v \notin N$ and Property (v) of good triples we get $\mathcal{O}^f(\phi_h(v)) \cap \mathcal{O}^f(\phi_h(w')) = \emptyset$. Using this and the definition of $\overline{\phi}_h$ we have $\mathcal{O}^f(\overline{\phi}_h(h(\overline{v}))) = \mathcal{O}^f(\phi_h(v))$ and $\mathcal{O}^f(\overline{\phi}_h(v)) \cap \mathcal{O}^f(\overline{\phi}_h(w')) = \mathcal{O}^f(\phi_h(v)) \cap \mathcal{O}^f(\phi_h(w')) = \emptyset$ so we are done.

(vi) If $h, h' \in \mathcal{K}$ and $h|_{\overline{M}^*} = h'|_{\overline{M}^*}$ then in particular $h(\overline{v}) = h'(\overline{v})$ and $h|_{M^*} = h'|_{M^*}$ so $\phi_h = \phi_{h'}$. Then by definition $\overline{\phi}_h = \overline{\phi}_{h'}$.

(vii) If $h \in \mathcal{K}$ and $w \notin N$ is a vertex and for some $i \in \omega \setminus \{0\}$ we have $(h\circ\overline{g})^i(w) = w$ then at least one of the points $\{w, \dots, (h\circ\overline{g})^{i-1}(w)\}$ is not in the domain of $g$, otherwise the triple $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ would already violate this property of good conditions. In other words, $v \in \{w, \dots, (h\circ\overline{g})^{i-1}(w)\}$. Moreover, $\{w, \dots, (h \circ \overline{g})^{i-1}(w)\} \subset \mathrm{dom}(\overline{g})$ and clearly $(h \circ \overline{g})(\{w, \dots, (h \circ \overline{g})^{i-1}(w)\}) = \{w, \dots, (h \circ \overline{g})^{i-1}(w)\}$, so $(h \circ \overline{g})(v) \in \mathrm{dom}(\overline{g})$, that is, $h(\overline{v}) \in \mathrm{dom}(\overline{g})$. But we know that $\mathrm{dom}(\overline{g}) = \{v\} \cup \mathrm{dom}(g) \subset \cap_h \mathrm{dom}(\phi_h) \cup \mathrm{dom}(g) \subset M$. Therefore $h(\overline{v}) \in M$, contradicting the assumption that $d_{\mathcal{K}}(\overline{v}, M) > 3$.

(viii) Since $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ is a good triple, this condition can fail for $(\overline{g}, (\overline{\phi}_h)_{h\in\mathcal{K}}, \overline{M})$, an $h$ and $w$ only if $w \in \mathrm{dom}(\overline{\phi}_h) \setminus \mathrm{dom}(\phi_h)$, in other words $w = h(\overline{v})$. So suppose $h^{-1}(w) = h'^{-1}(w)$, that is, $\overline{v} = h^{-1}(h(\overline{v})) = h'^{-1}(h(\overline{v}))$. This means that $h'(\overline{v}) = h(\overline{v})$, but then by (ii) we have $w = h'(\overline{v}) \in \mathrm{dom}(\overline{\phi}_{h'})$ as well.

(ix) Suppose that there exists an $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, x, y)$ ugly situation. If $h|_{M^*} = h'|_{M^*}$ and $h(\overline{v}) = h'(\overline{v})$ then $\overline{\phi}_h = \overline{\phi}_{h'}$ which contradicts properties (U1) and (U2).

Now if $h|_{M^*} \neq h'|_{M^*}$ or $h(\overline{v}) \neq h'(\overline{v})$, then since $\overline{v}$ is a splitting point we have $h(\overline{v}) \neq h'(\overline{v})$. Consequently, $h^{-1}(h(\overline{v})) \neq h'^{-1}(h(\overline{v}))$ and also by $d_{\mathcal{K}}(\overline{v}, M) > 3$ clearly $d_{\mathcal{K}}(h(\overline{v}), M) > 2$ and $d_{\mathcal{K}}(h'(\overline{v}), M) > 2$. Then we can apply Lemma 6.15 for $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, x, y)$ and $\{h(\overline{v})\} \supset \mathrm{dom}(\overline{\phi}_h) \setminus \mathrm{dom}(\phi_h)$, $\{h'(\overline{v})\} \supset \mathrm{dom}(\overline{\phi}_{h'}) \setminus \mathrm{dom}(\phi_{h'})$ by which there are no $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, x, y)$ ugly situations.

(x) Here the argument is similar. Suppose that there exists an $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, x, x', y)$ bad situation. If $h|_{M^*} = h'|_{M^*}$ and $h(\overline{v}) = h'(\overline{v})$ then $\overline{\phi}_h = \overline{\phi}_{h'}$ but then (B3) cannot be true. Now if $h|_{M^*} \neq h'|_{M^*}$ then as in the previous point we can use Lemma 6.15 for $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, x, x', y)$. Therefore, either $x = h(\overline{v})$, $x' = h'(\overline{v})$ or $y = h(\overline{v}) = h'(\overline{v})$. But the second option is impossible since $y = h(\overline{v}) = h'(\overline{v})$ contradicts that $\overline{v}$ was a splitting point. Now the first option is also impossible unless $x, x' \in N$: as $\overline{v} = \overline{g}(v)$, that is, $x = h(\overline{g}(v))$, contradicting (B1). But if $x, x' \in N \subset M$ then $d_{\mathcal{K}}(\overline{v}, M) \leq 1$, a contradiction again.

□

Now we prove a lemma which allows us to extend $g$ backwards. The proof is very similar to the proof of the forward extension, although to treat both cases in the same framework would have a great technical cost. For the sake of completeness we write down the proofs in detail.

**Lemma 6.18.** *Suppose that $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ is a good triple and $v \in M$ is a vertex so that for every $h \in \mathcal{K}$ we have $h(v) \in \mathrm{dom}(\phi_h)$. Then there exist extensions $\overline{g} \supset g$, $\overline{\phi}_h \supset \phi_h$ and $\overline{M} \supset M$ so that $(\overline{g}, (\overline{\phi}_h)_{h\in\mathcal{K}}, \overline{M})$ is a good triple and $v \in \mathrm{ran}(\overline{g})$.*

*Proof.* We will find a suitable vertex $\overline{v}$ and let $\overline{g} = g \cup \langle \overline{v}, v\rangle$.

Define a map $\tau_g : \mathrm{dom}(g) \to 2$ as follows:

$$\tau_g(w) = 1 \iff g(w)Rv, \tag{24}$$

and maps $\tau_h : \mathrm{dom}(\phi_h) \to 2$ for each $h \in \mathcal{K}$

$$\tau_h(w) = 1 \iff \phi_h(w)Rf^{-1}(\phi_h(h(v))). \tag{25}$$

**Claim 6.19.** *The maps $\tau_g, (\tau_h)_{h\in\mathcal{K}}$ are compatible, i. e., $\tau = \tau_g \cup \bigcup_{h\in\mathcal{K}} \tau_h$ is a function.*

*Proof of the Claim. $\tau_g$ and $\tau_h$ are compatible.* Let $h \in \mathcal{K}$ be arbitrary and $w \in \mathrm{dom}(\tau_g) \cap \mathrm{dom}(\tau_h) = \mathrm{dom}(g) \cap \mathrm{dom}(\tau_h)$. Clearly, by Property (ii) of good triples we have $w, h(g(w)) \in \mathrm{dom}(\phi_h)$. So we can use Property (iv) for $w$ and we get

$$(\phi_h \circ h \circ g)(w) = (f \circ \phi_h)(w).$$

From the definition of $\tau_h$ we obtain

$$\tau_h(w) = 1 \iff \phi_h(w)Rf^{-1}(\phi_h(h(v))) \iff f(\phi_h(w))R\phi_h(h(v)).$$

Putting together these equations and using that $\phi_h$ is an automorphism we obtain

$$\tau_h(w) = 1 \iff (\phi_h \circ h \circ g)(w)R\phi_h(h(v)) \iff g(w)Rv \iff \tau_g(w) = 1.$$

*$\tau_h$ and $\tau'_h$ are compatible.* Let $h, h' \in \mathcal{K}$ be arbitrary and $w \in \mathrm{dom}(\tau_h) \cap \mathrm{dom}(\tau_{h'})$. By the fact that $\tau_h$ and $\tau'_h$ are compatible with $\tau_g$ we can assume $w \notin \mathrm{dom}(\tau_g) = \mathrm{dom}(g)$.

We will use Property (x), that there are no bad situations. Let us consider the sequence $(h, h', \phi_h, \phi_{h'}, g, h(v), h'(v), w)$. Clearly, by $v \notin \mathrm{ran}(g)$ we have $b(h(v), h \circ g) = h(v)$ and similarly $b(h'(v), h' \circ g) = h'(v)$. Moreover, as $w \notin \mathrm{dom}(g)$, we have $e(w, h \circ g) = e(w, h' \circ g) = w$, so Property (B1) of Definition 6.10 holds. Obviously, $h^{-1}(h(v)) = h'^{-1}(h'(v))$, therefore Property (B2) is also true. By the assumptions of Lemma 6.18 clearly $h(v), w \in \mathrm{dom}(\phi_h)$ and $h'(v), w \in \mathrm{dom}(\phi_{h'})$, hence, as there are no bad situations Property (B3.b) must fail, consequently

$$\phi_h(h(v))Rf(\phi_h(w)) \iff \phi_{h'}(h'(v))Rf(\phi_{h'}(w)),$$

so, by definition of $\tau_h$ and $\tau_{h'}$ we get

$$\tau_h(w) = 1 \iff f^{-1}(\phi_h(h(v)))R\phi_h(w) \iff \phi_h(h(v))Rf(\phi_h(w)) \iff$$

$$\phi_{h'}(h'(v))Rf(\phi_{h'}(w)) \iff f^{-1}(\phi_{h'}(h'(v)))R\phi_h(w) \iff \tau_{h'}(w) = 1.$$

This finishes the proof of the claim. □

Now we return to the proof of Lemma 6.18. By Corollary 6.8 there exists a splitting point $\overline{v}$ for $M^*$ and $\mathcal{K}$ that realises $\tau$ and $d_{\mathcal{K}}(\overline{v}, M^*) > 3$. Let $\overline{g} = g \cup \langle \overline{v}, v\rangle$, $\overline{M} = M \cup \{\overline{v}\}$ and for every $h \in \mathcal{K}$ let $\overline{\phi}_h = \phi_h \cup \langle \overline{v}, f^{-1}(\phi_h(h(v)))\rangle$.

We claim that $(\overline{g}, (\overline{\phi}_h)_{h\in\mathcal{K}}, \overline{M})$ is a good triple.

(i) We check that $\overline{g}$ and $\overline{\phi}_h$ are partial automorphisms. Since $d_{\mathcal{K}}(\overline{v}, M) > 2$, $\mathrm{dom}(g) \subset M$ and $d_{\mathcal{K}}(\overline{v}, M) > 2$ the function $\overline{g}$ is injective.

We check the injectivity of functions $\overline{\phi}_h$.

If for some $w$ we have

$$\overline{\phi}_h(\overline{v}) = f^{-1}(\phi_h(h(v))) = \phi_h(w) \tag{26}$$

then using the facts that $\phi_h|_N = id|_N$ and that $N$ is the union of the finite orbits of $f$ we can conclude again that $w \in N$ would imply $\phi_h(h(v)) \in N$, and thus $v, h(v) \in N \subset \mathrm{ran}(g)$ which is impossible. So $w, h(v) \notin N$. But by (26) we have $\mathcal{O}^f(\phi_h(h(v))) = \mathcal{O}^f(\phi_h(w))$ so using Property (v) of good triples we obtain $\mathcal{O}^{h\circ g}(h(v)) = \mathcal{O}^{h\circ g}(w)$. Then as $v \notin \mathrm{ran}(g)$ clearly $w = (h \circ g)^k(h(v))$ for some $k \geq 0$. Suppose $k > 0$. Applying $\phi_h$ to both sides and using Property (iv) of good triples we get

$$\phi_h(w) = \phi_h((h \circ g)^k(h(v))) =$$

$$= f(\phi_h((h \circ g)^{k-1}(h(v)))) = \cdots = f^k(\phi_h(h(v))),$$

but then $f(\phi_h(w)) = f^{k+1}(\phi_h(h(v)))$, therefore by (26) we get $f^{k+1}(\phi_h(h(v))) = \phi_h(h(v))$, contradicting the fact that $f$ has only infinite orbits outside of $N$. Thus $k = 0$ and $\overline{v} = w$, so $\overline{\phi}_h$ is indeed injective.

