

# SET-THEORETICAL PROBLEMS CONCERNING HAUSDORFF MEASURES

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ABSTRACT. J. Zapletal asked if all the forcing notions considered in his monograph are homogeneous. Specifically, he asked if the forcing consisting of Borel sets of  $\sigma$ -finite 2-dimensional Hausdorff measure in  $\mathbb{R}^3$  (ordered under inclusion) is homogeneous. We give a partial negative answer to both questions by showing that this  $\sigma$ -ideal is not homogeneous.

Let  $\mathcal{N}_2^1$  be the  $\sigma$ -ideal of sets in the plane of 1-dimensional Hausdorff measure zero. D. H. Fremlin determined the position of the cardinal invariants of this  $\sigma$ -ideal in the Cichoń Diagram. This required proving numerous inequalities, and in all but three cases it was known that the inequalities can be strict in certain models. For one of the remaining ones Fremlin posed this as an open question in his monograph. We answer this by showing that consistently  $\text{cov}(\mathcal{N}_2^1) > \text{cov}(\mathcal{N})$ , where  $\mathcal{N}$  is the usual Lebesgue null ideal. We also prove that the remaining two inequalities can be strict. Moreover, we fit the cardinal invariants of the  $\sigma$ -ideal of sets of  $\sigma$ -finite Hausdorff measure into the diagram.

P. Humke and M. Laczkovich raised the following question. Is it consistent that there is an ordering of the reals in which all proper initial segments are Lebesgue null but for every ordering of the reals there is a proper initial segment that is not null with respect to the 1/2-dimensional Hausdorff measure? We determine the values of the cardinal invariants of the Cichoń Diagram as well as the invariants of the nullsets of Hausdorff measures in the first model mentioned in the previous paragraph, and as an application we answer this question of Humke and Laczkovich affirmatively.

## 1. INTRODUCTION

Throughout the paper, let  $n$  be a positive integer and let  $0 < r < n$  be a real number.

**Definition 1.1.** The  $r$ -dimensional Hausdorff measure of a set  $H \subset \mathbb{R}^n$  is

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$\mathcal{H}^r(H) = \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^r(H)$ , where

$$\mathcal{H}_\delta^r(H) = \inf \left\{ \sum_{k \in \omega} (\text{diam}(A_k))^r : H \subset \bigcup_{k \in \omega} A_k, \forall k \text{ diam}(A_k) \leq \delta \right\}.$$

**Remark 1.2.** It is easy to check that  $\mathcal{H}^r(H) = 0$  iff  $\mathcal{H}_\infty^r(H) = 0$ .

For more information on this notion see [2] or [9].

Let us define the following  $\sigma$ -ideal consisting of sets of  $\sigma$ -finite  $r$ -dimensional Hausdorff measure.

**Definition 1.3.**

$$\mathcal{I}_{n, \sigma\text{-fin}}^r = \{H \subset \mathbb{R}^n : \exists H_k \subset \mathbb{R}^n, \cup_{k \in \omega} H_k = H, \mathcal{H}^r(H_k) < \infty \text{ for every } k \in \omega\}.$$

Since it is not hard to see that every set of finite  $\mathcal{H}^r$ -measure is contained in a Borel, actually  $G_\delta$ , set of finite  $\mathcal{H}^r$ -measure, this  $\sigma$ -ideal has a Borel basis (that is, every member of the  $\sigma$ -ideal is contained in a Borel member of the  $\sigma$ -ideal).

Following the terminology of [12] let us define the following notion of forcing.

**Definition 1.4.**

$$\mathbb{P}_{\mathcal{I}_{n, \sigma\text{-fin}}^r} = \{B \subset \mathbb{R}^n : B \text{ is Borel, } B \notin \mathcal{I}_{n, \sigma\text{-fin}}^r\}, \text{ ordered under inclusion.}$$

For more information on forcing one can also consult [7] or [5].

In order to be able to formulate our first problem, we need some definitions.

**Definition 1.5.** A notion of forcing  $\mathbb{P}$  is called *homogeneous* if for every  $p \in \mathbb{P}$  the restriction of  $\mathbb{P}$  below  $p$  is forcing equivalent to  $\mathbb{P}$ .

In his monograph [12] J. Zapletal poses the following problem.

**Problem 1.6.** ([12, Question 7.1.3.]) “*Prove that some of the forcings presented in this book are not homogeneous.*”

In fact, we will actually work with the following very closely related notion.

**Definition 1.7.** A  $\sigma$ -ideal  $\mathcal{I}$  on a Polish space  $X$  is *homogeneous* if for every Borel set  $B \subset X$  with  $B \notin \mathcal{I}$  there is a Borel function  $f : X \rightarrow B$  such that  $I \in \mathcal{I}$  implies  $f^{-1}(I) \in \mathcal{I}$ .

J. Zapletal remarks that “In all cases encountered in this book the homogeneity of the forcing and the underlying ideal always come together”.

Then he also mentions: “A typical case is that of  $\mathcal{I}$  generated by sets of finite two-dimensional Hausdorff measure in  $\mathbb{R}^3$ .”

In Theorem 2.1 below we show that this  $\sigma$ -ideal  $\mathcal{I}_{3, \sigma\text{-fin}}^2$  is indeed non-homogeneous.

