RANKS ON THE BAIRE CLASS ξ FUNCTIONS

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Abstract. In 1990 Kechris and Louveau developed the theory of three very natural ranks on the Baire class 1 functions. A rank is a function assigning countable ordinals to certain objects, typically measuring their complexity. We extend this theory to the case of Baire class ξ functions, and generalize most of the results from the Baire class 1 case. We also show that their assumption of the compactness of the underlying space can be eliminated. As an application, we solve a problem concerning the so called solvability cardinals of systems of difference equations, arising from the theory of geometric decompositions. We also show that certain other very natural generalizations of the ranks of Kechris and Louveau surprisingly turn out to be bounded in ω_1. Finally, we prove a general result showing that all ranks satisfying some natural properties coincide for bounded functions.

1. Introduction

A real-valued function defined on a complete metric space is called *Baire class* 1 if it is the pointwise limit of a sequence of continuous functions. It is well-known that a function is of Baire class 1 iff the inverse image of every open set is F_σ iff there is a point of continuity relative to every non-empty closed set [7]. Baire class 1 functions play a central role in various branches of mathematics, most notably in Banach space theory, see e.g. [1] or [6]. A fundamental tool in the analysis of Baire class 1 functions is the theory of ranks, that is, maps assigning countable ordinals to Baire class 1 functions, typically measuring their complexity. In their seminal paper [8], Kechris and Louveau systematically investigated three very important ranks on the Baire class 1 functions. We will recall the definitions in Section 3 below, and only note here that they correspond to above three equivalent definitions of Baire class 1 functions. One can easily see that the theory has no straightforward generalization to the case of Baire class ξ functions. (Recall that f is of *Baire class* ξ if there exist sequences $\xi_n < \xi$ and f_n such that f_n is of Baire class ξ_n and $f_n \to f$ pointwise.)

Hence the following very natural but somewhat vague question arises.

Question 1.1. *Is there a natural extension of the theory of Kechris and Louveau to the case of Baire class ξ functions?*

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There is actually a very concrete version of this question that was raised by Elekes and Laczkovich in [3]. In order to be able to formulate this we need some preparation. For $\theta, \theta' < \omega_1$ let us define the relation $\theta \lesssim \theta'$ if $\theta' \leq \omega^\eta \implies \theta \leq \omega^\eta$ for every $1 \leq \eta < \omega_1$ (we use ordinal exponentiation here). Note that $\theta \leq \theta'$ implies $\theta \lesssim \theta'$, while $\theta \lesssim \theta'$, $\theta' > 0$ implies $\theta \leq \theta' \cdot \omega$. We will also use the notation $\theta \approx \theta'$ if $\theta \lesssim \theta'$ and $\theta' \lesssim \theta$. Then $\approx$ is an equivalence relation. Let us denote the set of Baire class ξ functions defined on $\mathbb{R}$ by $\mathcal{B}_\xi(\mathbb{R})$. The characteristic function of a set H is denoted by χ_H. A set is called *perfect* if it is closed and has no isolated points. Define the translation map $T_t : \mathbb{R} \to \mathbb{R}$ by $T_t(x) = x + t$ for every $x \in \mathbb{R}$.

Question 1.2. *(*[3, Question 6.7]*) Is there a map $\rho : \mathcal{B}_\xi(\mathbb{R}) \to \omega_1$ such that*

- *ρ is unbounded in ω_1, moreover, for every non-empty perfect set $P \subseteq \mathbb{R}$ and ordinal $\zeta < \omega_1$ there is a function $f \in \mathcal{B}_\xi(\mathbb{R})$ such that f is 0 outside of P and $\rho(f) \geq \zeta$,*
- *ρ is translation-invariant, i.e., $\rho(f \circ T_t) = \rho(f)$ for every $f \in \mathcal{B}_\xi(\mathbb{R})$ and $t \in \mathbb{R}$,*
- *ρ is essentially linear, i.e., $\rho(cf) \approx \rho(f)$ and $\rho(f+g) \lesssim \max\{\rho(f), \rho(g)\}$ for every $f, g \in \mathcal{B}_\xi(\mathbb{R})$ and $c \in \mathbb{R} \setminus \{0\}$,*
- *$\rho(f \cdot \chi_F) \lesssim \rho(f)$ for every closed set $F \subseteq \mathbb{R}$ and $f \in \mathcal{B}_\xi(\mathbb{R})$?*

The problem is not formulated in this exact form in [3], but a careful examination of the proofs there reveals that this is what they need for their results to go through. Actually, there are numerous equivalent formulations, for example we may simply replace $\lesssim$ by $\leq$ (indeed, just replace ρ satisfying the above properties by $\rho'(f) = \min\{\omega^\eta : \rho(f) \leq \omega^\eta\}$). However, it turns out, as it was already also the case in [8], that $\lesssim$ is more natural here.

Their original motivation came from the theory of paradoxical geometric decompositions (like the Banach-Tarski paradox, Tarski's problem of circling the square, etc.). It has turned out that the solvability of certain systems of difference equations plays a key role in this theory.

Definition 1.3. Let $\mathbb{R}^\mathbb{R}$ denote the set of functions from $\mathbb{R}$ to $\mathbb{R}$. A *difference operator* is a mapping $D : \mathbb{R}^\mathbb{R} \to \mathbb{R}^\mathbb{R}$ of the form

$$(Df)(x) = \sum_{i=1}^{n} a_i f(x + b_i),$$

where a_i and b_i are fixed real numbers.

Definition 1.4. A *difference equation* is a functional equation

$$Df = g,$$

where D is a difference operator, g is a given function and f is the unknown.

Definition 1.5. A *system of difference equations* is

$$D_i f = g_i \quad (i \in I),$$

where I is an arbitrary set of indices.

It is not very hard to show that a system of difference equations is solvable iff every *finite* subsystem is solvable. But if we are interested in continuous solutions then this result is no longer true. However, if every *countable* subsystem of a system has a continuous solution the the whole system has a continuous solution as well. This motivates the following definition, which has turned out to be a very useful tool for finding necessary conditions for the existence of certain solutions.