We have to check $\overline{g}$ preserves the relation, and again it is enough to check for $w \in \mathrm{dom}(g)$ and $w' \in \mathrm{dom}(\overline{g}) \setminus \mathrm{dom}(g)$, that is, $w' = \overline{v}$. Then by the fact that $w \in \mathrm{dom}(g) = \mathrm{dom}(\tau_g)$, (19) and the definition of $\tau$ we have

$$\overline{g}(w')R\overline{g}(w) \iff \overline{g}(\overline{v})R\overline{g}(w) \iff vRg(w) \iff \tau_g(w) = 1 \iff$$

$$wR\overline{v} \iff wRw',$$

so indeed, $\overline{g}$ preserves the relation.

Now if $w \in \mathrm{dom}(\phi_h)$ and $w' \in \mathrm{dom}(\overline{\phi}_h) \setminus \mathrm{dom}(\phi_h)$, that is, $w' = \overline{v}$ then we have

$$\overline{\phi}_h(w)R\overline{\phi}_h(w') \iff \phi_h(w)R\overline{\phi}_h(\overline{v})$$

which is by the definition of $\overline{\phi}_h$, $\tau$ and (25)

$$\iff \phi_h(w)Rf^{-1}(\phi_h(h(v))) \iff \tau_h(w) = 1 \iff wR\overline{v},$$

so we are done.

(ii) By the definition of $\overline{\phi}_h$ we have $\mathrm{dom}(\overline{\phi}_h) = \mathrm{dom}(\phi_h) \cup \{\overline{v}\} \supset \mathrm{rd}(h \circ g) \cup \{\overline{v}\}$ and using the fact that $d_{\mathcal{K}}(\overline{v}, \mathrm{dom}(\overline{g})) > 3$ we obtain that $h^{-1}(\overline{v}) \notin \mathrm{dom}(g)$, thus $\mathrm{rd}(h \circ g) \cup \{\overline{v}\} = \mathrm{rd}(h \circ \overline{g})$. Clearly, $\mathrm{rd}(\overline{g}) \cup \mathrm{dom}(\overline{\phi}_h) \subset \overline{M}$.

(iii) Obvious.

(iv) It is enough to check equality $\overline{\phi}_h \circ h \circ \overline{g}(v_0) = f \circ \overline{\phi}_h(v_0)$ for $v_0 = \overline{v}$, as for $v_0 \in \mathrm{dom}(g)$ we have $v_0 \in \mathrm{dom}(g) \subset \mathrm{dom}(\phi_h)$ so the equality holds because we started with a good triple. But using the definition of $\overline{\phi}_h$ and the fact that $h(v) \in \mathrm{dom}(\phi_h)$ we have $\overline{\phi}_h(h(\overline{g}(\overline{v}))) = \overline{\phi}_h(h(v)) = \phi_h(h(v)) = f(\overline{\phi}_h(v))$.

(v) Suppose that for vertices $w, w' \in \mathrm{dom}(\overline{\phi}_h) \setminus N$ we have that $\mathcal{O}^{h\circ\overline{g}}(w) \neq \mathcal{O}^{h\circ\overline{g}}(w')$. Then of course $w \neq w'$. We have extended $\phi_h$ only to $\overline{v}$ so it is enough to check the property with $w = \overline{v}$ and $w' \in \mathrm{dom}(\phi_h)$. But $\mathcal{O}^{h\circ\overline{g}}(\overline{v}) = \mathcal{O}^{h\circ\overline{g}}(h(\overline{g}(\overline{v}))) = \mathcal{O}^{h\circ\overline{g}}(h(v))$ thus $\mathcal{O}^{h\circ g}(h(v)) \cap \mathcal{O}^{h\circ g}(w') \subset \mathcal{O}^{h\circ\overline{g}}(\overline{v}) \cap \mathcal{O}^{h\circ\overline{g}}(w') = \emptyset$. Therefore, by the facts that $\overline{v} \notin N$ implies $h(v) \notin N$ and we started with a good triple, by Property (v) we obtain $\mathcal{O}^f(\phi_h(h(v))) \cap \mathcal{O}^f(\phi_h(w')) = \emptyset$. Using this and the definition of $\overline{\phi}_h$

we have $\mathcal{O}^f(f^{-1}(\overline{\phi}_h(h(v)))) = \mathcal{O}^f(\overline{\phi}_h(\overline{v}))$ thus $\mathcal{O}^f(\overline{\phi}_h(\overline{v})) \cap \mathcal{O}^f(\overline{\phi}_h(w')) = \mathcal{O}^f(\phi_h(h(v))) \cap \mathcal{O}^f(\phi_h(w')) = \emptyset$ so we are done.

(vi) If $h, h' \in \mathcal{K}$ and $h|_{\overline{M}^*} = h'|_{\overline{M}^*}$ then $h|_{M^*} = h'|_{M^*}$ thus $\phi_h = \phi_{h'}$, so by definition $\overline{\phi}_h = \overline{\phi}_{h'}$.

(vii) If $h \in \mathcal{K}$, and for some $i \in \omega \setminus \{0\}$ we have $(h \circ \overline{g})^i(w) = w$ then at least one of the points $\{w, \dots, (h \circ \overline{g})^{i-1}(w)\}$ is not in the domain of $g$, otherwise the triple $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ would violate this property of good conditions. In other words, $\overline{v} \in \{w, \dots, (h \circ \overline{g})^{i-1}(w)\}$. Moreover, $\{w, \dots, (h \circ \overline{g})^{i-1}(w)\} \subset \mathrm{dom}(\overline{g})$ and clearly $(h \circ \overline{g})^{-1}(\{w, \dots, (h \circ \overline{g})^{i-1}(w)\}) = \{w, \dots, (h \circ \overline{g})^{i-1}(w)\}$, so $\overline{v} \in \mathrm{ran}(h \circ \overline{g})$, that is, $h^{-1}(\overline{v}) \in \mathrm{ran}(\overline{g})$. But we know that $\mathrm{ran}(\overline{g}) = \{v\} \cup \mathrm{ran}(g) \subset M$. Therefore $h(\overline{v}) \in M$, contradicting the assumption that $d_{\mathcal{K}}(\overline{v}, M) > 3$.

(viii) Since $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ is a good triple, this condition can fail for $(\overline{g}, (\overline{\phi}_h)_{h\in\mathcal{K}}, \overline{M})$, an $h$ and $w$ only if $w \in \mathrm{dom}(\overline{\phi}_h) \setminus \mathrm{dom}(\phi_h)$, in other words $w = \overline{v}$. But $\overline{v} \in \mathrm{dom}(\overline{\phi}_{h'})$ for every $h' \in \mathcal{K}$ as well.

(ix) Suppose that there exists an $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, x, y)$ ugly situation. If $h|_{M^*} = h'|_{M^*}$ then $\overline{\phi}_h = \overline{\phi}_{h'}$ which contradicts properties (U1) and (U2).

Now if $h|_{M^*} \neq h'|_{M^*}$ then since $\overline{v}$ is a splitting point we have $h^{-1}(\overline{v}) \neq h'^{-1}(\overline{v})$ and also $d_{\mathcal{K}}(\overline{v}, M) > 2$. Then we can apply Lemma 6.15 for $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, x, y)$ and $\{\overline{v}\} \supset \mathrm{dom}(\overline{\phi}_h) \setminus \mathrm{dom}(\phi_h)$, $\{\overline{v}\} \supset \mathrm{dom}(\overline{\phi}_{h'}) \setminus \mathrm{dom}(\phi_{h'})$ so there are no $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, x, y)$ ugly situations.

(x) Suppose that there exists an $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, x, x', y)$ bad situation. If $h|_{M^*} = h'|_{M^*}$ then $\overline{\phi}_h = \overline{\phi}_{h'}$ but then (B3) cannot be true.

Now if $h|_{M^*} \neq h'|_{M^*}$, then as above we can use Lemma 6.15 for $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, x, x', y)$. Therefore, either $x = x' = \overline{v}$ or $y = \overline{v}$. The first option is impossible, as by (B2) we would obtain $h^{-1}(\overline{v}) = h^{-1}(x) = h'^{-1}(x') = h'^{-1}(\overline{v})$ contradicting the fact that $\overline{v}$ was a splitting point. We can exclude the second option, as $\overline{v} \in \mathrm{dom}(\overline{g})$, so we have $e(y, h \circ g) \neq y$ thus using (B1) we get $y \in N$, which is impossible again by $d_{\mathcal{K}}(\overline{v}, M) > 3$.

□

Now we prove a lemma which is the essence of the proof, namely that we can extend the maps $\phi_h$ forward as well.

**Lemma 6.20.** *Suppose that $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ is a good triple, $h \in \mathcal{K}$ and $v \in M$. Then there exists a vertex $z$ so that if for every $h' \in \mathcal{K}$ with $h'|_{M^*} = h|_{M^*}$ we extend $\phi_{h'}$ by letting $\overline{\phi}_{h'} = \phi_{h'} \cup \langle v, z\rangle$ then $(g, (\phi_{h'})_{h'\in\mathcal{K}, h'|_{M^*} \neq h|_{M^*}} \cup (\overline{\phi}_{h'})_{h\in\mathcal{K}, h'|_{M^*} = h|_{M^*}}, M)$ is a good triple.*

*Proof.* First find a vertex $z$ satisfying the following requirements (note that the below requirements depend solely on $h|_{M^*}$, hence these will be exactly the same for every $h' \in \mathcal{K}$ so that $h'|_{M^*} = h|_{M^*}$):

(1) $z \neg R f(z)$ and $z \notin \mathcal{O}^f(\mathrm{ran}(\phi_h))$,
(2) for every $w \in \mathrm{dom}(\phi_h)$ we have $zR\phi_h(w) \iff vRw$,
(3) if for some $h' \in \mathcal{K}$ and $x, x' \in V$ the sequence $(h, h', x, x', v)$ has Properties (B1) and (B2) of a bad situation then

(z.3.B) if $x \in \mathrm{dom}(\phi_h)$ and $x', v \in \mathrm{dom}(\phi_{h'})$ holds then

$$zRf^{-1}(\phi_h(x)) \iff f(z)R\phi_h(x) \iff \phi_{h'}(x')Rf(\phi_{h'}(v))$$

i. e., (B3.b) is false with $z = \overline{\phi}_h(v)$,

(z.3.U) if $x \in \mathrm{dom}(\phi_h), v = x'$ and $v \notin \mathrm{dom}(\phi_{h'})$ holds then $z \neg R f^{-1}(\phi_h(x))$,

i. e., (U2) is false with $z = \overline{\phi}_h(v)$,

(4) if for some $h' \in \mathcal{K}$ and $y, x' \in V$ the sequence $(h, h', v, x', y)$ has Properties (B1) and (B2) of a bad situation then

(z.4.B) if $y \in \mathrm{dom}(\phi_h)$ and $x', y \in \mathrm{dom}(\phi_{h'})$ holds then

$$z R f(\phi_h(y)) \iff \phi_{h'}(x') R f(\phi_{h'}(y)),$$

i. e., again, (B3) is false with $z = \overline{\phi}_h(v)$,

(z.4.U) if $y \in \mathrm{dom}(\phi_h), y = x'$ and $y \notin \mathrm{dom}(\phi_{h'})$ holds then $z \neg R f(\phi_h(y))$,

i. e., (U2) is false with $z = \overline{\phi}_h(v)$.

**Claim 6.21.** *There exists such a $z$.*

*Proof of the Claim.* Since $f$ has property $(*)_0$ it is enough to show that requirements on $z$ (2)-(4) are not contradicting. Obviously, by the injectivity of $\phi_h$ there is no contradiction between requirements of type (2), and by the fact that only non-relations are required between requirements of type z.3.U and of type z.4.U. Thus, it is enough to check that requirements of type z.3.B, z.3.B- z.3.U, z.4.B, z.4.B- z.4.U, and of type (2)-(3), (2)-(4) and (3)-(4) are not in a contradiction.