Our second problem concerns fitting the cardinal invariants of the  $\sigma$ -ideal of nullsets of the Hausdorff measures into the Cichoń Diagram. For more information on this diagram consult [1].

**Definition 1.8.** Let

$$\mathcal{N}_n^r = \{H \subset \mathbb{R}^n : \mathcal{H}^r(H) = 0\}.$$

D. H. Fremlin [3, 534B] showed that the picture is as follows.

$$\begin{array}{ccccccccc}
 \text{cov}(\mathcal{N}) & \rightarrow & \text{cov}(\mathcal{N}_n^r) & \rightarrow & \text{non}(\mathcal{M}) & \rightarrow & \text{cof}(\mathcal{M}) & \rightarrow & \text{cof}(\mathcal{N}_n^r) = \text{cof}(\mathcal{N}) \\
 \uparrow & & \uparrow & & \uparrow & \rightarrow & \uparrow & & \uparrow \\
 & & & & \mathfrak{b} & & \mathfrak{d} & & \\
 & & & & \uparrow & & \uparrow & & \\
 \text{add}(\mathcal{N}) & = & \text{add}(\mathcal{N}_n^r) & \rightarrow & \text{add}(\mathcal{M}) & \rightarrow & \text{cov}(\mathcal{M}) & \rightarrow & \text{non}(\mathcal{N}_n^r) \rightarrow \text{non}(\mathcal{N})
 \end{array}$$

All but three arrows (=inequalities) are known to be strict in the appropriate models (see e.g. [1] for the inequalities not involving  $\mathcal{N}_n^r$  and [11] for  $\text{non}(\mathcal{N}_n^r) < \text{non}(\mathcal{N})$ ). Fremlin, addressing one of these three questions, asked the following.

**Question 1.9.** ([3, 534Z, Problem (a)]) *Does  $\text{cov}(\mathcal{N}) = \text{cov}(\mathcal{N}_n^r)$  hold in ZFC?*

In Corollary 3.3 below we answer this question in the negative. The consistent strictness of the remaining two inequaities are proved in Section 4.

Our last problem was formulated in a recent preprint of P. Humke and M. Laczkovich [4]. Working on certain generalizations of results of Sierpiński and of Erdős they isolated the following definition.

**Definition 1.10.** For a  $\sigma$ -ideal  $\mathcal{I}$  on  $\mathbb{R}$  let us abbreviate the following statement as

$(*)_{\mathcal{I}} \iff$  there exists an ordering of  $\mathbb{R}$  such that all proper initial segments are in  $\mathcal{I}$ .

Using this notation our problem can be formulated as follows.

**Question 1.11.** ([4]) *Is it consistent that  $(*)_{\mathcal{N}}$  holds but  $(*)_{\mathcal{N}_1^{1/2}}$  fails?*

The following is easy to see and is also shown in [4].

**Claim 1.12.**  $\text{add}(\mathcal{I}) = \text{cov}(\mathcal{I}) \implies (*_{\mathcal{I}} \implies \text{cov}(\mathcal{I}) \leq \text{non}(\mathcal{I})$ .

Hence it suffices to answer the following question affirmatively.

**Question 1.13.** *Is it consistent that  $\text{add}(\mathcal{N}) = \text{cov}(\mathcal{N})$  and  $\text{cov}(\mathcal{N}_1^{1/2}) > \text{non}(\mathcal{N}_1^{1/2})$ ?*

In Corollary 3.4 below we answer this question affirmatively.

## 2. PARTIAL ANSWER TO ZAPLETAL'S QUESTION

**Theorem 2.1.** *The  $\sigma$ -ideal  $\mathcal{I}_{3,\sigma-fin}^2$  is not homogeneous, partially answering Zapletal's question.*

**Proof.** Let  $B \subset \mathbb{R}^3$  be an arbitrary Borel set with  $\dim_H(B) = \frac{5}{2}$ . Let  $f : \mathbb{R}^3 \rightarrow B$  be an arbitrary Borel map. Then [8, Theorem 1.4] states that for every Borel set  $A \subset \mathbb{R}^n$ , Borel map  $f : A \rightarrow \mathbb{R}^m$  and  $0 \leq d \leq 1$  there exists a Borel set  $D \subset A$  such that  $\dim_H D = d \cdot \dim_H A$  and  $\dim_H f(D) \leq d \cdot \dim_H f(A)$ . Applying this with  $n = m = 3$ ,  $A = \mathbb{R}^3$ , and  $d = \frac{11}{15}$  we obtain that there exists a Borel set  $D \subset \mathbb{R}^3$  with  $\dim_H(D) = \frac{11}{5}$  such that  $\dim_H(f(D)) \leq \frac{11}{15} \cdot \frac{5}{2} = \frac{11}{6}$ . Then  $\dim_H(D) > 2$  and  $\dim_H(f(D)) < 2$ , therefore  $f(D) \in \mathcal{I}_{3,\sigma-fin}^2$ , but  $f^{-1}(f(D)) \supset D \notin \mathcal{I}_{3,\sigma-fin}^2$ . Since  $f$  was arbitrary, the choice  $I = f(D)$  shows that  $\mathcal{I}_{3,\sigma-fin}^2$  is not homogeneous.  $\square$

**Remark 2.2.** The same proof actually yields that for every  $0 < r < n$  the  $\sigma$ -ideal  $\mathcal{I}_{n,\sigma-fin}^r$  is not homogeneous.