Definition 1.6. Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ be a class of real functions. The *solvability cardinal* of $\mathcal{F}$ is the minimal cardinal $sc(\mathcal{F})$ with the property that if every subsystem of size less than $sc(\mathcal{F})$ of a system of difference equations has a solution in $\mathcal{F}$ then the whole system has a solution in $\mathcal{F}$.

It was shown in [3] that the behavior of $sc(\mathcal{F})$ is rather erratic. For example, $sc(\text{polynomials}) = 3$ but $sc(\text{trigonometric polynomials}) = \omega_1$, $sc(\{f : f \text{ is continuous}\}) = \omega_1$ but $sc(\{f : f \text{ is Darboux}\}) = (2^\omega)^+$, and $sc(\mathbb{R}^{\mathbb{R}}) = \omega$.

It is also proved in their paper that $\omega_2 \leq sc(\{f : f \text{ is Borel}\}) \leq (2^\omega)^+$, therefore if we assume the Continuum Hypothesis then $sc(\{f : f \text{ is Borel}\}) = \omega_2$. Moreover, they obtained that $sc(\mathcal{B}_\xi) \leq (2^\omega)^+$ for every $2 \leq \xi < \omega_1$, and asked if $\omega_2 \leq sc(\mathcal{B}_\xi)$. They noted that a positive answer to Question 1.2 would yield a positive answer here.

For more information on the connection between ranks, solvability cardinals, systems of difference equations, liftings, and paradoxical decompositions consult [3], [10], [9] and the references therein.

In order to be able to answer the above questions we need to address one more problem. This is slightly unfortunate for us, but Kechris and Louveau have only worked out their theory in compact metric spaces, while it is really essential for our purposes to be able to apply the results in arbitrary Polish spaces.

Question 1.7. *Does the theory of Kechris and Louveau generalize from compact metric spaces to arbitrary Polish spaces?*

Now we describe our results and say a few words about the organization of the paper. First we review the results of Kechris and Louveau in quite some detail in Section 3, and also answer Question 1.7 in the affirmative. Most of the results in this section are not considered to be new, we only have to check that the proofs in [8] work in non-compact Polish spaces as well. A notable exception is Theorem 3.35 stating that the three ranks essentially coincide for bounded Baire class 1 functions, since our highly non-trivial proof for the case of general Polish spaces required completely new ideas. Next, in Section 4, we propose numerous very natural ranks on the Baire class ξ functions that surprisingly turn out to be bounded in ω_1! Then we answer Question 1.1 and Question 1.2 in the affirmative in Section 5. We actually define four ranks on every $\mathcal{B}_\xi$, but two of these turn out to be essentially equal, and the resulting three ranks are very good analogues of the original ranks of Kechris and Louveau. We are actually able to generalize most of their results to these new ranks. As a corollary, we also obtain that $\omega_2 \leq sc(\mathcal{B}_\xi)$, and hence if we assume the Continuum Hypothesis then $sc(\mathcal{B}_\xi) = \omega_2$ for every $2 \leq \xi < \omega_1$.

In Section 6 we prove that if a rank has certain natural properties then it coincides with α, β and γ on the bounded Baire class 1 functions. We also indicate how one could generalise this to the bounded Baire class ξ case.

Finally, we collect the open questions in Section 8.

2. Preliminaries

Most of the following notations and facts can be found in [7].

Throughout the paper, let (X, τ) be an uncountable *Polish* space, that is, a separable and completely metrizable topological space. We denote a compatible, complete metric for (X, τ) by d. A *Polish group* is a topological group whose topology is Polish.

$\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Delta}^0_\xi$ stand for the *ξ'th additive, multiplicative and ambiguous classes* of the Borel hierarchy. We say that a set H is *ambiguous* if $H \in \mathbf{\Delta}^0_2$.

If τ' is a topology on X then we denote the family of real valued functions defined on X that are of Baire class ξ with respect to τ' by $\mathcal{B}_\xi(\tau')$. In particular, $\mathcal{B}_\xi = \mathcal{B}_\xi(\tau)$. If Y is another Polish space (whose topology is clear from the context) then we also use the notation $\mathcal{B}_\xi(Y)$ for the family of Baire class ξ functions defined on Y. Similarly, $\mathbf{\Sigma}^0_\xi(\tau')$ and $\mathbf{\Sigma}^0_\xi(Y)$ are both the set of $\mathbf{\Sigma}^0_\xi$ subsets, with respect to τ', and in Y, respectively. We use the analogous notations for all the other pointclasses.

If Y is a Polish space then a subset $P \subseteq Y$ is *perfect* if it is closed and has no isolated points. A non-empty perfect subset of a Polish space with the subspace topology is an uncountable Polish space.

For a real valued function f on X and a real number c, we let $\{f < c\} = \{x \in X : f(x) < c\}$. We use the notations $\{f > c\}$, $\{f \le c\}$, $\{f \ge c\}$ and $\{f \ne c\}$ analogously.

It is well-known that a function is of Baire class ξ iff the inverse image of every open set is in $\mathbf{\Sigma}^0_{\xi+1}$ iff $\{f < c\}$ and $\{f > c\}$ are in $\mathbf{\Sigma}^0_{\xi+1}$ for every $c \in \mathbb{R}$. Moreover, the family of Baire class ξ functions is closed under uniform limits.

For a set H we denote the characteristic function, closure and complement of H by χ_H, $\overline{H}$, and H^c, respectively. For a set $H \subseteq X \times Y$ and an element $x \in X$ we denote the x-section of H by $H^x = \{y \in Y : (x, y) \in H\}$.