*The requirements in (3) are compatible,* (z.3.B). Suppose otherwise, namely, there is a contradiction between requirements of type (z.3.B). Then we have automorphisms $h'_1, h'_2 \in \mathcal{K}$, vertices $x_1, x_2, x'_1, x'_2$ showing the contradiction, that is, $(h, h'_1, x_1, x'_1, v)$ has properties (B1), (B2) of a bad situation and $x_1 \in \mathrm{dom}(\phi_h)$ and $x'_1, v \in \mathrm{dom}(\phi_{h'_1})$ and similarly for $(h, h'_2, x_2, x'_2, v)$ but

$$\phi_{h'_1}(x'_1) R f(\phi_{h'_1}(v)) \text{ and } \phi_{h'_2}(x'_2) \neg R f(\phi_{h'_2}(v)) \tag{27}$$

and $f^{-1}(\phi_h(x_1)) = f^{-1}(\phi_h(x_2))$, or equivalently, $x_1 = x_2$. We claim that there exists an $(h'_1, h'_2, x'_1, x'_2, v)$ bad situation which contradicts the fact that $(g, (\phi_h)_{h \in \mathcal{K}}, M)$ was a good triple:

(B1) $(h, h'_1, x_1, x'_1, v)$ and $(h, h'_2, x_2, x'_2, v)$ have property (B1), in particular $v = e(v, h'_1 \circ g) = e(v, h'_2 \circ g)$ or $v \in N$ and $x'_1 = b(x'_1, h'_1 \circ g)$ or $x'_1 \in N$ and $x'_2 = b(x'_2, h'_2 \circ g)$ or $x'_2 \in N$,

(B2) using (B2) for $(h, h'_1, x_1, x'_1, v)$ and $(h, h'_2, x_2, x'_2, v)$ we get $h^{-1}(x_1) = h'^{-1}_1(x'_1)$ and $h^{-1}(x_2) = h'^{-1}_2(x'_2)$, and using $x_1 = x_2$ we obtain $h'^{-1}_1(x'_1) = h'^{-1}_2(x'_2)$,

(B3) (27) shows that this holds.

*The requirements in (3) are compatible,* (z.3.B) *and* (z.3.U). Suppose that there is a contradiction between requirements of type (z.3.B) and (z.3.U). Then we have automorphisms $h'_1, h'_2 \in \mathcal{K}$, vertices $x_1, x'_1, x_2$ so that $(h, h'_1, x_1, x'_1, v)$ and $(h, h'_2, x_2, v, v)$ have properties (B1) and (B2), $x_1, x_2 \in \mathrm{dom}(\phi_h), x'_1, v \in \mathrm{dom}(\phi_{h'_1})$, $v \notin \mathrm{dom}(\phi_{h'_2})$ and

$$\phi_{h'_1}(x'_1) R f(\phi_{h'_1}(v)) \tag{28}$$

and $f^{-1}(\phi_h(x_1)) = f^{-1}(\phi_h(x_2))$, that is, $x_1 = x_2$. We claim that we have an $(h'_1, h'_2, x'_1, v)$ ugly situation:

(B1) follows from the fact that $(h, h'_1, x_1, x'_1, v)$ and $(h, h'_2, x_2, v, v)$ have Property (B1)

(B2) similarly, $(h, h_1', x_1, x_1', v)$ and $(h, h_2', x_2, v, v)$ have Property (B2) of bad situations and by $x_1 = x_2$ we get $h^{-1}(x_1) = h_1'^{-1}(x_1')$ and $h^{-1}(x_2) = h_2'^{-1}(v) = h^{-1}(x_1)$, thus, (B2) of bad situations holds for $(h_1', h_2', x_1', v, v)$,
(U1) since $(h, h_2', x_2, v, v)$ has property (U1), so $v \notin \mathrm{dom}(\phi_{h_2'})$,
(U2) finally, (28) shows that this holds as well.

*The requirements in (4) are compatible,* (z.4.B). Suppose that there is a contradiction between requirements of type (z.4.B). Then we have automorphisms $h_1', h_2' \in \mathcal{K}$, vertices $y_1, x_1', y_2, x_2'$ so that $(h, h_1', v, x_1', y_1)$ and $(h, h_2', v, x_2', y_2)$ have properties (B1), (B2) and $y_1, y_2 \in \mathrm{dom}(\phi_h), x_1', y_1' \in \mathrm{dom}(\phi_{h_1'})$ and $x_2', y_2' \in \mathrm{dom}(\phi_{h_2'})$ but

$$\phi_{h_1'}(x_1') R f(\phi_{h_1'}(y_1)) \text{ and } \phi_{h_2'}(x_2') \neg R f(\phi_{h_2'}(y_2)) \tag{29}$$

and $f(\phi_h(y_1)) = f(\phi_h(y_2))$, that is, $y_1 = y_2$. Then we have an $(h_1', h_2', x_1', x_2', y_1)$ bad situation:

(B1) follows from the fact that $(h, h_1', v, x_1', y_1)$ and $(h, h_2', v, x_2', y_2)$ have property (B1) and $y_1 = y_2$,
(B2) $(h, h_1', v, x_1', y_1)$ and $(h, h_2', v, x_2', y_2)$ have property (B2) so $h^{-1}(v) = h_1'^{-1}(x_1') = h_2'^{-1}(x_2')$ so this is also true,
(B3) (29) shows that this property holds.

*The requirements in (4) are compatible,* (z.4.B) *and* (z.4.U). Suppose that there is a contradiction between requirements of type (z.4.B) and (z.4.U). Then we have automorphisms $h_1', h_2' \in \mathcal{K}$, vertices $y_1, x_1', y_2$ so that $(h, h_1', v, x_1', y_1)$ and $(h, h_2', v, y_2, y_2)$ have properties (B1), (B2) and $y_1, y_2 \in \mathrm{dom}(\phi_h)$, $x_1', y_1 \in \mathrm{dom}(\phi_{h_1'})$, $y_2 \notin \mathrm{dom}(\phi_{h_2'})$ but

$$\phi_{h_1'}(x_1') R f(\phi_{h_1'}(y_1)) \tag{30}$$

and $f(\phi_h(y_1)) = f(\phi_h(y_2))$, that is, $y_1 = y_2$. We claim that we have an $(h_1', h_2', x_1', y_1)$ ugly situation:

(B1) follows from the fact that $(h, h_1', v, x_1', y_1)$ and $(h, h_2', v, y_2, y_2)$ have Property (B1) of bad situations and $y_1 = y_2$,
(B2) similarly, $(h, h_1', v, x_1', y_1)$ and $(h, h_2', v, y_2, y_2)$ have Property (B2) of bad situations we get $h^{-1}(v) = h_1'^{-1}(x_1)$ and $h^{-1}(v) = h_2'^{-1}(y_2) = h_2'^{-1}(y_1)$,
(U1) since $(h, h_2', v, y_2, y_2)$ has property (U1), so $y_2 \notin \mathrm{dom}(\phi_{h_2'})$ and $y_1 = y_2$,
(U2) finally, (30) shows that this condition is true as well.

*The requirements in (2) and (3) are compatible.* Otherwise there would be vertices $x, x'$ satisfying Property (B1) from Definition 6.10 and $w \in \mathrm{dom}(\phi_h)$ so that $f^{-1}(\phi_h(x)) = \phi_h(w)$.

Suppose first $x \notin N$. Then clearly $\mathcal{O}^f(\phi_h(x)) = \mathcal{O}^f(\phi_h(w))$ and $\phi_h(x), \phi_h(w) \notin N$. Using that $(g, (\phi_h)_{h \in \mathcal{K}}, M)$ is a good triple by Property (v) we obtain $\mathcal{O}^{h \circ g}(x) = \mathcal{O}^{h \circ g}(w)$ and by $\phi_h(x) \notin N$ the orbit $\mathcal{O}^f(\phi_h(x))$ is infinite. Since by Property (B1) of a bad situation $x = b(x, h \circ g)$ we get that $(h \circ g)^k(x) = w$ for some $k \geq 0$. Thus, by Properties (ii) and (iv) of good triples we get

$$\phi_h(w) = \phi_h((h \circ g)^k(x)) = f(\phi_h((h \circ g)^{k-1}(x))) = \cdots = f^k(\phi_h(x)).$$

But, this together with $f^{-1}(\phi_h(x)) = \phi_h(w)$ contradicts the fact that $\mathcal{O}^f(\phi_h(x))$ is infinite.

Now if $x \in N$ then clearly $\phi_h(x) = x$ so $f^{-1}(x) = \phi_h(w)$ is also is an element of $N$ by the fact that $N$ is the union of orbits of $f$, and therefore $\phi_h(w) = w$

by Property (iii) of good triples. Moreover, by (B2) we have $x' = h'(h^{-1}(x))$, so $h|_N = h'|_N = f|_N$ implies $x' \in N$ and $x = x'$. Thus, since the requirements are contradicting by our assumption we get

$$x \neg R f(\phi_{h'}(v)) \iff vRw \iff \phi_{h'}(v)Rw$$

but $w = f^{-1}(x)$ which is impossible.

*The requirements in (2) and (4) are compatible.* The argument here is similar. Otherwise there would be a vertex $y$ satisfying Property (B1) from Definition 6.10 and $w \in \mathrm{dom}(\phi_h)$ so that $f(\phi_h(y)) = \phi_h(w)$.

Suppose $y \notin N$. Then clearly $\mathcal{O}^f(\phi_h(y)) = \mathcal{O}^f(\phi_h(w))$, so $\phi_h(y), \phi_h(w) \notin N$ and $\mathcal{O}^f(\phi_h(y)) \cap N = \emptyset$, thus $\mathcal{O}^f(\phi_h(y))$ is infinite. Again, by Property (v) we obtain $\mathcal{O}^{h \circ g}(y) = \mathcal{O}^{h \circ g}(w)$. Since by Property (B1) of a bad situation $y = e(y, h \circ g)$ we get that $(h \circ g)^k(w) = y$ for some $k \geq 0$. But this, using Property (iv) contradicts $f(\phi_h(y)) = \phi_h(w)$ and the fact that the orbit $\mathcal{O}^f(\phi_h(y))$ is infinite.

Now if $y \in N$ then clearly $\phi_h(y) = \phi_{h'}(y) = y$ so $f(y) = \phi_h(w)$ is also an element of $N$ thus $\phi_h(w) = w$. Our requirements are contradicting, so

$$\phi_{h'}(x')Rf(\phi_{h'}(y)) \iff v \neg Rw.$$

But $y, f(y), w \in N$ so

$$\phi_{h'}(x')Rf(\phi_{h'}(y)) \iff f(y)R\phi_{h'}(x') \iff \phi_{h'}(f(y))R\phi_{h'}(x') \iff f(y)Rx'$$

using that $x' = h'(h^{-1}(v))$ and $f|_N = h|_N = h'|_N$

$$\iff f(y)Rh'(h^{-1}(v)) \iff h(h'^{-1}(f(y)))Rv \iff f(y)Rv.$$

But $w = f(y)$, so this gives

$$f(\phi_{h'}(y))R\phi_{h'}(x') \iff wRv,$$

showing that this is impossible.

*The requirements in (3) and (4) are compatible.* Suppose not, then we have sequences $(h, h'_1, x_1, x'_1, v)$ and $(h, h'_2, v, x'_2, y_2)$ having properties (B1) and (B2), $x_1, y_2 \in \mathrm{dom}(\phi_h)$ with $f^{-1}(\phi_h(x_1)) = f(\phi_h(y_2))$. Then $\mathcal{O}^f(\phi_h(x_1)) = \mathcal{O}^f(\phi_h(y_2))$.

Let $x_1 \notin N$, then $\phi_h(x_1), \phi_h(y_2) \notin N$. Again, by Property (v) we obtain $\mathcal{O}^{h \circ g}(x_1) = \mathcal{O}^{h \circ g}(y_2)$. But by $y_2 = e(y_2, h \circ g)$ we get that $(h \circ g)^k(x_1) = y_2$ for some $k \geq 0$. But this contradicts $f^2(\phi_h(y_2)) = \phi_h(x_1)$.