3. THE MODEL ANSWERING THE QUESTIONS OF FREMLIN AND  
HUMKE-LACZKOVICH

**Lemma 3.1.**  $\text{cov}(\mathcal{I}_{n,\sigma\text{-fin}}^r) = \text{cov}(\mathcal{N}_n^r)$

**Proof.** The inequality  $\text{cov}(\mathcal{I}_{n,\sigma\text{-fin}}^r) \leq \text{cov}(\mathcal{N}_n^r)$  is clear by  $\mathcal{N}_n^r \subset \mathcal{I}_{n,\sigma\text{-fin}}^r$ . In order to prove the opposite inequality let  $\{I_\alpha\}_{\alpha < \text{cov}(\mathcal{I}_{n,\sigma\text{-fin}}^r)}$  be a cover of  $\mathbb{R}^n$  by sets of  $\sigma$ -finite  $\mathcal{H}^r$ -measure. We can assume that they are actually of finite  $\mathcal{H}^r$ -measure, and also that they are Borel (even  $G_\delta$ ). By the Isomorphism Theorem of Measures [6, Thm. 17.41] a Borel set of finite  $\mathcal{H}^r$ -measure can be covered by  $\text{cov}(\mathcal{N})$  many  $\mathcal{H}^r$ -nullsets. Therefore  $\mathbb{R}^n$  can be covered by  $\text{cov}(\mathcal{I}_{n,\sigma\text{-fin}}^r) \cdot \text{cov}(\mathcal{N})$  many  $\mathcal{H}^r$ -nullsets. But  $\mathcal{I}_{n,\sigma\text{-fin}}^r \subset \mathcal{N}$  implies  $\text{cov}(\mathcal{I}_{n,\sigma\text{-fin}}^r) \geq \text{cov}(\mathcal{N})$ , hence  $\mathbb{R}^n$  can be covered by  $\text{cov}(\mathcal{I}_{n,\sigma\text{-fin}}^r)$  many  $\mathcal{H}^r$ -nullsets, proving  $\text{cov}(\mathcal{N}_n^r) \leq \text{cov}(\mathcal{I}_{n,\sigma\text{-fin}}^r)$ .  $\square$

The following theorem describes the values of all the cardinal invariants of the above diagram in a specific model of *ZFC*.

**Theorem 3.2.** *It is consistent with ZFC that  $\text{cov}(\mathcal{N}) = \mathfrak{d} = \text{non}(\mathcal{N}) = \omega_1$  and  $\text{cov}(\mathcal{N}_n^r) = \mathfrak{c} = \omega_2$ .*

**Proof.** Most ingredients of this proof are actually present in [12]. Let  $V$  be a ground model satisfying the Continuum Hypothesis, and let  $W$  be obtained by the countable support iteration of  $\mathbb{P}_{\mathcal{I}_{n,\sigma\text{-fin}}^r}$  of length  $\omega_2$ . Since the forcing  $\mathbb{P}_{\mathcal{I}_{n,\sigma\text{-fin}}^r}$  is proper by [12, 4.4.2] and adds a generic real avoiding the Borel members of  $\mathcal{I}_{n,\sigma\text{-fin}}^r$  coded in  $V$ , we obtain that  $\text{cov}(\mathcal{I}_{n,\sigma\text{-fin}}^r) = \mathfrak{c} = \omega_2$  in  $W$ . Hence,  $\text{cov}(\mathcal{N}_n^r) = \mathfrak{c} = \omega_2$  in  $W$  by Lemma 3.1. By [12, 4.4.8]  $\mathbb{P}_{\mathcal{I}_{n,\sigma\text{-fin}}^r}$  adds no splitting reals, hence no Random reals, and this is well-known to be preserved by the iteration, thus the Borel nullsets coded in  $V$  cover the reals of  $W$ , therefore  $\text{cov}(\mathcal{N}) = \omega_1$  in  $W$ . Moreover, by [12, Ex. 3.6.4]  $\mathcal{I}_{n,\sigma\text{-fin}}^r$  is polar, which is preserved by the iteration, therefore it preserves outer Lebesgue measure, hence the ground model is not null, thus  $\text{non}(\mathcal{N}) = \omega_1$  in  $W$ . Finally, the forcing is  $\omega^\omega$ -bounding by [12, 4.4.8], hence the same holds for the iteration, therefore  $\mathfrak{d} = \omega_1$  in  $W$ .  $\square$

The following are immediate.

**Corollary 3.3.** *Consistently  $\text{cov}(\mathcal{N}) < \text{cov}(\mathcal{N}_n^r)$ , answering Fremlin's question.*

**Corollary 3.4.** *The answer to Question 1.13 is affirmative, hence so is the answer to the question of Humke and Laczkovich.*

4. FURTHER RESULTS

First, for the sake of completeness, let us now determine the position of the cardinal invariants of the  $\sigma$ -ideal  $\mathcal{I}_{n,\sigma\text{-fin}}^r$  in the diagram.