If $\mathcal{H}$ is a family of sets then

$$\mathcal{H}_\sigma = \left\{ \bigcup_{n\in\mathbb{N}} H_n : H_n \in \mathcal{H} \right\} \text{ and } \mathcal{H}_\delta = \left\{ \bigcap_{n\in\mathbb{N}} H_n : H_n \in \mathcal{H} \right\}.$$

For $\theta, \theta' < \omega_1$ we use the relation $\theta \lesssim \theta'$ if $\theta' \le \omega^\eta \implies \theta \le \omega^\eta$ for every $1 \le \eta < \omega_1$ (we use ordinal exponentiation here). Note that $\theta \le \theta'$ implies $\theta \lesssim \theta'$ and $\theta \lesssim \theta'$, $\theta' > 0$ implies $\theta \le \theta' \cdot \omega$. We write $\theta \approx \theta'$ if $\theta \lesssim \theta'$ and $\theta' \lesssim \theta$. Then $\approx$ is an equivalence relation. For every ordinal θ we have $2\theta < \theta + \omega$, and since ω^η is a limit ordinal for every $\eta \ge 1$ we obtain that $2\theta \approx \theta$ for every ordinal θ.

A rank $\rho : \mathcal{B}_\xi \to \omega_1$ is called *additive* if $\rho(f + g) \le \max\{\rho(f), \rho(g)\}$ for every $f, g \in \mathcal{B}_\xi$. It is called *linear* if it is additive and $\rho(cf) = \rho(f)$ for every $f \in \mathcal{B}_\xi$ and $c \in \mathbb{R} \setminus \{0\}$. If X is a Polish group then the left and right translation operators are defined as $L_{x_0}(x) = x_0 \cdot x$ $(x \in X)$ and $R_{x_0}(x) = x \cdot x_0$ $(x \in X)$. A rank $\rho : \mathcal{B}_\xi \to \omega_1$ is called *translation-invariant* if $\rho(f \circ L_{x_0}) = \rho(f \circ R_{x_0}) = \rho(f)$ for every $f \in \mathcal{B}_\xi$ and $x_0 \in X$. We say that it is *essentially additive*, *essentially linear*, and *essentially translation-invariant* if the corresponding inequalities and equations hold with $\lesssim$

and $\approx$. Moreover, ρ is additive, essentially additive etc. *for bounded functions*, if the corresponding relations hold whenever f and g are bounded.

Let $(F_\eta)_{\eta<\lambda}$ be a (not necessarily strictly) decreasing sequence of sets. Let us assume that $F_0 = X$ and that the sequence is *continuous*, that is, $F_\eta = \bigcap_{\theta<\eta} F_\theta$ for every limit η. We also use the convention that $F_\eta = \emptyset$ if $\eta \geq \lambda$. We say that a set H is the *transfinite difference of* $(F_\eta)_{\eta<\lambda}$ if $H = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1})$. It is well-known that a set is in $\mathbf{\Delta}^0_{\xi+1}$ iff it is a transfinite difference of $\mathbf{\Pi}^0_\xi$ sets see e.g. [7, 22.27]. We have to point out here that the monograph [7] does *not* assume that the decreasing sequences are continuous, but when proving that every set in $\mathbf{\Delta}^0_{\xi+1}$ has a representation as a transfinite difference they actually construct continuous sequences, hence this issue causes no difficulty here.

The set of sequences of length k whose terms are elements of the set $\{0,\dots,n-1\}$ is denoted by n^k. For $s \in n^k$ we denote the i-th term of s by $s(i)$. If $l \in \{0,\dots,n-1\}$ then $s^\wedge l$ denotes the sequence in n^{k+1} whose first k terms agree with those of s and whose $k+1$st term is l.

3. Ranks on the Baire class 1 functions without compactness

In this section we summarize some results concerning ranks on the Baire class 1 functions, following the work of Kechris and Louveau. We do not consider the results in this section as original, we basically just carefully check that the results of Kechris and Louveau hold without the assumption of compactness of X. This is inevitable, since they assumed compactness throughout their paper but we will need these results in Section 5 for arbitrary Polish spaces.

A notable exception is Theorem 3.35 stating that the three ranks essentially coincide for bounded Baire class 1 functions. Since our highly non-trivial proof for the case of general Polish spaces required completely new ideas, we consider this result as original in the non-compact case.

The definitions of the ranks will use the notion of a *derivative operation.*

Definition 3.1. A *derivative* on the closed subsets of X is a map $D : \mathbf{\Pi}^0_1(X) \to \mathbf{\Pi}^0_1(X)$ such that $D(A) \subseteq A$ and $A \subseteq B \Rightarrow D(A) \subseteq D(B)$ for every $A, B \in \mathbf{\Pi}^0_1(X)$.

Definition 3.2. For a derivative D we define the *iterated derivatives* of the closed set F as follows:

$$\begin{aligned}
D^0(F) &= F,\\
D^{\eta+1}(F) &= D(D^\eta(F)),\\
D^\eta(F) &= \bigcap_{\theta<\eta} D^\theta(F) \text{ if } \eta \text{ is a limit.}
\end{aligned}$$

Definition 3.3. Let D be a derivative. The *rank* of D is the smallest ordinal η, such that $D^\eta(X) = \emptyset$, if such ordinal exists, ω_1 otherwise. We denote the rank of D by $\mathrm{rk}(D)$.

Remark 3.4. In all our applications D satisfies $D(F) \subsetneqq F$ for every non-empty closed set F, and since in a Polish space there is no strictly decreasing sequence of closed sets of length ω_1 (see e.g. [7, 6.9]), the rank of a derivative is always a countable ordinal.

Proposition 3.5. *If the derivatives D_1 and D_2 satisfy $D_1(F) \subseteq D_2(F)$ for every closed subset $F \subseteq X$ then* $\mathrm{rk}(D_1) \leq \mathrm{rk}(D_2)$.

Proof. It is enough to prove that $D_1^\eta(X) \subseteq D_2^\eta(X)$ for every ordinal η. We prove this by transfinite induction on η. For $\eta = 0$ this is obvious, since $D_1^0(X) = D_2^0(X) = X$. Now suppose this holds for η and we prove it for $\eta + 1$. Since $D_1^\eta(X) \subseteq D_2^\eta(X)$ and D_1 is a derivative, we have $D_1(D_1^\eta(X)) \subseteq D_1(D_2^\eta(X))$. Using this observation and the condition of the proposition for the closed set $D_2^\eta(X)$, we have $D_1^{\eta+1}(X) = D_1(D_1^\eta(X)) \subseteq D_1(D_2^\eta(X)) \subseteq D_2(D_2^\eta(X)) = D_2^{\eta+1}(X)$.