Now, if $x_1 \in N$ then $y_2 \in N$ holds as well. Then, as we have seen before $\phi_h(y_2) = \phi_{h'_2}(y_2) = y_2$ and also $x'_1 = x_1$. Again, by $h|_N = h'_2|_N = f|_N$ we get

$$f(y_2)R\phi_{h'}(x'_2) \iff f(y_2)Rx'_2 \iff$$

$$f(y_2)Rh'_2(h^{-1}(v)) \iff h(h'^{-1}_2(f(y_2)))Rv \iff f(y_2)Rv.$$

Moreover, using again $x'_1 = x_1 \in N$, $f^{-1}(x_1) \in N$, $h|_N = h'_1|_N = f|_N$ and $\phi_{h'_1}|_N = id_N$ we obtain

$$\phi_{h'_1}(x'_1)Rf(\phi_{h'_1}(v)) \iff f^{-1}(x_1)R\phi_{h'_1}(v) \iff f^{-1}(x_1)Rv,$$

so recalling that $f^2(y_2) = x_1$ we can conclude that the requirements are not in a contradiction.

□

We return to the proof of Lemma 6.20. Extend $\phi_h$ to $v$ defining $\overline{\phi}_h = \phi_h \cup \langle v, z\rangle$ for some $z$ having properties (1)-(4). We check that $(g, (\phi_{h'})_{h'\in\mathcal{K}, h'|_{M^*}\neq h|_{M^*}} \cup (\overline{\phi}_{h'})_{h\in\mathcal{K}, h'|_{M^*}=h|_{M^*}}, M)$ is still a good triple, going through the definition of good triples.

(i) We have to check that $\overline{\phi}_h$ is a partial automorphism, but this is exactly property (2) of $z$.
(ii) Obvious, as $\overline{\phi}_h$ is the extension of $\phi_h$ to a point $v$ already in $M$.
(iii) Obvious.
(iv) If $(\overline{\phi}_h \circ h \circ g)(v_0) = (f \circ \overline{\phi}_h)(v_0)$, became false after the extension for some $v_0$, then either $(h \circ g)(v_0) = v$ or $v_0 = v$. We can exclude both of the possibilities, as by Property (ii) of the good triples $\mathrm{dom}(\phi_h) \supset rd(h \circ g)$, so $\phi_h$ would have been already defined on $v$.
(v) If for $w, w' \in \mathrm{dom}(\overline{\phi}_h) \setminus N$ we have that $\mathcal{O}^{h\circ g}(w) \neq \mathcal{O}^{h\circ g}(w')$ then clearly $w \neq w'$. Also, either $w, w' \in \mathrm{dom}(\phi_h)$, in which case we are done, or say, $w = v$. But by property (1) of $z$ we have $\mathcal{O}^f(z) \cap \mathcal{O}^f(\phi_h(w')) = \emptyset$ and clearly $\mathcal{O}^f(\overline{\phi}_h(v)) = \mathcal{O}^f(z)$ .
(vi) Obvious, since we defined the extension of $\phi_h$ the same for a set of $h \in \mathcal{K}$ with the same restriction to $M^*$.
(vii) This property does not use the functions $\phi_h$.
(viii) By property (1) of $z$ we have $z \neg R f(z)$, so whenever $f(\overline{\phi}_h(w)) R \overline{\phi}_h(w)$ then clearly $w \neq v$, so $w \in \mathrm{dom}(\phi_h)$ and $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ was a good triple, thus, $(g, (\overline{\phi}_h)_{h\in\mathcal{K}}, M)$ also has Property (viii).
(ix) If there exists an $(h_1, h_1', \overline{\phi}_{h_1}, \overline{\phi}_{h_1'}, g, x, y)$ ugly situation then one of $h_1, h_1'$ must coincide with $h$ on $M^*$, so as the definition of ugly situation depends only on $h|_{M^*}$, we can suppose that one of the functions is $h$.

Note that $h_1|_{M^*} = h_1'|_{M^*}$ would imply $\overline{\phi}_h = \overline{\phi}_{h'}$ contradicting (U1) and (U2). Hence we can suppose that $\{h_1, h_1'\} = \{h, h'\}$ for some $h'|_{M^*} \neq h|_{M^*}$. Moreover, if $h_1' = h$ or $x, y \in \mathrm{dom}(\phi_h)$ by $\overline{\phi_{h'}} = \phi_{h'}$ and (U1) we would already have an $(h', h, \phi_{h'}, \phi_h, g, x, y)$ or $(h, h', \phi_h, \phi_{h'}, g, x, y)$ ugly situation, which is impossible as we have started with a good triple.

Thus, $h_1 = h$ and $x = v$ or $y = v$ and by the definition of the ugly situation $(h, h', \overline{\phi}_h, \phi_{h'}, x, y, y)$ has Properties (B1), (B2) and (U1) and

$$\overline{\phi}_h(x) R f(\overline{\phi}_h(y)). \tag{*U}$$

Now suppose that $x \in \mathrm{dom}(\overline{\phi}_h) \setminus \mathrm{dom}(\phi_h)$, that is, $x = v$ and $y \in \mathrm{dom}(\phi_h)$. Then since $(h, h', \phi_h, \phi_{h'}, g, x, y, y)$ has Properties (B1), (B2) and (U1) the requirement (z.4.U) on $z$ ensures that

$$z \neg R f(\phi_h(y)) \text{ or, equivalently } \overline{\phi}_h(x) \neg R f(\phi_h(y)),$$

a contradicting (*U).

Suppose $y \in \mathrm{dom}(\overline{\phi}_h) \setminus \mathrm{dom}(\phi_h)$, $y = v$ and $x \in \mathrm{dom}(\phi_h)$. Then again, $(h, h', \phi_h, \phi_{h'}, g, x, y, y)$ has (B1), (B2) and (U1), now we use the requirement (z.3.U) on $z$:

$$z \neg R f^{-1}(\phi_h(x)) \text{ or, equivalently by } v = y$$
$$\overline{\phi}_h(v) \neg R f^{-1}(\phi_h(x)) \iff f(\overline{\phi}_h(y)) \neg R \overline{\phi}_h(x),$$

a contradiction.

Finally, if $x, y \in \mathrm{dom}(\overline{\phi}_h) \setminus \mathrm{dom}(\phi_h)$, that is $x = y = v$. Then we obtain that $\overline{\phi}_h(x) R f(\overline{\phi}_h(y))$ means $zRf(z)$, but this contradicts Property (1) of $z$.

(x) Suppose that there exists an $(h_1, h_1', \overline{\phi}_{h_1}, \overline{\phi}_{h_1'}, g, x, x', y)$ bad situation. Again, we can suppose that at least one of $h_1$ and $h_1'$ equals to $h$ on $M^*$ and by symmetry there exists an $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, x, x', y)$ bad situation.

Note that using $x, x' \in M$ and $h^{-1}(x), h'^{-1}(x') \in \mathcal{K}^{-1}(M) \subset M^*$ we obtain that $h_1|_{M^*} = h_1'|_{M^*}$ would imply $x = x'$ and $\overline{\phi}_{h_1} = \overline{\phi}_{h_1'}$ which contradict (B3.b). Thus, $h'|_{M^*} \neq h|_{M^*}$ so $\overline{\phi}_{h'} = \phi_{h'}$.

Clearly, at least one of vertices $x, x', y$ must be equal to $v$, hence otherwise there would be an $(h, h', \phi_h, \phi_{h'}, g)$ bad situation.

Suppose that $y = v$ and $x \neq v$. Then by requirement (z.3.B) on $z$ and $z = \overline{\phi}_h(v)$ we obtain

$$f(\overline{\phi}_h(v)) R \phi_h(x) \iff \phi_{h'}(x') R f(\phi_{h'}(v)),$$

showing that this is impossible.

Suppose now $x = v$ and $y \neq v$. Then by requirement (z.4.B) on $z$ we get

$$zRf(\phi_h(y)) \iff \phi_{h'}(x') R f(\phi_{h'}(y)),$$

or reformulating the statement

$$\overline{\phi}_f(v) R f(\phi_h(y)) \iff \phi_{h'}(x') R f(\phi_{h'}(y)),$$

again, showing that $(h, h', \overline{\phi}_h, \phi_{h'}, g, x, x', y)$ is not a bad situation.

Finally, if $x = y = v$, property (B3) would give that

$$zRf(z) \iff \phi_{h'}(x') R f(\phi_{h'}(v)),$$

is not true. By Property (1) of $z$ we get $z \neg Rf(z)$ so

$$\phi_{h'}(x') R f(\phi_{h'}(v)). \tag{31}$$

Now we claim that there is an $(h', h, \phi_{h'}, \phi_h, g, x', v)$ ugly situation:

(B1) follows from the facts that $(h, h', \overline{\phi}_h, \phi_{h'}, g, x, x', y)$ is a bad situation, $x = v$ and $y = v$,

(B2) again, as $(h, h', \overline{\phi}_h, \phi_{h'}, g, x, x', y)$ is a bad situation, we have $h^{-1}(x) = h'^{-1}(x')$, so by $x = v$ we have $h'^{-1}(x') = h^{-1}(v)$ which shows this property,

(U1) clear since $v \notin \mathrm{dom}(\phi_h)$,

(U2) (31) is exactly what is required.

This contradicts the fact that $(g, (\phi_h)_{h \in \mathcal{K}}, M)$ was a good triple.

□

**Corollary 6.22.** *Suppose that $v$ is a vertex, $(g, (\phi_h)_{h \in \mathcal{K}}, M)$ is a good triple and $v \in M$. Then there exist extensions $\overline{\phi}_h \supset \phi_h$ so that $(g, (\overline{\phi}_h)_{h \in \mathcal{K}}, \overline{M})$ is a good triple and $v \in \bigcap_{h \in \mathcal{K}} \mathrm{dom}(\overline{\phi}_h)$.*

*Proof.* First notice that by the compactness of $\mathcal{K}$ the set $\{h|_{M^*} : h \in \mathcal{K}, v \notin \mathrm{dom}(\phi_h)\}$ is finite. By Lemma 6.20 we can define the extensions one-by-one for every element of $\{h|_{M^*} : h \in \mathcal{K}, v \notin \mathrm{dom}(\phi_h)\}$. □

Finally, before we prove our main result we need a lemma about backward extension of the functions $\phi_h$.

**Lemma 6.23.** *Suppose that $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ is a good triple, $h \in \mathcal{K}$ and $z$ a vertex. Then for every $h \in \mathcal{K}$ there exists extensions $\overline{\phi}_h \supset \phi_h$ and $\overline{M} \supset M$ so that $(g, (\overline{\phi}_h)_{h\in\mathcal{K}}, \overline{M})$ is a good triple and $z \in \bigcap_{h\in\mathcal{K}} \mathcal{O}^f(\mathrm{ran}(\overline{\phi}_h))$.*

*Proof.* Clearly, the set $\{h|_{M^*} : h \in \mathcal{K}, z \notin \mathcal{O}^f(\mathrm{ran}(\phi_h))\}$ is finite. Let $\tau_{h|_{M^*}} : M^* \to 2$ so that

$$\tau_{h|_{M^*}}(w) = 0 \iff \phi_{h|_{M^*}}(w) \neg R z \tag{32}$$

and define $\tau_{h|_{M^*}}$ on $M^* \setminus \mathrm{dom}(\phi_{h|_{M^*}})$ arbitrarily.

We claim that there exists a finite set of vertices $\{v_{h|_{M^*}} : h \in \mathcal{K}, z \notin \mathrm{ran}(\phi_h)\}$ which are splitting points for $M^*$ and $\mathcal{K}$, $v_{h|_{M^*}}$ realises $\tau_{h|_{M^*}}$ and

$$d_{\mathcal{K}}(v_{h|_{M^*}}, M \cup \{v_{h'|_{M^*}} : h'|_{M^*} \neq h|_{M^*}\}) > 2 : \tag{33}$$

in order to see this, by the fact that the set $\{h|_{M^*} : h \in \mathcal{K}\}$ is finite, we can enumerate it as $\{p_0, \dots, p_k\}$. Now by Corollary 6.8 we can choose inductively for every $i \leq k$ a $v_{p_i}$ splitting point for $M^*$ and $\mathcal{K}$ so that $d(v_{p_i}, M^* \cup \{v_{p_j} : j < i\}) > 2$. Now, if $h$ is given then $h|_{M^*} = p_i$ for some $i$. If $d(v_{p_i}, M^* \cup \{v_{h'|_{M^*}} : h'|_{M^*} \neq h|_{M^*}\}) \leq 2$ then since $d(v_{p_i}, M^*) > 2$ there was an $i' \neq i$ so that $d(v_{p_{i'}}, v_{p_i}) \leq 2$. But this is impossible by $d(v_{p_i}, \{v_{p_j} : j < i\}) > 2$.