**Proposition 4.1.** *In ZFC,*

$$\begin{aligned} \text{add}(\mathcal{I}_{n,\sigma\text{-fin}}^r) &= \omega_1, \\ \text{cov}(\mathcal{I}_{n,\sigma\text{-fin}}^r) &= \text{cov}(\mathcal{N}_n^r), \\ \text{non}(\mathcal{I}_{n,\sigma\text{-fin}}^r) &= \text{non}(\mathcal{N}_n^r), \\ \text{cof}(\mathcal{I}_{n,\sigma\text{-fin}}^r) &= \mathfrak{c}. \end{aligned}$$

**Proof.** Let  $\{B_\alpha\}_{\alpha < \omega_1}$  be a disjoint family of Borel sets of positive finite  $\mathcal{H}^r$ -measure, then clearly  $\bigcup_{\alpha < \omega_1} B_\alpha \notin \mathcal{I}_{n,\sigma\text{-fin}}^r$  showing  $\text{add}(\mathcal{I}_{n,\sigma\text{-fin}}^r) = \omega_1$ .

$\text{cov}(\mathcal{I}_{n,\sigma\text{-fin}}^r) = \text{cov}(\mathcal{N}_n^r)$  is just Lemma 3.1.

In order to prove  $\text{non}(\mathcal{I}_{n,\sigma\text{-fin}}^r) = \text{non}(\mathcal{N}_n^r)$ , let us assume to the contrary that  $\text{non}(\mathcal{N}_n^r) = \kappa < \lambda = \text{non}(\mathcal{I}_{n,\sigma\text{-fin}}^r)$ . Let  $H \notin \mathcal{N}_n^r$  be such that  $|H| = \kappa$ . Then  $H$  is of  $\sigma$ -finite  $\mathcal{H}^r$ -measure, that is  $H = \cup_{k \in \omega} H_k$  such that  $\mathcal{H}^r(H_k) < \infty$  for every  $k \in \omega$ . Fix  $k$  such that  $\mathcal{H}^r(H_k) > 0$ . Every set of finite  $\mathcal{H}^r$ -measure is contained in a Borel (actually  $G_\delta$ ) set of finite  $\mathcal{H}^r$ -measure, therefore there exists a Borel set  $B \supset H_k$  of positive finite  $\mathcal{H}^r$ -measure. Clearly,  $|H_k| \leq \kappa$ . By the Isomorphism Theorem of Measures [6, 17.41] this implies that  $\text{non}(\mathcal{N}) \leq \kappa$ . But  $\mathcal{I}_{n,\sigma\text{-fin}}^r \subset \mathcal{N}$  yields  $\lambda = \text{non}(\mathcal{I}_{n,\sigma\text{-fin}}^r) \leq \text{non}(\mathcal{N}) \leq \kappa$ , a contradiction.

Finally, let  $\{B_\alpha\}_{\alpha < \mathfrak{c}}$  be a disjoint family of Borel sets of positive finite  $\mathcal{H}^r$ -measure. Since every set of  $\sigma$ -finite  $\mathcal{H}^r$ -measure can contain at most countably many of them, it is easy to see that  $\text{cof}(\mathcal{I}_{n,\sigma\text{-fin}}^r) = \mathfrak{c}$ .  $\square$

Next we show that the remaining two inequalities in the above extended Cichoń Diagram are also strict in certain models.

Recall that, as usual in set theory, each natural number is identified with the set of its predecessors, i.e.  $k = \{0, \dots, k-1\}$ . Also recall that  $[k]^m = \{A \subset k : |A| = m\}$ .

**Theorem 4.2.** *It is consistent with ZFC that  $\text{cov}(\mathcal{N}_n^r) < \text{non}(\mathcal{M})$ .*

**Proof.** Let  $W$  be the Laver model, that is, the model obtained by iteratively adding  $\omega_2$  Laver reals with countable support over a model  $V$  satisfying the Continuum Hypothesis, see [1] for the definitions and basic properties of this model. For example, it is well-known that  $\text{non}(\mathcal{M}) = \omega_2$  in this model.

On the other hand,  $W$  satisfies the so called Laver property, an equivalent form of which is the following:

If  $0 < r < n$  and  $x \in \prod_{k \in \omega} 2^{kn} \cap W$  then there is

$$T \in \prod_{k \in \omega} [2^{kn}]^{\leq 2^{k\frac{r}{2}}} \cap V$$

such that  $x(k) \in T(k)$  for all  $k \in \omega$ . (Here  $[A]^{\leq d}$  is the system of subsets of  $A$  of cardinality at most  $d$ .) This follows from [1, Lemma 6.3.32] by letting  $f(k) = 2^{kn}$ ,  $S(k) = \{x(k)\}$ , and using an arbitrary positive rational number  $s < \frac{r}{2}$ .

The following argument takes place in  $W$ . For every  $k \in \omega$  let  $\psi_k$  be a bijection from  $2^{kn}$  to the set of all cubes of the form

$$\left[ \frac{j_0}{2^k}, \frac{j_0+1}{2^k} \right] \times \dots \times \left[ \frac{j_{n-1}}{2^k}, \frac{j_{n-1}+1}{2^k} \right],$$

where  $j_i \in 2^k$  for each  $i \in n$ .