For limit η the claim is an easy consequence of the continuity of the sequences, hence the proof is complete. □

Proposition 3.6. *Let $n \geq 1$ and let D, $D_0, \dots, D_{n-1}$ be derivative operations on the closed subsets of X. Suppose that they satisfy the following conditions for arbitrary closed sets F and F':*

$$D(F) \subseteq \bigcup_{k=0}^{n-1} D_k(F), \tag{3.1}$$

$$D(F \cup F') \subseteq D(F) \cup D(F'). \tag{3.2}$$

Then for these derivatives

$$\mathrm{rk}(D) \lesssim \max_{k<n} \mathrm{rk}(D_k). \tag{3.3}$$

Proof. We will prove by induction on η that

$$D^{\omega^\eta}(F) \subseteq \bigcup_{k=0}^{n-1} D_k^{\omega^\eta}(F) \tag{3.4}$$

for every closed set F. It is easy to see that proving (3.4) is enough, since if η is an ordinal satisfying $\mathrm{rk}(D_k) \leq \omega^\eta$ for every $k < n$ then we have $\mathrm{rk}(D) \leq \omega^\eta$.

Now we prove (3.4). The case $\eta = 0$ is exactly (3.1). For limit η the statement is obvious, since the sequences are decreasing and continuous. Hence, it remains to prove (3.4) for $\eta + 1$ if it holds for η. For this it is enough to show that for every $m \in \omega$

$$D^{\omega^\eta \cdot m \cdot n}(F) \subseteq \bigcup_{k=0}^{n-1} D_k^{\omega^\eta \cdot m}(F), \tag{3.5}$$

indeed,

$$D^{\omega^{\eta+1}}(F) = \bigcap_{m \in \omega} D^{\omega^\eta \cdot m \cdot n}(F) \subseteq \bigcap_{m \in \omega} \left(\bigcup_{k=0}^{n-1} D_k^{\omega^\eta \cdot m}(F) \right),$$

hence $x \in D^{\omega^{\eta+1}}(F)$ implies that without loss of generality $x \in D_0^{\omega^\eta \cdot m}(F)$ for infinitely many m, but the sequence $D_0^{\omega^\eta \cdot m}(F)$ is decreasing, hence $x \in \bigcap_{m \in \omega} D_0^{\omega^\eta \cdot m}(F) = D_0^{\omega^{\eta+1}}(F)$.

Now we prove (3.5). Let $F_\emptyset = F$, and for $m \in \mathbb{N}$, $s \in n^m$ and $k < n$ let

$$F_{s^\wedge k} = D_k^{\omega^\eta}(F_s).$$

It is enough that for $m \geq 1$

$$D^{\omega^\eta \cdot m}(F) \subseteq \bigcup_{s \in n^m} F_s, \tag{3.6}$$

since it is easy to see that

$$\bigcup_{s \in n^{m \cdot n}} F_s \subseteq \bigcup_{k=0}^{n-1} \bigcup \{F_s : s \in n^{m \cdot n} \text{ and } |\{i : s(i) = k\}| \geq m\},$$

yielding (3.5), as

$$\bigcup \{F_s : s \in n^{m \cdot n} \text{ and } |\{i : s(i) = k\}| \geq m\} \subseteq D_k^{\omega^\eta \cdot m}(F).$$

It remains to prove (3.6) by induction on m. For $m = 1$, this is only the induction hypothesis of (3.4) for η. By supposing (3.6) for m, we have

$$D^{\omega^\eta \cdot (m+1)}(F) = D^{\omega^\eta}\left(D^{\omega^\eta \cdot m}(F)\right) \subseteq D^{\omega^\eta}\left(\bigcup_{s \in n^m} F_s\right) \subseteq$$
$$\subseteq \bigcup_{s \in n^m} D^{\omega^\eta}(F_s) \subseteq \bigcup_{s \in n^{m+1}} F_s,$$

where we used (3.2) ω^η many times for the second containment, and for the last one we used the induction hypothesis, that is (3.4) for η. This finishes the proof. $\square$

3.1. **The separation rank.** This rank was first introduced by Bourgain [2].

Definition 3.7. Let A and B be two subsets of X. We associate a derivative with them by

$$D_{A,B}(F) = \overline{F \cap A} \cap \overline{F \cap B}. \tag{3.7}$$

It is easy to see that $D_{A,B}(F)$ is closed, $D_{A,B}(F) \subseteq F$ and $D_{A,B}(F) \subseteq D_{A,B}(F')$ for every pair of sets A and B and every pair of closed sets $F \subseteq F'$, hence $D_{A,B}$ is a derivative. We use the notation $\alpha(A, B) = \mathrm{rk}(D_{A,B})$.

Definition 3.8. The *separation rank* of a Baire class 1 function f is defined as

$$\alpha(f) = \sup_{\substack{p<q \\ p,q \in \mathbb{Q}}} \alpha(\{f \leq p\}, \{f \geq q\}). \tag{3.8}$$

Remark 3.9. Actually,

$$\alpha(f) = \sup_{\substack{x<y \\ x,y \in \mathbb{R}}} \alpha(\{f \leq x\}, \{f \geq y\}),$$

since if $x < p < q < y$ then $\alpha(\{f \leq x\}, \{f \geq y\}) \leq \alpha(\{f \leq p\}, \{f \geq q\})$, since any set $H \in \mathbf{\Delta}_2^0(X)$ separating the level sets $\{f \leq p\}$ and $\{f \geq q\}$ also separates $\{f \leq x\}$ and $\{f \geq y\}$.