Let $\overline{\phi}_h = \phi_h \cup \langle v_{h|_{M^*}}, z\rangle$ if $z \notin \mathcal{O}^f(\mathrm{ran}(\phi_h))$ and $\overline{\phi}_h = \phi_h$ otherwise. Let $\overline{M} = M \cup \{v_{h|_{M^*}} : h \in \mathcal{K}\}$. In order to prove the lemma it is enough to show that $(g, (\overline{\phi}_h)_{h\in\mathcal{K}}, \overline{M})$ is a good triple. Note that by (vi) of good triples we have that $h|_{M^*} = h'|_{M^*}$ implies $\phi_h = \phi'_h$, but by the definition $v_{h|_{M^*}}$'s we also have that $h|_{M^*} = h'|_{M^*}$ implies $\overline{\phi}_h = \overline{\phi}'_h$.

(i) For $h \in \mathcal{K}$ we check that the extension is still an automorphism, but for every $w \in \mathrm{dom}(\phi_h)$ we have by (32)

$$w R v_{h|_{M^*}} \iff \tau_{h|_{M^*}}(w) = 1 \iff$$
$$\phi_{h|_{M^*}}(w) R z \iff \overline{\phi}_{h|_{M^*}}(w) R \overline{\phi}_{h|_{M^*}}(v_{h|_{M^*}}).$$

(ii) Clearly, $\bigcup_{h\in\mathcal{K}} \mathrm{dom}(\overline{\phi}_h) \subset \bigcup_{h\in\mathcal{K}} \mathrm{dom}(\phi_h) \cup \{v_{h|_{M^*}} : h \in \mathcal{K}\} \subset \overline{M}$.

(iii) Obvious.

(iv) If $(\overline{\phi}_h \circ h \circ g)(v_0) = (f \circ \overline{\phi}_h)(v_0)$, became false after the extension for some $v_0$, then either $(h \circ g)(v_0) = v_{h|_{M^*}}$ or $v_0 = v_{h|_{M^*}}$. Both cases are impossible, as $v_0 \in \mathrm{dom}(g) \subset M$ and $d_{\mathcal{K}}((h \circ g)(v_0), M) \leq 1$ so they would imply $d_{\mathcal{K}}(v_{h|_M}, M) \leq 1$ which contradicts (33).

(v) Let $h \in \mathcal{K}$. If for $w, w' \in \mathrm{dom}(\overline{\phi}_h) \setminus N$ we have that $\mathcal{O}^{h\circ g}(w) \neq \mathcal{O}^{h\circ g}(w')$ then clearly $w \neq w'$. Also, either $w, w' \in \mathrm{dom}(\phi_h)$, in which case we are done, or say, $w = v_{h|_M}$ and $w' \in \mathrm{dom}(\phi_h)$. But $z \notin \mathcal{O}^f(\mathrm{ran}(\phi_h))$ by the definition of the functions $\overline{\phi}_h$. Therefore, using $\mathcal{O}^f(\overline{\phi}_h(v_{h|_{M^*}})) = \mathcal{O}^f(z)$ and $\mathcal{O}^f(z) \cap \mathcal{O}^f(\phi_h(w')) = \emptyset$ we are done.

(vi) As mentioned above, already $h|_{M^*} = h'|_{M^*}$ implies $\overline{\phi}_h = \overline{\phi}'_h$, let alone $h|_{\overline{M}^*} = h'|_{\overline{M}^*}$.

(vii) This property does not use the functions $\phi_h$.

(viii) Fix an $h \in \mathcal{K}$. By the fact that $v_{h|_{M^*}}$ was a splitting point for $M$ and $\mathcal{K}$ we have that $h^{-1}(v_{h|_{M^*}}) = h'^{-1}(v_{h|_{M^*}})$ implies $h|_{M^*} = h'|_{M^*}$. But then $v_{h'|_{M^*}} = v_{h|_{M^*}} \in \mathrm{dom}(\overline{\phi}_{h'})$ as well, so this condition cannot be violated by $w = v_{h|_{M^*}}$, therefore, $w \in \mathrm{dom}(\phi_h)$. By the fact that $(g, (\phi_h)_{h\in\mathcal{K}}, M)$ is a good triple clearly $w \in \mathrm{dom}(\phi_{h'}) \subset \mathrm{dom}(\overline{\phi}_{h'})$.

(ix) Suppose that there exists an $h, h' \in \mathcal{K}$ and vertices $x, y$ forming an $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, g, x, y)$ ugly situation. Notice first that if $h|_{M^*} = h'|_{M^*}$ implies $\overline{\phi}_h = \overline{\phi}_{h'}$ and this contradicts the conjunction of (U1) and (U2).

Therefore, we have $h|_{M^*} \neq h'|_{M^*}$. Then we claim that Lemma 6.15 can be used for $\overline{\phi}_h, \overline{\phi}_{h'}$ and $v = v_{h|_{M^*}}$ and $v' = v_{h'|_{M^*}}$. Indeed, since $h|_{M^*} \neq h'|_{M^*}$ and $v = v_{h|_{M^*}}$ is a splitting points for $\mathcal{K}$ and $M^*$ clearly $h^{-1}(v) \neq h'^{-1}(v)$ (33) shows that the other condition of Lemma 6.15 holds as well. So there is no $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, g, x, y)$ ugly situation.

(x) Let us consider an $(h, h', \overline{\phi}_h, \overline{\phi}_{h'}, g, x, x', y)$ bad situation. Again, if $h|_{M^*} = h'|_{M^*}$ then $\overline{\phi}_h = \overline{\phi}_{h'}$ and $x = x'$. But then (B3) must fail.

So $h|_{M^*} \neq h'|_{M^*}$. Then again, the assumptions of Lemma 6.15 hold for $\overline{\phi}_h, \overline{\phi}_{h'}$ and $v = v_{h|_{M^*}}$ and $v' = v_{h'|_{M^*}}$. Using part (2) we obtain that either $x = v_{h|_{M^*}}$, $x' = v_{h'|_{M^*}}$ or $y = v_{h|_{M^*}} = v_{h'|_{M^*}}$. But from (B2) we have $d(x, x') < 2$, so $d(v_{h|_{M^*}}, v_{h'|_{M^*}}) \leq 2$, so in both cases we are in a contradiction with (33).

□

Now we are ready to prove the main theorem of this section.

*Proof of Theorem 6.6.* Choose a vertex from each orbit of $f$ and enumerate these vertices as $\{z_0, z_1, \dots\}$ and recall that we have fixed an enumeration of $V$, $\{v_0, v_1, \dots\}$.

By Lemma 6.14 the triple $(g_0, (\phi_{0,h})_{h\in\mathcal{K}}, M_0) = (id_N, (id_N)_{h\in\mathcal{K}}, N)$ is good.

Suppose that we have already defined a good triple $(g_i, (\phi_{i,h})_{h\in\mathcal{K}}, M_i)$ for every $i \leq n$ with the following properties:

(1) $M_0 \subset M_1 \subset \cdots \subset M_n$, $g_0 \subset g_1 \subset \cdots \subset g_n$ and $\forall h \in \mathcal{K}$ we have $\phi_{h,0} \subset \phi_{h,1} \subset \cdots \subset \phi_{h,n}$,

(2) if $2k < n$ then

$$\{v_0, v_1, \dots v_k\} \subset \operatorname{ran}(g_{2k}) \cap \operatorname{dom}(g_{2k}),$$

(3) if $2k + 1 \leq n$

$$\{z_0, z_1, \dots z_k\} \subset \bigcap_{h\in\mathcal{K}} \mathcal{O}^f(\operatorname{ran}(\phi_{h,2k+1})),$$

We do the inductive step for an even $n+1$. Choose the minimal index $k$ (which is by the inductive assumption is $\geq \frac{n-1}{2}$) so that $v_k \notin \operatorname{ran}(g_n) \cap \operatorname{dom}(g_n)$.

First, by Remark 6.13 we can extend $M_n$ to $M'_n \supset \{v_k, h(v_k) : h \in \mathcal{K}\}$ so that $(g_n, (\phi_{h,n})_{h\in\mathcal{K}}, M_n)$ is still a good triple. By Corollary 6.22 there exists an extension $g'_n \supset g_n$, $\phi'_{h,n} \supset \phi_{h,n}$ and $M''_n \supset M'_n$ so that $\{v_k, h(v_k) : h \in \mathcal{K}\} \subset \bigcap_{h\in\mathcal{K}} \operatorname{dom}(\phi'_{h,n})$ and the extended triple is still good.

Second, by Lemma 6.16 applied firstly and Lemma 6.18 applied secondly we get extensions $g_{n+1} \supset g'_n$, $\phi_{h,n+1} \supset \phi'_{h,n}$ and $M_{n+1} \supset M''_n$ so that $(g_{n+1}, (\phi_{h,n+1})_{h\in\mathcal{K}}, M_{n+1})$ is a good triple and $v_k \in \operatorname{ran}(g_{n+1}) \cap \operatorname{dom}(g_{n+1})$. This extension obviously satisfies the inductive hypothesis.

Now we do the inductive step for an odd $n+1$ as follows: choose the minimal index $k$ ($\geq \frac{n}{2}$) so that $z_k \notin \bigcap_{h\in\mathcal{K}} \mathcal{O}^f(\operatorname{ran}(\phi_{h,n}))$. By Lemma 6.23 there exist extensions $g_{n+1} \supset g_n$, $\phi_{h,n+1} \supset \phi_{h,n+1}$ and $M_{n+1} \supset M_n$ so that $z_k \in \bigcap_{h\in\mathcal{K}} \mathcal{O}^f(\operatorname{ran}(\phi_{h,n+1}))$. This triple satisfies the inductive assumptions as well.

Thus the induction can be carried out. We claim that $g = \bigcup_n g_n$ and $\phi_h = \bigcup_n \phi_{h,n}$ are automorphisms of $\mathcal{R}$ and for every $h \in \mathcal{K}$ we have

$$\phi_h \circ h \circ g = f \circ \phi_h.$$

Indeed, as $g$ and $\phi_h$ are increasing unions of partial automorphisms, they are partial automorphisms as well. Moreover by assumption (2) of the induction $V = \{v_0, v_1, \dots\} \subset \mathrm{rd}(g)$, thus $g \in \mathrm{Aut}(\mathcal{R})$. By (ii) of good triples we have $\mathrm{dom}(g_n) \subset \mathrm{dom}(\phi_{h,n})$ so

$$V = \bigcup_{n\in\omega} \mathrm{dom}(g_n) \subset \bigcup_{n\in\omega} \mathrm{dom}(\phi_{h,n}) = \mathrm{dom}(\phi_h).$$

By (iv) we obtain

$$\phi_h \circ h \circ g = f \circ \phi_h.$$

We have seen that $g \in \mathrm{Aut}(\mathcal{R})$, so $\mathrm{ran}(h \circ g) = V$, therefore from the above equality we get

$$\mathrm{ran}(\phi_h) = f(\mathrm{ran}(\phi_h)),$$

so the set $\mathrm{ran}(\phi_h)$ is $f$ invariant, consequently contains full orbits of $f$. But by assumption (3) of the induction $\mathrm{ran}(\phi_h)$ intersects each $f$ orbit, so $\phi_h \in \mathrm{Aut}(\mathcal{R})$ as well.

The second part of the theorem is obvious, as $id_N = g_0 \subset g$ and for every $h \in \mathcal{K}$ also $id_N = \phi_{h,0} \subset \phi_h$. □

6.4. **Translation of compact sets, general case.** Now we give a complete characterization of the non-Haar null conjugacy classes in $\mathrm{Aut}(\mathcal{R})$. Interestingly enough, a variant of the following property has already been isolated by Truss [18].

**Definition 6.24.** Let $f \in \mathrm{Aut}(\mathcal{R})$. We say that $f$ has property $(*)$ if

- $f$ has only finitely many finite orbits and infinitely many infinite orbits,
- for every finite set $M \subset V$ and $\tau : M \to 2$ there exists a $v$ that realises $\tau$ and $v \notin \mathcal{O}^f(M)$.