For every  $T \in \prod_{k \in \omega} [2^{kn}]^{\leq 2^{k\frac{r}{2}}}$  define

$$N_T = \bigcap_{k \in \omega} \bigcup_{j \in T(k)} \psi_k(j).$$

First we show that  $N_T \in \mathcal{N}_n^r$ . Note that the diameter of a cube of side-length  $\frac{1}{2^k}$  is  $\sqrt{n} \frac{1}{2^k}$ . Clearly, for every  $k \in \omega$  we have  $\mathcal{H}_\infty^r(N_T) \leq \mathcal{H}_\infty^r\left(\bigcup_{j \in T(k)} \psi_k(j)\right) \leq |T(k)| \left(\sqrt{n} \frac{1}{2^k}\right)^r \leq 2^{k\frac{r}{2}} \left(\sqrt{n} \frac{1}{2^k}\right)^r = \sqrt{n}^r 2^{-k\frac{r}{2}}$ , which tends to 0 as  $k$  tends to  $\infty$ , therefore  $\mathcal{H}_\infty^r(N_T) = 0$  and consequently, by Remark 1.2,  $\mathcal{H}^r(N_T) = 0$ .

Next we finish the proof by showing that  $\{N_T : T \in \prod_{k \in \omega} [2^{kn}]^{\leq 2^{k\frac{r}{2}}} \cap V\}$  is a cover of  $[0, 1]^n$  (note that  $|V| = \omega_1$  in  $W$ , and also that if  $\omega_1$  members of  $\mathcal{N}_n^r$  cover

the unit cube then the same holds for  $\mathbb{R}^n$ , hence this implies  $\text{cov}(\mathcal{N}_n^r) = \omega_1$ . So let  $z \in [0, 1]^n$ , then there exists  $x \in \prod_{k \in \omega} 2^{kn}$  such that  $z \in \psi_k(x(k))$  for each  $k \in \omega$ . Let  $T \in \prod_{k \in \omega} [2^{kn}]^{\leq 2^{k \frac{r}{2}}} \cap V$  be such that  $x(k) \in T(k)$  for all  $k \in \omega$ , then it is easy to check that  $z \in N_T$ , finishing the proof.  $\square$

Next we turn to the consistency of  $\text{cov}(\mathcal{M}) < \text{non}(\mathcal{N}_n^r)$ . First we need some preparation.

For each  $k \in \omega$  let  $M_k \in \omega$  be so large that

$$(4.1) \quad 2^k \left( \frac{\sqrt{n}}{M_k} \right)^r < \frac{1}{2^k}.$$

**Definition 4.3.** Let  $C_k$  be the set of all cubes of the form

$$\left[ \frac{j_0}{M_k}, \frac{j_0 + 1}{M_k} \right] \times \dots \times \left[ \frac{j_{n-1}}{M_k}, \frac{j_{n-1} + 1}{M_k} \right],$$

where  $j_i \in M_k$  for each  $i \in n$ . Let  $\mathbb{C}_k$  consist of all sets that can be written as the union of  $2^k$  elements of  $C_k$ .

**Lemma 4.4.** *For every partition  $\mathbb{C}_k = \bigcup_{i \in 2^k} X_i$  there is some  $i \in 2^k$  such that  $\bigcup X_i = [0, 1]^n$ .*

**Proof.** Otherwise, pick  $x_i \notin \bigcup X_i$  and cubes  $Q_i \in C_k$  containing  $x_i$ , then  $\bigcup_{i \in 2^k} Q_i \in \mathbb{C}_k$  belongs to one of the  $X_i$ , yielding a contradiction.  $\square$

**Definition 4.5.** Now we define the *norm function*  $\nu: \bigcup_{k \in \omega} \mathcal{P}(\mathbb{C}_k) \rightarrow \omega$  as follows. For  $X \subset \mathbb{C}_k$  define  $\nu(X) \geq 1$  if  $\bigcup X = [0, 1]^n$  and define  $\nu(X) \geq j + 1$  if for every partition  $X = X_0 \cup X_1$  there is  $i \in 2$  such that  $\nu(X_i) \geq j$ .

**Lemma 4.6.**  $\nu(\mathbb{C}_k) \geq k + 1$ .

*Proof.* Otherwise, we could iteratively split  $\mathbb{C}_k$  into pieces so that at stage  $m$  we have a partition into  $2^m$  many sets each with norm at most  $k - m$ , hence eventually we could have a partition into  $2^k$  many sets none of which covers  $[0, 1]^n$ , contradicting the previous lemma.  $\square$

**Lemma 4.7.** *If  $X \subset \mathbb{C}_k$  and  $\nu(X) \geq j$  and  $y \in [0, 1]^n$  then  $\nu(\{H \in X : y \in H\}) \geq j - 1$ .*

*Proof.* We may assume  $j > 1$ . Let  $X_0 = \{H \in X : y \in H\}$  and  $X_1 = \{H \in X : y \notin H\}$ . Then either  $\nu(X_0) \geq j - 1 \geq 1$  or  $\nu(X_1) \geq j - 1 \geq 1$ . But note that  $\nu(X_1) \not\geq 1$  since  $y \notin \bigcup X_1$ .  $\square$

In this paper a *finite sequence* will mean a function defined on a natural number, the *length* of the sequence  $t$ , denoted by  $|t|$  is simply  $\text{dom}(t)$ . Moreover, a *tree* will mean a set of finite sequences closed under initial segments. Then for  $t, s \in T$  we have  $t \subset s$  iff  $s$  end-extends  $t$  and this partial order is indeed a tree in the usual sense. For a  $t \in T$  let us denote by  $\text{succ}_T(t)$  the set of immediate successors of  $t$  in  $T$ .

