Transfinite Sequences of Continuous and Baire Class 1 Functions

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Abstract

The set of continuous or Baire class 1 functions defined on a metric space $X$ is endowed with the natural pointwise partial order. We investigate how the possible lengths of well-ordered monotone sequences (with respect to this order) depend on the space $X$.

Introduction

Any set $\mathcal{F}$ of real valued functions defined on an arbitrary set $X$ is partially ordered by the pointwise order; that is, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. Then, $f < g$ iff $f \leq g$ and $g \nleq f$; equivalently, $f(x) \leq g(x)$ for all $x \in X$ and $f(x) < g(x)$ for at least one $x \in X$. Our aim will be to investigate the possible lengths of the increasing or decreasing well-ordered sequences of functions in $\mathcal{F}$ with respect to this order. A classical theorem (see Kuratowski [5], §24.III, Theorem 2') asserts that if $\mathcal{F}$ is the set of Baire class 1 functions (that is, pointwise limits of continuous functions) defined on a Polish space $X$ (that is, a complete separable metric space), then there exists a monotone sequence of length $\xi$ in $\mathcal{F}$ iff $\xi < \omega_1$. P. Komjáth [3] proved that the corresponding question concerning Baire class $\alpha$ functions for $2 \leq \alpha < \omega_1$ is independent of ZFC.

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1 Sequences of Continuous Functions

In the present paper we investigate what happens if we replace the Polish space $X$ by an arbitrary metric space.

Section 1 considers chains of continuous functions. We show that for any metric space $X$, there exists a chain in $C(X, \mathbb{R})$ of order type $\xi$ iff $|\xi| \leq d(X)$. Here, $|A|$ denotes the cardinality of the set $A$, while $d(X)$ denotes the density of the space $X$, that is

$$d(X) = \max(\min\{|D| : D \subseteq X & \overline{D} = X\}, \omega).$$

In particular, for separable $X$, every well-ordered chain has countable length, just as for Polish spaces.

Section 2 considers chains of Baire class 1 functions on separable metric spaces. Here, the situation is entirely different from the case of Polish spaces, since on some separable metric spaces, there are well-ordered chains of every order type less than $\omega_2$. Furthermore, the existence of chains of type $\omega_2$ and longer is independent of ZFC + $\neg$CH. Under MA, there are chains of all types less than $c^+$, whereas in the Cohen model, all chains have type less than $\omega_2$.

We note here that instead of examining well-ordered sequences, which is a classical problem, we could try to characterize all the possible order types of linearly ordered subsets of the partially ordered set $\mathcal{F}$. This problem was posed by M. Laczkovich, and is considered in detail in [2].

1 Sequences of Continuous Functions

Lemma 1.1 For any topological space $X$: If there is a well-ordered sequence of length $\xi$ in $C(X, \mathbb{R})$, then $\xi < d(X)^+$. 

Proof. Let $\{f_\alpha : \alpha < \xi\}$ be an increasing sequence in $C(X, \mathbb{R})$, and let $D \subseteq X$ be a dense subset of $X$ such that $d(X) = \max(|D|, \omega)$. By continuity, the $f_\alpha|D$ are all distinct; so, for each $\alpha < \xi$, choose a $d_\alpha \in D$ such that $f_\alpha(d_\alpha) < f_{\alpha+1}(d_\alpha)$. For each $d \in D$ the set $E_d = \{\alpha : d_\alpha = d\}$ is countable, because every well-ordered subset of $\mathbb{R}$ is countable. Since $\xi = \bigcup_{d \in D} E_d$, we have $|\xi| \leq \max(|D|, \omega) = d(X)$.  

The converse implication is not true in general. For example, if $X$ has the countable chain condition (ccc), then every well-ordered chain in $C(X, \mathbb{R})$ is countable (because $X \times \mathbb{R}$ is also ccc). However, the converse is true for metric spaces:
Lemma 1.2 If $(X, \varrho)$ is any metric space and $\prec$ is any total order of the cardinal $d(X)$, then there is a chain in $C(X, \mathbb{R})$ which is isomorphic to $\prec$.

Proof. First, note that every countable total order is embeddable in $\mathbb{R}$, so if $d(X) = \omega$, then the result follows trivially using constant functions. In particular, we may assume that $X$ is infinite, and then fix $D \subseteq X$ which is dense and of size $d(X)$. For each $n \in \omega$, let $D_n$ be a subset of $D$ which is maximal with respect to the property $\forall d, e \in D_n \ [d \neq e \rightarrow \varrho(d, e) \geq 2^{-n}]$. Then $\bigcup_n D_n$ is also dense, so we may assume that $\bigcup_n D_n = D$. We may also assume that $\prec$ is a total order of the set $D$. Now, we shall produce $f_d \in C(X, \mathbb{R})$ for $d \in D$ such that $f_d < f_e$ whenever $d \prec e$.