**Theorem 6.25.** *Suppose that $f$ has property $(*)$. Then the conjugacy class of $f$ is compact biter. If $f$ has no finite orbits, then the conjugacy class of $f$ is compact catcher.*

Our strategy is to reduce this theorem to the special case that has been proven in Theorem 6.6.

**Claim 6.26.** *Suppose that $f$ has property $(*)$. Let $N$ be the union of the finite orbits of $f$ and $\tau : N \to 2$. Then either*

*(1) for every $\overline{N} \supset N$ finite and $\overline{\tau} \supset \tau$, $\overline{\tau} : \overline{N} \to 2$ there exists a vertex $v$ that realises $\overline{\tau}$, so that $v \notin \mathcal{O}^f(\overline{N})$ and $v \neg Rg(v)$ or*

*(2) for every $\overline{N} \supset N$ finite and $\overline{\tau} \supset \tau$, $\overline{\tau} : \overline{N} \to 2$ there exists a vertex $v$ that realises $\overline{\tau}$, so that $v \notin \mathcal{O}^f(\overline{N})$ and $vRg(v)$.*

*(The possibilities are not mutually exclusive.)*

*Proof.* Suppose that neither of these holds. In other words, there exist finite sets $\overline{N}, \overline{N}' \supset N$ and $\overline{\tau} : \overline{N} \to 2$, $\overline{\tau}' : \overline{N}' \to 2$ extending $\tau$ so that for every $v$ that realises $\overline{\tau}$ and $v \notin \mathcal{O}^f(\overline{N})$ we have $vRf(v)$ and $v \neg Rf(v)$ that realises $\overline{\tau}'$ and $v \notin \mathcal{O}^f(\overline{N}')$.

Notice that as $f$ is an automorphism the fact that for every $v$ that realises $\overline{\tau}$ and $v \notin \mathcal{O}^f(\overline{N})$ we have $vRf(v)$ implies that for every $k$ if $v$ realises $\overline{\tau} \circ f^{-k}$ and $v \notin \mathcal{O}^f(f^k(\overline{N}))$ then $vRf(v)$.

Let $M = \overline{N} \setminus N$ and $n \in \omega$ so that the length of each orbit in $N$ divides $n$. As $f$ has only infinite orbits outside of $N$, for large enough $k$ we have $f^{kn}(M) \cap \overline{N}' = \emptyset$. Moreover, by the condition on $n$ we have that $\overline{\tau} \circ f^{-kn}$ coincides with $\tau$ on $N$. But then $\overline{\tau} \circ f^{-kn} \cup \overline{\tau}'$ is a function extending $\tau$. Since $f$ has property $(*)$ there exists a $v \notin \mathcal{O}^f(f^{kn}(\overline{N}) \cup \overline{N}')$ which realises $\overline{\tau} \circ f^{-kn} \cup \overline{\tau}'$. Then on the one hand $v$ realises $\overline{\tau} \circ f^{-kn}$ and $v \notin \mathcal{O}^f(f^{kn}(\overline{N}))$ so, as mentioned above, $vRf(v)$. On the other hand it also realises $\overline{\tau}'$ and $v \notin \mathcal{O}^f(\overline{N}')$ thus $v \neg Rf(v)$, a contradiction. □

*Proof of Theorem 6.25.* Let $N$ be the union of the finite orbits of $f$. Define a function $\sigma : \{\tau : N \to 2\} \to 2$ as follows: let $\sigma(\tau) = 0$ if condition (1) holds from Claim 6.26 and $\sigma(\tau) = 1$ otherwise. Moreover, define an equivalence relation $\simeq$ on $\{\tau : N \to 2\}$ by $\tau \simeq \tau'$ if there exists a $k \in \mathbb{Z}$ such that $\tau \circ f^k = \tau'$. Note that if $\tau \simeq \tau'$ then $\sigma(\tau) = \sigma(\tau')$: suppose that (1) holds for $\tau$ and $\tau' = \tau \circ f^k$ and let $\overline{\tau}' \supset \tau'$. Then, as $\overline{\tau}' \circ f^{-k} \supset \tau$, there exists a $v$ realizing $\overline{\tau}' \circ f^{-k}$, $v \notin \mathcal{O}^f(\mathrm{dom}(\overline{\tau}' \circ f^{-k})) = \mathcal{O}^f(\mathrm{dom}(\overline{\tau}'))$ and $v \neg Rf(v)$. But then $f^{-k}(v) \neg Rf^{-k+1}(v)$, $f^{-k}(v) \notin \mathcal{O}^f(\mathrm{dom}(\overline{\tau}'))$ and $f^{-k}(v)$ realises $\overline{\tau}'$. Thus, we can consider $\sigma$ as a $\{\tau : N \to 2\}/_{\simeq} \to 2$ map.

Let

$$V_{[\tau]} = \{v \in V \setminus N : v \text{ realises some } \tau' \simeq \tau\}.$$

Then clearly $V$ is the disjoint union of the sets $N$ and $V_{[\tau]}$ for $\simeq$ equivalence classes of maps $\tau : N \to 2$. The idea is to switch the edges and non-edges in every set $V_{[\tau]}$ according to $\sigma$: let us define an edge relation $R'$ on the vertices $V$ as follows: for every distinct $v, w \in V$ if $v, w \in V_{[\tau]}$ for some $\tau$ and $\sigma([\tau]) = 1$ let $vR'w \iff v \neg Rw$, otherwise let $vR'w \iff vRw$.

**Claim 6.27.** *There exists an isomorphism $S : (V, R') \to (V, R)$ so that $S|_N = id|_N$ and for every $\tau$ we have $S(V_{[\tau]}) = V_{[\tau]}$. Moreover, the subgroup $G_f = \{h \in \mathrm{Aut}(\mathcal{R}) : h|_N = f^k|_N$ for some $k \in \mathbb{Z}\}$ is invariant under conjugating with $S$ (we consider $S$ here as an element of $Sym(V)$, which is typically not an automorphism of $\mathcal{R}$) and for every $h \in G_f$ we have $h(N) = N$ and $h(V_{[\tau]}) = V_{[\tau]}$ for each map $\tau : N \to 2$.*

*Proof.* We define $S$ by induction, using a standard back-and-forth argument. Let us start with $S_0|_N = id|_N$ and suppose that we have already defined $S_n$ a partial isomorphism that respects the sets $V_{[\tau]}$ so that $\{v_0, v_1, \ldots, v_n\} \subset \mathrm{ran}(S_n) \cap \mathrm{dom}(S_n)$. Now we want to extend $S_n$ to $v_{n+1}$. Let $\tau$ be so that $v_{n+1} \in V_{[\tau]}$ and $v_{n+1}$ realises $\tau$. Let us define $\rho : \mathrm{ran}(S_n) \to 2$ as $\rho(z) = 1 \iff S_n^{-1}(z) R v_{n+1}$. Clearly, in order to prove that $S_n$ can be extended it is enough to check that there exists a $z_{n+1} \in V_{[\tau]}$ realizing $\rho$ with respect to the relation $R'$. Let us define $\rho'$ as

$$\rho'(z) = \begin{cases} 1 - \rho(z), & \text{if } z \in V_{[\tau]} \\ \rho(z), & \text{otherwise.} \end{cases}$$

Then, by property $(*)$ of $f$ there exists a vertex $z_{n+1}$ that realises $\rho'$ with respect to $R$ and also $\rho' \supset \tau$ so $z_{n+1} \in V_{[\tau]}$. But by the definition of $R'$, as $R'$ was obtained by switching the edges within the sets $V_{[\tau]}$, clearly $z_{n+1}$ realises $\rho$ with respect to $R'$. The "back" part can be proved similarly.

In order to prove the second claim suppose that $h|_N = f^k|_N$ for some $k$. It is clear that since $N$ is the union of orbits of $f$ it must be the case for $h$ as well, so

$h(N) = N$. First, we claim that for every $\tau : N \to 2$ we have $h(V_{[\tau]}) = V_{[\tau]}$: let $v \in V_{[\tau]}$ and $\tau' \simeq \tau$ so that $v$ realises $\tau$. Then $h(v)$ realises $\tau \circ h^{-1}$, but we have $h^{-1}|_N = f^{-k}|_N$ thus $h(v)$ realises $\tau \circ f^{-k}$, so by definition $h(v) \in V_{[\tau]}$.

Now we check that $S^{-1}hS$ is an automorphism of $\mathcal{R}$. Take arbitrary vertices $x, y \in V$. If for some $\tau$ we have $x, y \in V_{[\tau]}$ then $xRy \iff x\neg R'y$ and $xRy \iff S(x)R'S(y)$ and since $h$ and $S$ fix the sets $V_{[\tau]}$ we have $S(x), S(y) \in V_{[\tau]}$ and $h$ is an automorphism,

$$xRy \iff S(x)R'S(y) \iff S(x)\neg RS(y) \iff h(S(x))\neg Rh(S(y)) \iff$$
$$h(S(x))R'h(S(y)) \iff S^{-1}(h(S(x)))RS^{-1}(h(S(y))).$$

If $x$ and $y$ are in different parts of the partition $V = N \cup \bigcup_{[\tau]} V_{[\tau]}$, then the statement is obvious, as in this case $R$ coincides with $R'$.

□

Thus, conjugating with $S$ induces an automorphism $\overline{S}$ of the group $G_f$.

**Claim 6.28.** *$\overline{S}(f)$ has property $(*)_0$ from Definition 6.5, $\overline{S}(f)|_N = f|_N$ and $N$ is the union of finite orbits of $\overline{S}(f)$.*

*Proof.* The second part of the claim is obvious: conjugating does not change the cardinality of orbits so $\overline{S}(f)$ has infinitely many infinite orbits and finitely many finite ones and also $S|_N = id|_N$ so $S^{-1}fS|_N = f|_N$.

Now take a finite set $M$ and a map $\tau : M \to 2$. Without loss of generality we can suppose $N \subset M$. Then, define $\rho : S(M) \to 2$ as follows:

$$\rho(w) = \begin{cases} 1 - \tau(S^{-1}(w)), & \text{if } w \in V_{[\tau|_N]} \text{ and } \sigma(V_{[\tau|_N]}) = 1 \\ \tau(S^{-1}(w)), & \text{otherwise.} \end{cases}$$

Then there exists a $v_0 \notin \mathcal{O}^f(S(M))$ so that $v_0$ realises $\rho$ and

$$v_0 \neg Rf(v_0) \text{ if } \sigma(V_{[\tau|_N]}) = 0 \text{ and } v_0 Rf(v_0) \text{ if } \sigma(V_{[\tau|_N]}) = 1. \tag{34}$$

Since $v_0$ realises $\tau|_N$ and $\tau|_N \simeq \tau|_N \circ f^{-1}$ we have that $f(v_0)$ realises $\tau|_N \circ f^{-1}$ thus $f(v_0) \in V_{[\tau|_N]}$. Let $v = S^{-1}(v_0)$, since $S$ fixes the sets $V_{[\tau|_N]}$ we have $v \in V_{[\tau|_N]}$ as well.

We show that $v$ realises $\tau$. Let $w \in M$ be arbitrary. Suppose first that $\sigma(V_{[\tau|_N]}) = 0$ or $w \notin V_{[\tau|_N]}$. Then from the fact that $v_0 = S(v) \in V_{[\tau|_N]}$ we have

$$vRw \iff vR'w \iff S(v)RS(w) \iff v_0RS(w) \tag{35}$$

by the fact that $w \notin V_{[\tau|_N]}$ or $\sigma(V_{[\tau|_N]}) = 0$

$$\iff \rho(S(w)) = 1 \iff \tau(S^{-1}(S(w))) = \tau(w) = 1.$$

Now, if $\sigma(V_{[\tau|_N]}) = 1$ and $w \in V_{\tau|_N}$ then from the definition of $\rho$ clearly

$$vRw \iff v\neg R'w \iff S(v)\neg RS(w) \iff v_0\neg RS(w) \iff \tag{36}$$
$$\rho(S(w)) = 0 \iff \tau(S^{-1}(S(w))) = \tau(w) = 1.$$

Moreover, using Claim 6.27 we get that $S$ and $f$ fixes the sets $V_{[\tau|_N]}$ and $v \in V_{[\tau|_N]}$ so clearly $(S^{-1} \circ f \circ S)(v) \in V_{[\tau|_N]}$. Thus, by equations (35) and (36) used for $w = (S^{-1} \circ f \circ S)(v)$ we obtain that $vR(S^{-1} \circ f \circ S)(v)$ is true if and only if either

$$v_0RS((S^{-1} \circ f \circ S)(v)) \text{ and } \sigma(V_{[\tau|_N]}) = 0$$

or

$$v_0 \neg RS((S^{-1} \circ f \circ S)(v)) \text{ and } \sigma(V_{[\tau|_N]}) = 1$$

holds.