Proposition 3.10. *If f is a Baire class 1 function then $\alpha(f) < \omega_1$.*

Proof. From the definition of the rank and Remark 3.4 it is enough to prove that for any pair of rational numbers $p < q$ and non-empty closed set $F \subseteq X$, $D_{A,B}(F) \subsetneq F$, where $A = \{f \leq p\}$ and $B = \{f \geq q\}$. Since f is of Baire class 1, it has a point of continuity restricted to F, hence A and B cannot be both dense in F. Consequently, $D_{A,B}(F) = \overline{F \cap A} \cap \overline{F \cap B} \subsetneq F$, proving the proposition. $\square$

Next we prove that $\alpha(A,B) < \omega_1$ iff A and B can be separated by a transfinite difference of closed sets.

Definition 3.11. If the sets A and B can be separated by a transfinite difference of closed sets then let $\alpha_1(A,B)$ denote the length of the shortest such sequence, otherwise let $\alpha_1(A,B) = \omega_1$. We define the *modified separation rank* of a Baire class 1 function f as

$$\alpha_1(f) = \sup_{\substack{p<q \\ p,q\in\mathbb{Q}}} \alpha_1(\{f \leq p\}, \{f \geq q\}). \tag{3.9}$$

Proposition 3.12. *Let A and B two subsets of X. Then*

$$\alpha(A,B) \leq \alpha_1(A,B) \leq 2\alpha(A,B), \text{ hence } \alpha(A,B) \approx \alpha_1(A,B).$$

Proof. For the first inequality we can assume that $\alpha_1(A,B) < \omega_1$, so A and B can be separated by a transfinite difference of closed sets. Let $(F_\eta)_{\eta<\lambda}$ be such a sequence, where $\lambda = \alpha_1(A,B)$. Now we have

$$A \subseteq \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1}) \subseteq B^c.$$

It is enough to prove that $D^\eta_{A,B}(X) \subseteq F_\eta$ for every η. We prove this by induction. For $\eta = 0$ this is obvious, since $D^0_{A,B}(X) = F_0 = X$.

Now suppose that $D^\eta_{A,B}(X) \subseteq F_\eta$. We show that $D^{\eta+1}_{A,B}(X) = \overline{D^\eta_{A,B}(X) \cap A} \cap \overline{D^\eta_{A,B}(X) \cap B} \subseteq F_{\eta+1}$. If η is even then

$$D^\eta_{A,B}(X) \setminus F_{\eta+1} \subseteq F_\eta \setminus F_{\eta+1} \subseteq B^c,$$

hence $D^\eta_{A,B}(X) \cap B \subseteq F_{\eta+1}$. Since $F_{\eta+1}$ is closed, we obtain $\overline{D^\eta_{A,B}(X) \cap B} \subseteq F_{\eta+1}$, hence $D^{\eta+1}_{A,B} \subseteq F_{\eta+1}$. If η is odd then $F_\eta \setminus F_{\eta+1}$ is disjoint from $\bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1})$, hence $F_\eta \setminus F_{\eta+1} \subseteq A^c$, and an argument analogous to the above one yields $\overline{D^\eta_{A,B}(X) \cap A} \subseteq F_{\eta+1}$, hence $D^{\eta+1}_{A,B} \subseteq F_{\eta+1}$.

If η is limit and $D^\theta_{A,B}(X) \subseteq F_\theta$ for every $\theta < \eta$ then $D^\eta_{A,B}(X) \subseteq F_\eta$ because the sequences $D^\eta_{A,B}(X)$ and F_η are continuous.

For the second inequality we suppose that $\alpha(A,B) < \omega_1$, that is, the sequence $D^\eta_{A,B}(X)$ terminates at the empty set at some countable ordinal. Let

$$F_{2\eta} = D^\eta_{A,B}(X), \quad F_{2\eta+1} = \overline{D^\eta_{A,B}(X) \cap B}.$$

Clearly, $F_0 = X$ and $F_{2\eta} \supseteq F_{2\eta+1}$ for every η. It is easily seen from the definition of $D^{\eta+1}_{A,B}(X)$ that $F_{2\eta+1} \supseteq F_{2\eta+2}$ for every η. Moreover, the sequence $F_{2\eta} = D^\eta_{A,B}(X)$ is continuous. This implies that the sequence formed by the F_η's is decreasing and continuous.

Now we show that the transfinite difference of this sequence separates A and B.

Every ring of the form $F_{2\eta} \setminus F_{2\eta+1}$ is disjoint from B, so we only need to prove that A is contained in the union of these rings. We show that A is disjoint from the complement of this union by proving that

$$(F_{2\eta+1} \setminus F_{2\eta+2}) \cap A = \left(\overline{D^{\eta}_{A,B}(X) \cap B} \setminus D^{\eta+1}_{A,B}(X)\right) \cap A = \emptyset$$

for every η. From the definition of the derivative, $D^{\eta+1}_{A,B}(X) = \overline{D^{\eta}_{A,B}(X) \cap A} \cap \overline{D^{\eta}_{A,B}(X) \cap B}$. Using that $D^{\eta}_{A,B}(X)$ is closed, for a point $x \in A \cap \overline{D^{\eta}_{A,B}(X) \cap B}$ we have $x \in \overline{D^{\eta}_{A,B}(X) \cap A}$, hence $x \in D^{\eta+1}_{A,B}(X)$. □

Remark 3.13. It is claimed in [8] that if X is compact and $\alpha(A,B) = \lambda + n$ with λ limit and $0 < n \in \omega$ then $\alpha_1(A,B)$ is either $\lambda + 2n$ or $\lambda + 2n - 1$. However, this does not seem to be true. For a counterexample, let X be the $2n+1$-dimensional cube in $\mathbb{R}^{2n+1}$. Let $A = (F_0 \setminus F_1) \cup (F_2 \setminus F_3) \cup \dots \cup (F_{2n} \setminus F_{2n+1})$, where F_i is a $(2n+1-i)$-dimensional face of X, and $F_{i+1} \subseteq F_i$ for $i \leq 2n$. Let $B = X \setminus A$. The definition of A shows that $\alpha_1(A,B) \leq 2n+2$.