For each $n$, if $c \in D_n$, define $\varphi^n_c(x) = \max(0, 2^{-n} - \varrho(x, c))$. For each $d \in D$, let $\psi^n_d = \{\varphi^n_c : c \in D_n \& c \prec d\}$. Since every $x \in X$ has a neighborhood on which all but at most one of the $\varphi^n_c$ vanish, we have $\psi^n_d \in C(X, [0, 2^{-n}])$, and $\psi^n_d \leq \psi^n_e$ whenever $d \prec e$. Thus, if we let $f_d = \sum_{n<\omega} \psi^n_d$, we have $f_d \in C(X, [0, 2])$, and $f_d \leq f_e$ whenever $d \prec e$. But also, if $d \in D_n$ and $d \prec e$, then $\psi^n_d(d) = 0 < 2^{-n} = \psi^n_e(d)$, so actually $f_d < f_e$ whenever $d \prec e$. \[\square\]

Putting these lemmas together, we have:

Theorem 1.3 Let $(X, \varrho)$ be a metric space. Then there exists a well-ordered sequence of length $\xi$ in $C(X, \mathbb{R})$ iff $\xi < d(X)^{+}$.

Corollary 1.4 A metric space $(X, \varrho)$ is separable iff every well-ordered sequence in $C(X, \mathbb{R})$ is countable.

2 Sequences of Baire Class 1 Functions

If we replace continuous functions by Baire class 1 functions, then Corollary 1.4 becomes false, since on some separable metric spaces, we can get well-ordered sequences of every type less than $\omega_2$. To prove this, we shall apply some basic facts about $\subseteq^*$ on $\mathcal{P}(\omega)$. As usual, for $x, y \subseteq \omega$, we say that $x \subseteq^* y$ iff $x \setminus y$ is finite. Then $x \subseteq^* y$ iff $x \setminus y$ is finite and $y \setminus x$ is infinite. This $\subseteq^*$ partially orders $\mathcal{P}(\omega)$.

Lemma 2.1 If $X \subseteq \mathcal{P}(\omega)$ is a chain in the order $\subseteq^*$, then on $X$ (viewed as a subset of the Cantor set $2^\omega \cong \mathcal{P}(\omega)$), there is a chain of Baire class 1 functions which is isomorphic to $(X, \subset^*)$. 
Proof. Note that for each \( x \in X \),
\[
\{ y \in X : y \subseteq^* x \} = \bigcup_{m \in \omega} \{ y \in X : \forall n \geq m \ [y(n) \leq x(n)] \}
\]
which is an \( F_\sigma \) set in \( X \). Likewise, the sets \( \{ y \in X : y \supseteq^* x \} \), \( \{ y \in X : y \subset^* x \} \),
and \( \{ y \in X : y \supset^* x \} \), are all \( F_\sigma \) sets in \( X \), and hence also \( G_\delta \) sets. It follows
that if \( f_\chi : X \to \{0, 1\} \) is the characteristic function of \( \{ y \in X : y \subset^* x \} \), then \( f_\chi : X \to \mathbb{R} \) is a Baire class 1 function. Then, \( \{ f_\chi : x \in X \} \) is the required chain. \( \square \)

**Lemma 2.2** For any infinite cardinal \( \kappa \), suppose that \( (\mathcal{P}(\omega), \subset^*) \) contains a
chain \( \{ x_\alpha : \alpha < \kappa \} \) (i.e., \( \alpha < \beta \to x_\alpha \subset^* x_\beta \}). Then \( (\mathcal{P}(\omega), \subset^*) \) contains a
chain \( X \) of size \( \kappa \) such that every ordinal \( \xi < \kappa^+ \) is embeddable into \( X \).