Now $v_0 = S(v)$ so we get that $vR(S^{-1} \circ f \circ S)(v)$ holds if and only if either

$$v_0 RSf(v_0) \text{ and } \sigma(V_{[\tau|_N]}) = 0$$

or

$$v_0 \neg Rf(v_0) \text{ and } \sigma(V_{[\tau|_N]}) = 1$$

holds. From this, using (34) we get that $v \neg R(S^{-1} \circ f \circ S)(v)$.

Finally, we prove $v \notin \mathcal{O}^{S^{-1} \circ f \circ S}(M)$. Suppose the contrary, let $w \in M$ so that $(S^{-1} f S)^k(w) = v$. Then $(S^{-1} f S)^k(w) = S^{-1} f^k S(w)$ so $S(v) = v_0 \in \mathcal{O}^f(S(M))$, contradicting the choice of $v_0$.

Thus, $S^{-1} f S$ has property $(*)_0$. □

Now we are ready to finish the proof of the theorem. Let $\mathcal{K}_0 \subset \mathrm{Aut}(\mathcal{R})$ be an arbitrary non-empty compact set. We will translate a non-empty portion of $\mathcal{K}_0$ into the conjugacy class of $f$. First, translating $\mathcal{K}_0$ we can suppose that there exists a non-empty portion $\mathcal{K}$ of $\mathcal{K}_0$ so that for every $h \in \mathcal{K}$ we have that $h|_N = f|_N$ (note that if $f$ has no finite orbits then $\mathcal{K} = \mathcal{K}_0$ is a suitable choice). In particular, $\mathcal{K} \subset G_f$. By Claim 6.27 $G_f$ is invariant under conjugating by $S$ and such a map is clearly an auto-homeomorphism of $G_f$, so $\overline{S}(\mathcal{K})$ is also compact. Using Claim 6.28 $\overline{S}(f)$ has property $(*)_0$ and we can apply the second part of Theorem 6.6 and we get a $g$ so that $g|_N = id|_N$ and for each $h \in \overline{S}(\mathcal{K})$ an automorphism $\phi_h$ such that $\phi_h \circ h \circ g \circ \phi_h^{-1} = \overline{S}(f)$ and $\phi_h|_N = id|_N$. In particular, all the automorphisms $g$ and $\phi_h$ are in $G_f$. We will show that $\overline{S}^{-1}(g)$ translates $\mathcal{K}$ into the conjugacy class of $f$. Let $h \in \mathcal{K}\overline{S}^{-1}(g)$ be arbitrary. Then of course $h = h' S g S^{-1}$ for some $h' \in \mathcal{K}$ and $S^{-1} h S = S^{-1} h' S g$ so, as $S^{-1} h' S \in \overline{S}(\mathcal{K})$ we get

$$S^{-1} h S = \phi_{h'}^{-1} S^{-1} f S \phi_{h'}.$$

Thus,

$$h = S \phi_{h'}^{-1} S^{-1} f S \phi_{h'} S^{-1}$$

and as $\phi_{h'} \in G_f$ and $G_f$ is $\overline{S}$ invariant we have $S \phi_{h'} S^{-1} \in G_f \subset \mathrm{Aut}(\mathcal{R})$. Therefore, $h$ is a conjugate of $f$ which finishes the proof.

□

From Theorem 6.25 and Proposition 4.10 we can deduce the complete characterization of the non-Haar null conjugacy classes of $\mathrm{Aut}(\mathcal{R})$:

**Theorem 6.29.** *For almost every element $f$ of* $\mathrm{Aut}(\mathcal{R})$

*(1) for every pair of finite disjoint sets, $A, B \subset V$ there exists $v \in V$ such that $(\forall x \in A)(xRv)$ and $(\forall y \in B)(y \neg Rv)$ and $v \notin \mathcal{O}^f(A \cup B)$, i. e., the union of orbits of the elements of $A \cup B$,*

*(2) (from Theorem 4.14) $f$ has only finitely many finite orbits.*

*These properties characterise the non-Haar null conjugacy classes, i. e., a conjugacy class is non-Haar null if and only if one (or equivalently each) of its elements has properties* (1) *and* (2)*.*

*Moreover, every non-Haar null conjugacy class is compact biter and those non-Haar null classes in which the elements have no finite orbits are compact catchers.*

*Proof of Theorem 6.29.* The facts that the classes of elements having properties 1 and 2 and that these classes are compact biters (or catchers, when there are no finite orbits) is exactly Theorem 6.25.

The only remaining thing is to show that the union of the conjugacy classes of elements not having properties 1 and 2 is Haar null. The collection of automorphisms having infinitely many finite orbits is Haar null by Theorem 4.14.

Now consider the set $\mathcal{C}_0 = \{f \in \mathrm{Aut}(\mathcal{R}) : f \text{ has property } 1\}$. Recall that in Proposition 4.10 we have shown that the set $\mathcal{C}$ is co-Haar null for every $G$ having the $FACP$, in particular, for $\mathrm{Aut}(\mathcal{R})$ the set

$$\mathcal{C} = \{f \in \mathrm{Aut}(\mathcal{R}) : \forall F \subset V \text{ finite } \forall v \in V \text{ (if } \mathrm{Aut}(\mathcal{R})_{(F)}(v) \text{ is infinite}$$

$$\text{then it is not covered by finitely many orbits of } f)\}$$

is co-Haar null. Thus, it is enough to show that $\mathcal{C}_0 \supset \mathcal{C}$ or equivalently $\mathrm{Aut}(\mathcal{R}) \setminus \mathcal{C}_0 \subset \mathrm{Aut}(\mathcal{R}) \setminus \mathcal{C}$. But this is obvious: if $f \notin \mathcal{C}_0$ then there exist disjoint finite sets $A$ and $B$ such that the set $U = \{v : (\forall x \in A)(xRv) \text{ and } (\forall y \in B)(y \neg Rv)\}$ can be covered by the $f$ orbit of $A \cup B$. So, letting $F = A \cup B$ and noting that $U$ is infinite and $\mathrm{Aut}(\mathcal{R})_{(F)}$ acts transitively on $U \setminus F$ we get that for every $v \in U \setminus F$ the orbit $\mathrm{Aut}(\mathcal{R})_{(F)}(v) \subset \mathcal{O}^f(F)$, showing that $f \notin \mathcal{C}$. □

## 7. Applications

In this section we present two applications of our results. First, we use Theorem 4.13 to show that a large family of automorphism groups of countable structures can be decomposed into the union of a Haar null and a meagre set.

**Corollary 7.1.** *Let $G$ be a closed subgroup of $S_\infty$ satisfying the $FACP$ and suppose that the set $F = \{g \in G : \mathrm{Fix}(g) \text{ is infinite}\}$ is dense in $G$. Then $G$ can be decomposed into the union of an (even conjugacy invariant) Haar null and a meagre set.*

*Proof.* Clearly, $F$ is conjugacy invariant, and since it can be written as $F = \{g \in G : \forall n \in \omega\ \exists m > n\ (g(m) = m)\}$, $F$ is $G_\delta$. Using the assumptions of this corollary, it is dense $G_\delta$, hence co-meagre. Using Theorem 4.13, it is Haar null, hence $F \cup (G \setminus F)$ is an appropriate decomposition of $G$. □

**Corollary 7.2.** $\mathrm{Aut}(\mathcal{R})$, $\mathrm{Aut}(\mathbb{Q}, <)$ *and* $\mathrm{Aut}(\mathcal{B}_\infty)$ *can be decomposed into the union of an (even conjugacy invariant) Haar null and a meagre set.*

*Proof.* In order to show that the set of elements in these groups with infinitely many fixed points is dense, in each case it is enough to show that if $p$ is a finite, partial automorphism then there is another partial automorphism $p'$ extending $p$ such that $p' \supset p \cup (x, x)$ with $x \notin \mathrm{dom}(p)$.

For $\mathrm{Aut}(\mathbb{Q}, <)$, let $x$ be greater than each element in $\mathrm{dom}(p) \cup \mathrm{ran}(p)$, then it is easy to see that $p \cup (x, x)$ is also a partial automorphism.

For $\mathrm{Aut}(\mathcal{R})$, let $x$ be an element different from each of $\mathrm{dom}(p) \cup \mathrm{ran}(p)$ with the property that $x$ is connected to every vertex in $\mathrm{dom}(p) \cup \mathrm{ran}(p)$. Then it is easy to see that $p \cup (x, x)$ is a partial automorphism.

For $\mathrm{Aut}(\mathcal{B}_\infty)$, let $a_0 \cup a_1 \cup \cdots \cup a_{n-1}$ be a partition of $\mathbf{1}$ with the property that $\mathrm{dom}(p) \cup \mathrm{ran}(p)$ is a subset of the algebra generated by $A = \{a_0, a_1, \ldots, a_{n-1}\}$. Then $p$ can be described by a permutation $\pi$ of $\{0, 1, \ldots, n-1\}$, that is, $p(a_i) = p_{\pi(i)}$ for every $i$. Let us write each $a_i$ as a disjoint union $a_i = a_i' \cup a_i''$ of non-zero elements.

Again, a partial permutation can be described by a permutation of the elements $\{a'_1, \dots, a'_n\} \cup \{a''_1, \dots, a''_n\}$. Hence, let $p'$ be defined by $p'(a'_i) = a'_{\pi(i)}$, $p'(a''_i) = a''_{\pi(i)}$. Then $p'$ is a partial automorphism extending $p$ with a new fixed point $\bigcup_{i<n} a'_i$. $\square$

Let us now turn to our second application. Applying our results and methods about $\mathrm{Aut}(\mathcal{R})$ one can prove a version of a theorem of Truss [18]. Truss has shown first that if $f, g \in \mathrm{Aut}(\mathcal{R})$ are non-identity elements then $f$ can be expressed as a product of five, later that it can be expressed as the product of three conjugates of $g$ [20]. Using the methods developed in Section 5 and the characterisation of the non-Haar null classes of $\mathrm{Aut}(\mathcal{R})$ one can prove this statement with four conjugates.

**Theorem 7.3.** *Let $C \subset Aut(\mathcal{R})$ be the conjugacy class of a non-identity element. Then $C^4 (= \{f_1 f_2 f_3 f_4 : f_1, f_2, f_3, f_4 \in C\}) = \mathrm{Aut}(\mathcal{R})$.*

The full proof of this theorem will be omitted, as this statement has been already known and writing down the new proof in detail would be comparable in length to the original proof. So, we split the proof into two propositions from which only the first one will be shown.

A certain conjugacy class plays an important role in the proof.

**Definition 7.4.** Let $C_0$ be the collection of elements $f \in \mathrm{Aut}(\mathcal{R})$ with the following properties

(1) there are infinitely many infinite orbits and no finite ones,
(2) for every pair of finite disjoint sets, $A, B \subset V$ there exists $v \in V$ such that $v \notin \mathcal{O}^f(A \cup B)$, $(\forall x \in A)(xRv)$, and $(\forall y \in \mathcal{O}^f(A \cup B) \setminus A)(y \neg Rv)$, (in particular, $(\forall y \in B)(y \neg Rv)$),
(3) for every $v \in V$ and $k \in \mathbb{Z}$ we have $v \neg R f^k(v)$,
(4) for every $v, w$ the set $\{k \in \mathbb{Z} : vRf^k(w)\}$ is finite.

Theorem 7.3 clearly follows from the following two propositions.

**Proposition 7.5.** *$C_0$ is a conjugacy class and $C_0^2 = \mathrm{Aut}(\mathcal{R})$.*

**Proposition 7.6.** *Let $C$ be the conjugacy class of a non-identity element. Then $C^2 \supset C_0$.*

We will prove Proposition 7.5, because it shows how our characterisation can be used, and after that we only sketch the proof of Proposition 7.6.