Now let us define the following forcing notion.

**Definition 4.8.** Let  $T \in \mathbb{P}$  iff

- (1)  $T$  is a non-empty tree,
- (2) for every  $t \in T$  and  $k < |t|$  we have  $t(k) \in \mathbb{C}_k$ ,

- (3) for every  $t \in T$  we have  $\text{succ}_T(t) \neq \emptyset$ ,
- (4) for every  $t \in T$  there exists  $s \in T$ ,  $s \supset t$  with  $|\text{succ}_T(s)| > 1$ ,
- (5) for every  $K \in \omega$  the set  $\{t \in T : |\text{succ}_T(t)| > 1 \text{ and } \nu(\text{succ}_T(t)) \leq K\}$  is finite.

If  $T, T' \in \mathbb{P}$  then define

$$T \leq_{\mathbb{P}} T' \iff T \subset T'.$$

We will usually simply write  $\leq$  for  $\leq_{\mathbb{P}}$ . Clearly,  $1_{\mathbb{P}}$  is the set of all finite sequences satisfying (2).

**Remark 4.9.** A  $t \in T$  with  $|\text{succ}_T(t)| > 1$  is called a *branching node*. For  $t \in T$  define  $T[t] = \{s \in T : s \subset t \text{ or } s \supset t\}$ . It is easy to see that if  $t \in T \in \mathbb{P}$  then  $T[t] \in \mathbb{P}$  and  $T[t] \leq T$ .

**Remark 4.10.** Forcing notions of this type are discussed in [10] in great generality. However, in order to keep the paper relatively self-contained we also include the rather standard proofs here, but note that most of the techniques below can already be found in [10].

**Lemma 4.11.**  $\mathbb{P}$  is proper.

*Proof.* Let  $\mathfrak{M}$  be a countable elementary submodel, and recall that  $T \in \mathbb{P}$  is  $\mathfrak{M}$ -generic if for every dense open subset  $D \subset \mathbb{P}$  with  $D \in \mathfrak{M}$  we have  $T \Vdash \dot{G} \cap D \cap \mathfrak{M} \neq \emptyset$ , where  $\dot{G}$  is a name for the generic filter. Also recall that properness means that whenever a condition  $T \in \mathfrak{M}$  is given then there exists an  $\mathfrak{M}$ -generic  $T' \leq T$ . We construct this  $T'$  by a so called fusion argument.

Let the sequence  $D_0, D_1, \dots$  enumerate the dense open subsets of  $\mathbb{P}$  that are in  $\mathfrak{M}$ . During the construction we make sure that all objects we pick ( $t, s, t'_s, s', L_k, L_k^+, S_s$ , etc.) are in  $\mathfrak{M}$ . The whole construction, and hence  $T'$ , will typically not be in  $\mathfrak{M}$ .

We define the set of branching notes of  $T'$  ‘level-by level’ as follows. Let  $t \in T$  be a branching node with  $\nu(\text{succ}_T(t)) > 0$  and set  $L_0 = \{t\}$ . Also define  $L_0^+ = \text{succ}_T(t)$ . Moreover, for every  $s \in L_0^+$  also fix a  $S_s \leq T[s]$  with  $S_s \in D_0$  (this is possible, since  $D_0$  is dense). This finishes the 0th step of the fusion.

Now, if  $L_k, L_k^+$ , and for every  $s \in L_k^+$  a condition  $S_s \leq T[s]$  have already been defined then for every  $s \in L_k^+$  we pick a  $t'_s \in S_s$  with  $\nu(\text{succ}_{S_s}(t'_s)) > k + 1$ . Let  $L_{k+1} = \{t'_s : s \in L_k^+\}$ , and define  $L_{k+1}^+ = \bigcup_{s \in L_k^+} \text{succ}_{S_s}(t'_s)$ . Now, for every  $s' \in L_{k+1}^+$  pick a  $S_{s'} \leq S_s[s']$  with  $S_{s'} \in D_{k+1}$ . This finishes the  $k + 1$ st step of the fusion.

Finally, define  $T'$  as the closure of  $\bigcup_{k \in \omega} L_k$  under initial segments (this is the same as the closure of  $\bigcup_{k \in \omega} L_k^+$  under initial segments). It is easy to check that  $T' \in \mathbb{P}$  and  $T' \leq T$ . It remains to show that  $T'$  is  $\mathfrak{M}$ -generic. So let  $k \in \omega$  be fixed, and we need to show that  $T' \Vdash \dot{G} \cap D_k \cap \mathfrak{M} \neq \emptyset$ .