**Proof.** Let \( S = \bigcup_{1 \leq n < \omega} \kappa^n \). For \( s = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) \in S \), let \( s^+ = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n + 1) \). Starting with the \( x_{(\alpha)} = x_\alpha \), choose \( x_s \in \mathcal{P}(\omega) \) by
induction on length \( (s) \) so that \( x_s = x_{s^0} \subset^* x_{s^\alpha} \subset^* x_{s^\beta} \subset^* x_{s^+} \) whenever \( s \in S \) and \( 0 < \alpha < \beta < \kappa \). Let \( X = \{ x_s : s \in S \} \). Then, whenever \( x, y \in X \)
with \( x \subset^* y \), the ordinal \( \kappa \) is embeddable in \( (x, y) = \{ z \in X : x \subset^* z \subset^* y \} \).
From this, one easily proves by induction on \( \xi < \kappa^+ \) (using cf(\( \xi \)) \( \leq \kappa \)) that \( \xi \)
is embeddable in each such interval \( (x, y) \). \( \square \)

Since \( \mathcal{P}(\omega) \) certainly contains a chain of type \( \omega_1 \), these two lemmas yield:

**Theorem 2.3** There is a separable metric space \( X \) on which, for every \( \xi < \omega_2 \),
there is a well-ordered chain of length \( \xi \) of Baire class 1 functions.

Under \( CH \), this is best possible, since there will be only \( 2^{\omega_1} = \omega_1 \) Baire class 1 functions on a separable metric space, so there could not be a chain
of length \( \omega_2 \). Under \( \neg CH \), the existence of longer chains of Baire class 1 functions depends on the model of set theory. It is consistent with \( \mathfrak{c} = 2^{\omega_1} \) being
arbitrarily large that there is a chain in \( (\mathcal{P}(\omega), \subset^*) \) of type \( \mathfrak{c} \); for example, this
is true under \( MA \) (see [1]). In this case, there will be a separable \( X \) with
well-ordered chains of all lengths less than \( \mathfrak{c}^+ \). However, in the Cohen model,
where \( \mathfrak{c} \) can also be made arbitrarily large, we never get chains of type \( \omega_2 \). We
shall prove this by using the following lemma, which relates it to the rectangle problem:
Lemma 2.4 Suppose that there is a separable metric space \( Y \) with an \( \omega_2 \)-chain of Borel subsets, \( \{ B_\alpha : \alpha < \omega_2 \} \) (so, \( \alpha < \beta \rightarrow B_\alpha \subseteq \overline{B_\beta} \)). Then in \( \omega_2 \times \omega_2 \), the well-order relation \( < \) is in the \( \sigma \)-algebra generated by the set of all rectangles, \( \{ S \times T : S, T \in \mathcal{P}(\omega_2) \} \).

Proof. Each \( B_\alpha \) has some countable Borel rank. Since there are only \( \omega_1 \) ranks, we may, by passing to a subsequence, assume that the ranks are bounded. Say, each \( B_\alpha \) is a \( \Sigma^0_\mu \) set for some fixed \( \mu < \omega_1 \).

Let \( J = \omega^\omega \), and let \( A \subseteq Y \times J \) be a universal \( \Sigma^0_\mu \) set; that is, \( A \) is \( \Sigma^0_\mu \) in \( Y \times J \) and every \( \Sigma^0_\mu \) subset of \( Y \) is of the form \( A^j = \{ y : (y, j) \in A \} \) for some \( j \in J \) (see [5], §31). Now, for \( \alpha, \beta < \omega_2 \), fix \( y_\alpha \in B_{\alpha+1} \setminus B_\alpha \), and fix \( j_\beta \in J \) such that \( A^{j_\beta} = B_\beta \). Then \( \alpha < \beta \) iff \( (y_\alpha, j_\beta) \in A \). Thus, \( \{(y_\alpha, j_\beta) : \alpha < \beta < \omega_2 \} \) is a Borel subset of \( \{ y_\alpha : \alpha < \omega_2 \} \times \{ j_\beta : \beta < \omega_2 \} \), and is hence in the \( \sigma \)-algebra generated by open rectangles, so \( < \), as a subset of \( \omega_2 \times \omega_2 \), is in the \( \sigma \)-algebra generated by rectangles. \( \square \)

Theorem 2.5 Assume that \( V[G] \) is an extension of \( V \) by \( \geq \omega_2 \) Cohen reals, where the ground model, \( V \), satisfies CH. Then in \( V[G] \), no separable metric space can have a chain of length \( \omega_2 \) of Baire class 1 functions.

Proof. By [4], in \( V[G] \), the well-order relation in \( \omega_2 \times \omega_2 \) is not in the \( \sigma \)-algebra generated by all rectangles. Now, suppose that \( \{ f_\alpha : \alpha < \omega_2 \} \) is a chain of Baire class one functions on the separable metric space \( X \). Let \( B_\alpha = \{ (x, r) \in X \times \mathbb{R} : r \leq f_\alpha(x) \} \). Then the \( B_\alpha \) form an \( \omega_2 \)-chain of Borel subsets of the separable metric space \( X \times \mathbb{R} \), so we have a contradiction by Lemma 2.4. \( \square \)

References


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