*Proof of Proposition 7.5.* Suppose that $f, f' \in C_0$. We first show that $f$ and $f'$ are conjugate by building an automorphism $\varphi$ so that $\varphi \circ f = f' \circ \varphi$.

Suppose that we have an $R$-preserving map $\varphi$ such that $\mathrm{dom}(\varphi)$ is the union of finitely many $f$ orbits, $\mathrm{ran}(\varphi)$ is the union of finitely many $f'$ orbits and $\varphi \circ f = f' \circ \varphi$ holds where both sides are defined. We extend $\mathrm{dom}(\varphi)$ and $\mathrm{ran}(\varphi)$ to every vertex back-and-forth.

Recall that $\{v_0, v_1 \dots\}$ is an enumeration of the vertices of $\mathcal{R}$. Take the minimal $i$ with $v_i \notin \mathrm{dom}(\varphi)$. Then, by condition (4) on the map $f$, $v_i$ is only connected to finitely many vertices from $\mathrm{dom}(\varphi)$, let us denote these vertices by $\{w_1, \dots, w_k\}$ and choose one element $\{w_{k+1}, \dots, w_l\}$ from every $f$ orbit in the domain of $\varphi$ that is different from $\mathcal{O}^f(w_i)$ for every $i \leq k$.

Then, since condition (2) holds for $f'$ there exists a vertex $v'$ so that $v' \notin \bigcup_{i \leq l} \mathcal{O}^{f'}(\varphi(w_i))$, $v' R \varphi(w_i)$ for $i \leq k$ and $v' \neg R w$ whenever $w \in (\bigcup_{i \leq l} \mathcal{O}^{f'}(\varphi(w_i))) \setminus \{\varphi(w_i) : i \leq k\}$. Let $\varphi(v_i) = v'$ and extend $\varphi$ to $\mathcal{O}^f(v_i)$ defining $\varphi(f^n(v_i)) =$

$f'^n(\varphi(v_i))$. Using condition (3) it is easy to see that the extended $\varphi$ will be a partial automorphism.

Now take the minimal $i$ with $v_i \notin \mathrm{ran}(\varphi)$ and follow the procedure outlined above, etc. Clearly, this process yields an automorphism $\varphi$ that witnesses that $f$ and $f'$ are conjugate, hence $C_0$ is a conjugacy class.

Now we prove the second part of the proposition. By conditions (1), (2) and Theorem 6.29 the class $C_0$ is compact catcher. Moreover, observe that conditions (1)-(4) are invariant under taking inverses, hence $C_0 = C_0^{-1}$. Now let $h \in \mathrm{Aut}(\mathcal{R})$ be arbitrary. Then, using the fact that $C_0$ is compact catcher for the compact set $\{\mathrm{id}_{\mathcal{R}}, h\}$ there exists a $g \in \mathrm{Aut}(\mathcal{R})$ such that $g, gh \in C_0$, in other words, $h \in C_0^{-1}C_0 = C_0^2$, so $C_0^2 = \mathrm{Aut}(\mathcal{R})$ holds.

□

In order to show Proposition 7.6 one can use the methods from Section 5. By the conjugacy invariance of $C^2$ it is enough to prove that for every $h \in C$ there exists a $g$ such that $g^{-1}hgh \in C_0$. Define $g$ inductively, maintaining the following properties: whenever we extend $g$ (or $g^{-1}$) to some $v$ vertex the vertex $g(v)$ should be far enough in $d_{\{h\}}$ from every vertex already used in the induction, be a splitting point for the compact set $\{h, \mathrm{id}_{\mathcal{R}}\}$ (note that this is equivalent to saying that $v$ is not a fixed point of $h$) and not connected to every already used vertex - except for those to which it is necessary in order for $g$ to be an automorphism.

## 8. Various behaviours

We have seen that in natural Polish groups we may encounter very different behaviours of conjugacy classes with respect to the ideal of Haar null sets. We were particularly interested in answering the following two natural questions:

- How many non-Haar null conjugacy classes are there?
- Is the union of the Haar null classes Haar null?

Note that these questions make perfect sense in the locally compact case as well. In this section we construct a couple of examples.

If $(A, +)$ is an abelian group we will denote by $\phi$ the automorphism of $A$ defined by $a \mapsto -a$.

**Proposition 8.1.** *Let $(A, +)$ be an abelian Polish group such that for every $a \in A$ there exists an element $b$ with $2b = a$. Observe that $\phi \in \mathrm{Aut}(A)$, $\phi^2 = id_A$ and $(\mathbb{Z}_2 \ltimes_\phi A, \cdot)$ can be partitioned into $\{0\} \times A$ and $\{1\} \times A$. Moreover, in the group $\mathbb{Z}_2 \ltimes_\phi A$ the conjugacy class of every element of $\{0\} \times A$ is of cardinality at most 2, whereas the set $\{1\} \times A$ is a single conjugacy class.*

*Proof.* Let $(0, a) \in \{0\} \times A$ and $(i, b) \in \mathbb{Z}_2 \ltimes_\phi A$ arbitrary. We claim that the conjugacy class of $(0, a)$ is $\{(0, a), (0, -a)\}$. If $i = 0$ then $(0, a)$ and $(i, b)$ commute, so let $i = 1$. By definition

$$(1, b)^{-1} \cdot (0, a) \cdot (1, b) = (1, b) \cdot (1, b + a) = (0, b - (b + a)) = (0, -a),$$

which shows our claim.

Now let $(1, a), (1, a') \in \mathbb{Z}_2 \ltimes_\phi A$ be arbitrary. Now for an arbitrary element $(1, b)$ we get

$$(1, b)^{-1} \cdot (1, a) \cdot (1, b) = (1, b) \cdot (0, -b + a) = (1, b - (-b + a)) = (1, 2b - a),$$

thus, choosing $b$ so that $2b = a' + a$ we obtain

$$(1, b)^{-1} \cdot (1, a) \cdot (1, b) = (1, a').$$

□

**Corollary 8.2.** *Let $A = \mathbb{Z}_3^\omega$ or $A = (\mathbb{Q}_d)^\omega$, (that is, the countable infinite power of the rational numbers with the discrete topology). Then $\mathbb{Z}_2 \ltimes_\phi A$ has a non-empty clopen conjugacy class, namely $\{(1, a) : a \in A\}$ and every other conjugacy class has cardinality at most 2. Hence, the union of the Haar null classes $\{(0, a) : a \in A\}$ is also non-empty clopen.*

**Lemma 8.3.** *Suppose that $G_1$ and $G_2$ are Polish groups and $A_1 \subset G_1$ is Borel and $U \subset G_2$ is non-empty and open. Then $A_1 \times U$ is Haar null in $G_1 \times G_2$ iff $A_1$ is Haar null.*

*Proof.* Suppose first that $A_1$ is Haar null witnessed by a measure $\mu_1$. Then, if $\mu'$ is the same measure copied to $G_1 \times \{1\}$, it is easy to see that $\mu'$ witnesses the Haar nullness of $A_1 \times G_2$, in particular, the Haar nullness of $A_1 \times U$.

Conversely, suppose that $A_1 \times U$ is Haar null witnessed by the measure $\mu$. Clearly, as countably many translates of $U$ cover $G_2$, countably many translates of $A_1 \times U$ cover $A_1 \times G_2$, hence $A_1 \times G_2$ is Haar null as well, and this is also witnessed by the measure $\mu$. Let $\mu_1 = \mathrm{proj}_{G_1 *} \mu$, then $\mu_1$ witnesses the Haar nullness of $A_1$. □

**Proposition 8.4.** *If $G$ is a Polish group with $\kappa$ many non-Haar null conjugacy classes then $G \times (\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega)$ has $\kappa$ many non-Haar null conjugacy classes and the union of the Haar null conjugacy classes is not Haar null.*

*Proof.* Clearly, the conjugacy classes of $G \times (\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega)$ are of the form $C_1 \times C_2$ where $C_1$ is a conjugacy class in $G$ and $C_2$ is a conjugacy class in $\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega$. By Corollary 8.2 we have that every conjugacy class in the latter group is finite with one exception, this exceptional conjugacy class is clopen; let us denote it by $U$. Now, since the finite sets are Haar null in $\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega$ by Lemma 8.3, the set of non-Haar null conjugacy classes in $G \times (\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega)$ is equal to $\{C \times U : C \text{ is a non-Haar null conjugacy class in } G\}$, hence the cardinality of the non-Haar null classes is $\kappa$. Moreover, the union of the Haar null conjugacy classes contains $G \times ((\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}_3^\omega) \setminus U)$, which is non-empty and open, consequently it is not Haar null.

□

Finally, we would like to recall the following well known theorem.

**Theorem 8.5** (HNN extension, [9])**.** *There exists a countably infinite group with two conjugacy classes.*

We denote such a group by HNN, and consider it as a discrete Polish group.

Combining Proposition 8.4, Corollaries 3.4, 3.5, 8.2, Lemma 8.3 and Theorems 1.3 and 8.5 we obtain the examples which we summarise in Table 1. Recall that $C$, $LC \setminus C$ and $NLC$ stand for compact, locally compact non-compact, and non-locally compact, respectively.

| | The union of the Haar null classes is Haar null | | |
|---|---|---|---|
| | C | LC $\setminus$ C | NLC |
| $0$ | $-$ | $-$ | $-$ |
| $n$ | $\mathbb{Z}_n$ | HNN | ??? |
| $\aleph_0$ | ??? | $\mathbb{Z}$ | $S_\infty$ |
| $\mathfrak{c}$ | $-$ | $-$ | $\mathrm{Aut}(\mathbb{Q},<)$; $\mathrm{Aut}(\mathcal{R})$ |
| | The union of the Haar null classes is not Haar null | | |
| | C | LC $\setminus$ C | NLC |
| $0$ | $2^\omega$ | $\mathbb{Z}\times 2^\omega$ | $\mathbb{Z}^\omega$ |
| $n$ | $\mathbb{Z}_n\times(\mathbb{Z}_2\ltimes_\phi \mathbb{Z}_3^\omega)$ | HNN $\times(\mathbb{Z}_2\ltimes_\phi \mathbb{Z}_3^\omega)$ | $\mathbb{Z}_n\times(\mathbb{Z}_2\ltimes_\phi \mathbb{Q}_d^\omega)$ |
| $\aleph_0$ | ??? | $\mathbb{Z}\times(\mathbb{Z}_2\ltimes_\phi \mathbb{Z}_3^\omega)$ | $S_\infty\times(\mathbb{Z}_2\ltimes_\phi \mathbb{Z}_3^\omega)$ |
| $\mathfrak{c}$ | $-$ | $-$ | $\mathrm{Aut}(\mathbb{Q},<)\times(\mathbb{Z}_2\ltimes_\phi \mathbb{Z}_3^\omega)$ |

## 9. Open problems

We finish with a couple of open questions. In Section 8 we produced several groups with various numbers of non-Haar null conjugacy classes. However, our examples are somewhat artificial.

**Question 9.1.** *Are there natural examples of automorphism groups with given cardinality of non-Haar null conjugacy classes?*

The following question is maybe the most interesting one from the set theoretic viewpoint.

**Question 9.2.** *Suppose that a Polish group has uncountably many non-Haar null conjugacy classes. Does it have continuum many non-Haar null conjugacy classes?*

The answer is of course affirmative under e.g. the Continuum Hypothesis. Since the definition of Haar null sets is complicated (the collection of non-Haar null closed sets can already be $\mathbf{\Sigma}^1_1$-hard and $\mathbf{\Pi}^1_1$-hard [17]), it is unlikely that this question can be answered with an absoluteness argument.

The characterisation result of Section 4 and the similarity between Theorems 5.5 and 6.29 suggest that similarly to the results of Truss, Kechris and Rosendal the behaviour of the random automorphism can be treated in a broader context.

**Problem 9.3.** *Formulate necessary and sufficient model theoretic conditions which characterise the measure theoretic behaviour of the conjugacy classes.*

In particular, it would be very interesting to find a unified proof of the description of the non-Haar null classes of $\mathrm{Aut}(\mathbb{Q},<)$ and $\mathrm{Aut}(\mathcal{R})$.

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