Before the proof let us make three remarks. First, it is easy to see from the construction that if  $T'' \leq T'$  then for every  $k \in \omega$  there exists  $s \in L_k^+ \cap T''$ . Second, it can also be seen from the construction that  $T'[s] \leq S_s$  for every  $s$ . Third, if  $S \in D \cap \mathfrak{M}$  then obviously  $S \Vdash \dot{G} \cap D \cap \mathfrak{M} \neq \emptyset$ , since  $S \Vdash S \in \dot{G} \cap D \cap \mathfrak{M}$ .

Now we prove  $T' \Vdash \dot{G} \cap D_k \cap \mathfrak{M} \neq \emptyset$ . We prove this by showing that for every  $T'' \leq T'$  there exists  $T''' \leq T''$  forcing this. Let  $T'' \leq T'$  be given. Then, by the above remark there exists  $s \in L_k^+ \cap T''$ . Set  $T''' = T''[s]$ , then clearly  $T''' \leq T''$ . Finally,  $T''' = T''[s] \leq T'[s] \leq S_s \in D_k \cap \mathfrak{M}$ , hence  $S_s \Vdash \dot{G} \cap D_k \cap \mathfrak{M} \neq \emptyset$ , hence  $T'''$  forces the same, finishing the proof.  $\square$

**Lemma 4.12.** *If  $(H_k)_{k \in \omega} \in \prod_{k \in \omega} \mathbb{C}_k$  then  $\mathcal{H}^r(\bigcap_{m \in \omega} \bigcup_{k \geq m} H_k) = 0$ .*

*Proof.* For every  $m \in \omega$ , using (4.1), we have

$$\begin{aligned} \mathcal{H}_\infty^r \left( \bigcap_{m \in \omega} \bigcup_{k \geq m} H_k \right) &\leq \mathcal{H}_\infty^r \left( \bigcup_{k \geq m} H_k \right) \leq \sum_{k \geq m} \mathcal{H}_\infty^r(H_k) \leq \\ &\leq \sum_{k \geq m} 2^k \left( \frac{\sqrt{n}}{M_k} \right)^r \leq \sum_{k \geq m} \frac{1}{2^k} = \frac{1}{2^{m-1}}, \end{aligned}$$

hence  $\mathcal{H}_\infty^r(\bigcap_{m \in \omega} \bigcup_{k \geq m} H_k) = 0$ , therefore  $\mathcal{H}^r(\bigcap_{m \in \omega} \bigcup_{k \geq m} H_k) = 0$ .  $\square$

**Remark 4.13.** In the usual way, by slight abuse of notation, the generic filter  $G$  can be thought of as a sequence  $G = (G_k)_{k \in \omega} \in \prod_{k \in \omega} \mathbb{C}_k$ . What we will formally need is that if a generic filter  $G$  is given, then  $\bigcap_{T \in G} T$  defines such a sequence, hence  $G_k$  makes sense.

**Lemma 4.14.** *If  $G$  is a generic filter over a ground model  $V$  then  $V[G] \models V \cap [0, 1]^n \subset \bigcap_{m \in \omega} \bigcup_{k \geq m} \dot{G}_k$ .*

*Proof.* Fix  $y \in V \cap [0, 1]^n$ . In order to show that  $1_{\mathbb{P}} \Vdash "V \cap [0, 1]^n \subset \bigcap_{m \in \omega} \bigcup_{k \geq m} \dot{G}_k"$  we show that for every  $T \in \mathbb{P}$  there is  $T' \leq T$  forcing this. So let  $T$  be given, and define  $T'$  as follows. Starting from the root of  $T$ , we recursively thin out  $T$  such that for every  $t \in T$  with  $\nu(\text{succ}_T(t)) \geq 1$  we cut off all the nodes  $s \in \text{succ}_T(t)$  with  $y \notin s(|s| - 1)$ . One can easily check using Lemma 4.7 that  $T' \in \mathbb{P}$  and  $T' \leq T$ . So it suffices to show that for every  $m \in \omega$  we have  $T' \Vdash "y \in \bigcup_{k \geq m} \dot{G}_k"$ . Hence let  $T'' \leq T'$  be given, we need to find  $T''' \leq T''$  forcing this. Pick  $t \in T''$  with  $|t| \geq m$  and  $\nu(\text{succ}_{T''}(t)) \geq 1$ . This implies that the successors of  $t$  were thinned out, hence  $y \in s(|s| - 1)$  for every  $s \in \text{succ}_{T''}(t)$ . Fix such an  $s$ , and define  $T''' = T''[s]$ . Then  $T''' = T''[s] \Vdash "\dot{G}_{|s|-1} = s(|s| - 1) \ni y"$ , finishing the proof.  $\square$

**Lemma 4.15.**  *$\mathbb{P}$  is  $\omega^\omega$ -bounding.*

*Proof.* For  $f, g \in \omega^\omega$  we write  $f \leq g$  if  $f(n) \leq g(n)$  for every  $n \in \omega$ . Let  $\dot{f} \in \omega^\omega$  be a name. We claim that  $1_{\mathbb{P}} \Vdash "\exists g \in V \cap \omega^\omega$  such that  $\dot{f} \leq g"$ . It suffices to show that for every  $T$  there exists  $T' \leq T$  and  $g \in V \cap \omega^\omega$  such that  $T' \Vdash "\dot{f} \leq g"$ . We will construct this  $T'$  by a fusion argument similar to that of Lemma 4.11.