Now $D^0_{A,B}(X) = X = F_0$, and by induction, $D^i_{A,B}(X) = F_i$ for $0 \leq i \leq 2n+1$, since $D^i_{A,B}(X) = D(D^{i-1}_{A,B}(X)) = D_{A,B}(F_{i-1}) = \overline{F_{i-1} \cap A} \cap \overline{F_{i-1} \cap B} = F_i$. Now we have $D^{2n+2}_{A,B}(X) = D_{A,B}(D^{2n+1}_{A,B}(X)) = D_{A,B}(F_{2n+1}) = \emptyset$, proving that in this case $\alpha(A,B) = 2n+2$. Using Proposition 3.12 this shows that $\alpha_1(A,B) = \alpha(A,B) = 2n+2$.

We leave the proof of the following corollary to the reader.

Corollary 3.14. *If f is a Baire class 1 function then*

$$\alpha(f) \leq \alpha_1(f) \leq 2\alpha(f), \textit{ hence } \alpha(f) \approx \alpha_1(f).$$

Corollary 3.15. *If f is a Baire class 1 function then $\alpha_1(f) < \omega_1$.*

Proof. It is an easy consequence of the previous corollary and Proposition 3.10. □

3.2. **The oscillation rank.** This rank was investigated by numerous authors, see e.g. [6].

First, we define the oscillation of a function, then turn to the oscillation rank.

Definition 3.16. The *oscillation* of a function $f : X \to \mathbb{R}$ at a point $x \in X$ restricted to a closed set $F \subseteq X$ is

$$\omega(f,x,F) = \inf\left\{ \sup_{x_1,x_2 \in U \cap F} |f(x_1) - f(x_2)| : U \text{ open, } x \in U \right\}. \tag{3.10}$$

Definition 3.17. For each $\varepsilon > 0$ consider the derivative defined by

$$D_{f,\varepsilon}(F) = \{x \in F : \omega(f,x,F) \geq \varepsilon\}. \tag{3.11}$$

It is obvious that $D_{f,\varepsilon}(F)$ is closed, $D_{f,\varepsilon}(F) \subseteq F$ and $D_{f,\varepsilon}(F) \subseteq D_{f,\varepsilon}(F')$ for every function $f : X \to \mathbb{R}$, every $\varepsilon > 0$ and every pair of closed sets $F \subseteq F'$, hence $D_{f,\varepsilon}$ is a derivative. Let us denote the rank of $D_{f,\varepsilon}$ by $\beta(f,\varepsilon)$.

Definition 3.18. The *oscillation rank* of a function f is

$$\beta(f) = \sup_{\varepsilon>0} \beta(f,\varepsilon). \tag{3.12}$$

Proposition 3.19. *If f is a Baire class 1 function then $\beta(f) < \omega_1$.*

Proof. Using Remark 3.4, it is enough to prove $D_{f,\varepsilon}(F) \subsetneq F$ for every $\varepsilon > 0$ and every non-empty closed set $F \subseteq X$. And this is easy, since f restricted to F is continuous at a point $x \in F$, and thus $x \notin D_{f,\varepsilon}(F)$, hence $D_{f,\varepsilon}(F) \subsetneq F$. $\square$

3.3. **The convergence rank.** Now we turn to the convergence rank following Zalcwasser [11] and Gillespie and Hurwitz [4].

Definition 3.20. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real valued continuous functions on X. The *oscillation* of this sequence at a point x restricted to a closed set $F \subseteq X$ is

$$\omega((f_n)_{n\in\mathbb{N}}, x, F) = \inf_{\substack{x\in U\\ U \text{ open}}} \inf_{N\in\mathbb{N}} \sup \left\{ |f_m(y) - f_n(y)| : n,m \geq N,\ y \in U \cap F \right\}. \tag{3.13}$$

Definition 3.21. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of real valued continuous functions, and for each $\varepsilon > 0$, define a derivative as

$$D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) = \left\{ x \in F : \omega((f_n)_{n\in\mathbb{N}}, x, F) \geq \varepsilon \right\}. \tag{3.14}$$

It is easy to see that $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F)$ is closed, $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) \subseteq F$ and $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) \subseteq D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F')$ for every sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$, every $\varepsilon > 0$ and every pair of closed sets $F \subseteq F'$, hence $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}$ is a derivative. Let us denote the rank of $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}$ by $\gamma((f_n)_{n\in\mathbb{N}}, \varepsilon)$.

Definition 3.22. For a Baire class 1 function f let the *convergence rank* of f be defined by

$$\gamma(f) = \min \left\{ \sup_{\varepsilon>0} \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) : \forall n\ f_n \text{ is continuous and } f_n \to f \text{ pointwise} \right\}. \tag{3.15}$$

Proposition 3.23. *If f is a Baire class 1 function then $\gamma(f) < \omega_1$.*

Proof. It suffices to show that $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) \subsetneq F$ for every $\varepsilon > 0$, every non-empty closed set $F \subseteq X$ and every sequence of pointwise convergent continuous functions $(f_n)_{n\in\mathbb{N}}$. Suppose the contrary, then for every N the set $G_N = \{x \in F : \exists n, m \geq N\ |f_n(x) - f_m(x)| > \frac{\varepsilon}{2}\}$ is dense in F. It is also open in F, hence by the Baire category theorem there is a point $x \in F$ such that $x \in G_N$ for every $N \in \mathbb{N}$, hence the sequence $(f_n)_{n\in\mathbb{N}}$ does not converge at x, contradicting our assumption. $\square$

3.4. **Properties of the ranks.**

Theorem 3.24. *If f is a Baire class 1 function then $\alpha(f) \leq \beta(f) \leq \gamma(f)$.*

Proof. For the first inequality, it is enough to prove that for every $p, q \in \mathbb{Q}$, $p < q$ we can find $\varepsilon > 0$ such that $\alpha(\{f \leq p\}, \{f \geq q\}) \leq \beta(f, \varepsilon)$. Let $A = \{f \leq p\}$, $B = \{f \geq q\}$ and $\varepsilon = p - q$. Using Proposition 3.5 it suffices to show that $D_{A,B}(F) \subseteq D_{f,\varepsilon}(F)$ for every $F \in \mathbf{\Pi}^0_1(X)$. If $x \in F \setminus D_{f,\varepsilon}(F)$ then x has a neighborhood U such that $\sup_{x_1,x_2 \in U\cap F} |f(x_1) - f(x_2)| < \varepsilon = p - q$, hence U cannot intersect both A and B. So $x \notin D_{A,B}(F)$, proving the first inequality.

For the second inequality, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions converging pointwise to a function f. It is enough to show that $\beta(f,\varepsilon) \leq \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon/3)$. Similarly to the first paragraph we show that $D_{f,\varepsilon}(F) \subseteq D_{(f_n)_{n\in\mathbb{N}},\varepsilon/3}(F)$ for every $F \in \mathbf{\Pi}^0_1(X)$. It is enough to show that if $x \in F \setminus D_{(f_n)_{n\in\mathbb{N}},\varepsilon/3}(F)$ then $x \notin D_{f,\varepsilon}(F)$. For such an x there is a neighborhood U of x and an $N \in \mathbb{N}$ such that for all $n, m \geq N$ and $x' \in F \cap U$, $|f_n(x') - f_m(x')| < \varepsilon/3$. Letting $m \to \infty$ we get $|f_n(x') - f(x')| \leq \varepsilon/3$ for all $n \geq N$ and $x' \in F \cap U$. Let $V \subseteq U$ be a neighborhood of x for which $\sup_V f_N - \inf_V f_N < \varepsilon/6$. Now for every $x', x'' \in V \cap F$ we have

$$|f(x') - f(x'')| \leq |f_N(x') - f_N(x'')| + 2\frac{\varepsilon}{3} < \frac{5}{6}\varepsilon < \varepsilon,$$

showing that $x \notin D_{f,\varepsilon}(F)$. □

Proposition 3.25. *If X is a Polish group then the ranks α, β and γ are translation invariant.*

Proof. Note first that for a Baire class 1 function f and $x_0 \in X$ the functions $f \circ L_{x_0}$ and $f \circ R_{x_0}$ are also of Baire class 1. Since the topology of a topological group is translation invariant, and the the definitions of the ranks depend only on the topology of the space, the proposition easily follows. □

Theorem 3.26. *The ranks are unbounded in ω_1, actually unbounded already on the characteristic functions.*

We postpone the proof, since later we will prove the more general Theorem 4.3.

Proposition 3.27. *If f is continuous then $\alpha(f) = \beta(f) = \gamma(f) = 1$.*

Proof. In order to prove $\alpha(f) = 1$, consider the derivative $D_{\{f\leq p\},\{f\geq q\}}$, where $p < q$ is a pair of rational numbers. Since the level sets $\{f \leq p\}$ and $\{f \geq q\}$ are disjoint closed sets, $D_{\{f\leq p\},\{f\geq q\}}(X) = \emptyset$.

For $\beta(f) = 1$, note that a continuous function f has oscillation 0 at every point restricted to every set, hence $D_{f,\varepsilon}(X) = \emptyset$ for every $\varepsilon > 0$.

And finally for $\gamma(f) = 1$ consider the sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$, for which $f_n = f$ for every $n \in \mathbb{N}$. It is easy to see that $\omega((f_n)_{n\in\mathbb{N}}, x, F) = 0$ for every point $x \in X$ and every closed set $F \subseteq X$. Now we have that $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(X) = \emptyset$ for every $\varepsilon > 0$, hence $\gamma(f) = 1$. □

Theorem 3.28. *If f is a Baire class 1 function and $F \subseteq X$ is closed then $\alpha(f \cdot \chi_F) \leq 1 + \alpha(f)$, $\beta(f \cdot \chi_F) \leq 1 + \beta(f)$ and $\gamma(f \cdot \chi_F) \leq 1 + \gamma(f)$.*

Proof. First we prove the statement for the ranks α and β. Let D be a derivative either of the form $D_{A,B}$ or of the form $D_{f,\varepsilon}$ where $A = \{f \leq p\}$ and $B = \{f \geq q\}$ for a pair of rational numbers $p < q$ and $\varepsilon > 0$. Let $\overline{D}$ be the corresponding derivative for the function $f \cdot \chi_F$, i.e. $\overline{D} = D_{A',B'}$ or $\overline{D} = D_{f\cdot\chi_F,\varepsilon}$, where $A' = \{f \cdot \chi_F \leq p\}$ and $B' = \{f \cdot \chi_F \geq q\}$.

Since the function $f \cdot \chi_F$ is constant 0 on the open set $X \setminus F$, it is easy to check that in both cases $\overline{D}(X) \subseteq F$. And since the functions f and $f \cdot \chi_F$ agree on F, we have by transfinite induction that $\overline{D}^{1+\eta}(X) \subseteq D^{\eta}(X)$ for every countable ordinal η, implying that $\alpha(f \cdot \chi_F) \leq 1 + \alpha(f)$ and also $\beta(f \cdot \chi_F) \leq 1 + \beta(f)$.

Now we prove the statement for γ. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions converging pointwise to f with $\sup_{\varepsilon>0} \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) = \gamma(f)$. Let $g_n(x) = 1 - \min\{1, n \cdot d(x, F)\}$ and set $f'_n(x) = f_n(x) \cdot g_n(x)$. It is easy to check that for every n the function f'_n is continuous and $f'_n \to f \cdot \chi_F$ pointwise. For every $x \in X \setminus F$ there is a neighborhood of x such that for large enough n the function f'_n is 0 on this neighborhood, hence $D_{(f'_n)_{n\in\mathbb{N}},\varepsilon}(X) \subseteq F$ for every $\varepsilon > 0$. From this point on the proof is similar to the previous cases, since the sequences of functions $(f_n)_{n\in\mathbb{N}}$ and $(f'_n)_{n\in\mathbb{N}}$ agree on F, hence, by transfinite induction $D^{1+\eta}_{(f'_n)_{n\in\mathbb{N}},\varepsilon}(X) \subseteq D^{\eta}_{(f_n)_{n\in\mathbb{N}},\varepsilon}(X)$ for every $\varepsilon > 0$. From this we have $\gamma((f'_n)_{n\in\mathbb{N}}, \varepsilon) \leq 1 + \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon)$ for every $\varepsilon > 0$, hence $\gamma(f \cdot \chi_F) \leq 1 + \gamma(f)$. Thus the proof of the theorem is complete. $\square$

Theorem 3.29. *The ranks β and γ are essentially linear.*

Proof. It is easy to see that $\beta(cf) = \beta(f)$ and $\gamma(cf) = \gamma(f)$ for every $c \in \mathbb{R} \setminus \{0\}$, hence it suffices to show that β and γ are essentially additive.