Let  $t \in T$  be a node with  $\nu(\text{succ}_T(t)) > 0$  and set  $L_0 = \{t\}$ . Also define  $L_0^+ = \text{succ}_T(t)$ . Moreover, for every  $s \in L_0^+$  also fix a  $S_s \leq T[s]$  and  $m_s \in \omega$  such that  $S_s \Vdash "\dot{f}(0) = m_s"$  (this is possible by the basic properties of forcing). This finishes the 0th step of the fusion.

Now, if  $L_k, L_k^+$ , and for every  $s \in L_k^+$  a condition  $S_s \leq T[s]$  have already been defined then for every  $s \in L_k^+$  we pick a  $t'_s \in S_s$  with  $\nu(\text{succ}_{S_s}(t'_s)) > k + 1$ . Let  $L_{k+1} = \{t'_s : s \in L_k^+\}$ , and define  $L_{k+1}^+ = \bigcup_{s \in L_k^+} \text{succ}_{S_s}(t'_s)$ . Now, for every  $s' \in L_{k+1}^+$  pick a  $S_{s'} \leq S_s[s']$  and  $m_{s'} \in \omega$  with  $S_{s'} \Vdash "\dot{f}(k+1) = m_{s'}"$ . This finishes the  $k+1$ st step of the fusion.

Finally, define  $T'$  as the closure of  $\bigcup_{k \in \omega} L_k^+$  under initial segments. It is easy to check that  $T' \in \mathbb{P}$  and  $T' \leq T$ . Define  $g(k) = \max\{m_s : s \in L_k^+\}$  (the maximum exists, since this set is finite). It remains to show that  $T' \Vdash "\dot{f} \leq g"$ . So let  $k \in \omega$  be fixed, and let  $T'' \leq T'$  be given. Pick  $s \in T'' \cap L_k^+$ , and define  $T''' = T''[s]$ .



Then  $T''' = T''[s] \leq S_s \Vdash \dot{f}(k) = m_s$ , hence  $T''' \Vdash \dot{f}(k) \leq g(k)$ , finishing the proof.  $\square$

**Theorem 4.16.** *It is consistent with ZFC that  $\text{cov}(\mathcal{M}) < \text{non}(\mathcal{N}_n^r)$ .*

*Proof.* Let  $V$  be a model satisfying the Continuum Hypothesis, and let  $V_{\omega_2}$  be the model obtained by an  $\omega_2$ -long countable support iteration of  $\mathbb{P}$ . Let  $(V_\alpha)_{\alpha \leq \omega_2}$  denote the intermediate models. Since  $\mathbb{P}$  is proper and adds a real, by standard arguments the continuum is  $\omega_2$  in  $V_{\omega_2}$ .

On the one hand,  $\mathbb{P}$  is  $\omega^\omega$ -bounding, hence so is its iteration. Therefore the iteration adds no Cohen reals, hence the meagre Borel sets coded in  $V$  cover  $V_{\omega_2} \cap \mathbb{R}^n$ , hence  $\text{cov}(\mathcal{M}) = \omega_1$ .

On the other hand, if  $H \in V_{\omega_2}$ ,  $|H| = \omega_1$  then, by a standard reflection argument,  $H \subset V_\alpha$  for some  $\alpha < \omega_2$ . Hence, by Lemma 4.14 and Lemma 4.12 we have  $V_{\alpha+1} \Vdash \mathcal{H}^r(H) = 0$ . Therefore, since  $\mathcal{H}^r(H) = 0$  means the existence of certain covers, and by absoluteness the corresponding covers exist in  $V_{\omega_2}$ , we obtain  $V_{\omega_2} \Vdash \mathcal{H}^r(H) = 0$ . Hence,  $\text{non}(\mathcal{N}_n^r) = \omega_2$ . Therefore the proof is complete.  $\square$

## 5. OPEN PROBLEMS

First we reiterate Zapletal's question [12, Question 7.1.3.] in its original form.

**Problem 5.1.** *Is the forcing notion  $\mathbb{P}_{\mathcal{I}_{3,\sigma\text{-fin}}^2}$  homogeneous?*

Moreover, Theorem 2.1 also leaves open the following.

**Problem 5.2.** *Does there exist a Borel set  $B \subset \mathbb{R}^3$  with  $B \notin \mathcal{I}_{3,\sigma\text{-fin}}^2$  such that  $\mathcal{I}_{3,\sigma\text{-fin}}^2$  is not homogeneous below  $B$ ?*

**Problem 5.3.** *Let  $0 < r < s < n$ . Does  $\text{non}(\mathcal{N}_n^r) = \text{non}(\mathcal{N}_n^s)$  hold in ZFC?*

Analogously,

**Problem 5.4.** *Let  $0 < r < s < n$ . Does  $\text{cov}(\mathcal{N}_n^r) = \text{cov}(\mathcal{N}_n^s)$  hold in ZFC?*

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