First we consider a modification of the definition of the rank β as follows. Let β_0 be the rank obtained by simply replacing $\sup_{x_1,x_2\in U\cap F} |f(x_1) - f(x_2)|$ in (3.10) by $\sup_{x_1\in U\cap F} |f(x) - f(x_1)|$ in the definition of β. Clearly, $\beta_0(f, \varepsilon) \leq \beta(f, \varepsilon) \leq \beta_0(f, \varepsilon/2)$, hence actually $\beta_0 = \beta$. Therefore it is sufficient to prove the theorem for β_0.

To prove the theorem for β_0, let $D_0 = D_{f,\varepsilon/2}$, $D_1 = D_{g,\varepsilon/2}$ and $D = D_{f+g,\varepsilon}$ (we use here the derivatives defining β_0). We show that the conditions of Proposition 3.6 hold for these derivatives.

For condition (3.1), let $x \in D_{f+g,\varepsilon}(F)$. Since $\omega(f + g, x, F) \geq \varepsilon$, we have $\omega(f, x, F)$ or $\omega(g, x, F) \geq \varepsilon/2$, hence $x \in D_{f,\varepsilon/2}(F) \cup D_{g,\varepsilon/2}(F)$.

Condition (3.2) is similar, let $x \in (F \cup F') \setminus (D_{f+g,\varepsilon}(F) \cup D_{f+g,\varepsilon}(F'))$. Since $x \notin D_{f+g,\varepsilon}(F)$, there is a neighborhood U of x with $|(f+g)(x)-(f+g)(x')| < \varepsilon' < \varepsilon$ for $x' \in U \cap F$. And similarly, there is a neighborhood U' with $|(f + g)(x) - (f + g)(x')| < \varepsilon'' < \varepsilon$ for $x' \in U' \cap F'$. Now the neighborhood $U \cap U'$ shows that $\omega(f + g, x, F \cup F') < \varepsilon$, proving that $x \notin D_{f+g,\varepsilon}(F \cup F')$.

The proposition yields that $\beta_0(f+g, \varepsilon) \lesssim \max\{\beta_0(f, \varepsilon/2), \beta_0(g, \varepsilon/2)\}$, hence $\beta_0(f+g) \lesssim \max\{\beta_0(f), \beta_0(g)\}$. This proves the statement for β_0, hence for β.

For γ, we do the same, prove the conditions of the proposition for $D_0 = D_{(f_n)_{n\in\mathbb{N}},\varepsilon/2}$, $D_1 = D_{(g_n)_{n\in\mathbb{N}},\varepsilon/2}$ and $D = D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}$, and use the conclusion of the proposition to finish the proof.

For condition (3.1), let $x \in F \setminus \big(D_{(f_n)_{n\in\mathbb{N}},\varepsilon/2}(F) \cup D_{(g_n)_{n\in\mathbb{N}},\varepsilon}(F)\big)$. Now we can choose a common open set $x \in U$ and a common $N \in \mathbb{N}$ such that for all $n, m \geq N$ and $y \in U \cap F$ we have $|f_n(y) - f_m(y)| \leq \varepsilon' < \varepsilon/2$ and $|g_n(y) - g_m(y)| \leq \varepsilon' < \varepsilon/2$ (again, with a common $\varepsilon' < \varepsilon/2$). But from this we have $|(f_n + g_n)(y) - (f_m + g_m)(y)| \leq 2\varepsilon' < \varepsilon$ for all $n, m \geq N$ and $y \in U \cap F$, so $x \notin D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F)$, yielding (3.1).

For (3.2) let $x \in (F \cup F') \setminus \big(D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F) \cup D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F')\big)$. For this x we have a neighborhood U of x, $N \in \mathbb{N}$ and $\varepsilon' < \varepsilon$, such that $|(f_n + g_n)(y) - (f_m + g_m)(y)| \leq \varepsilon'$ for every $n, m \geq N$ and $y \in U \cap F$. Similarly, we can find a neighborhood U', $N' \in \mathbb{N}$ and $\varepsilon'' < \varepsilon$, such that $|(f_n + g_n)(y) - (f_m + g_m)(y)| \leq \varepsilon''$ for every $n, m \geq N'$ and $y \in U' \cap F'$. From this, $\omega((f_n + g_n)_{n\in\mathbb{N}}, x, F \cup F') \leq \max\{\varepsilon', \varepsilon''\} < \varepsilon$, hence $x \notin D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F \cup F')$.

Therefore the proof of the theorem is complete. □

Remark 3.30. The analogous result does not hold for the rank α. To see this note first that $\alpha(A, A^c)$ can be arbitrarily large below ω_1 when A ranges over $\mathbf{\Delta}^0_2(X)$. This is a classical fact and we prove a more general result in Corollary 4.4.

First we check that for every $A \in \mathbf{\Delta}^0_2(X)$ the characteristic function χ_A can be written as the difference of two upper semicontinuous (usc) functions. Indeed, let $(K_n)_{n\in\omega}$ and $(L_n)_{n\in\omega}$ be increasing sequences of closed sets with $A = \bigcup_n K_n$ and $A^c = \bigcup_n L_n$, and let

$$f_0 = \begin{cases} 0 & \text{on } K_0 \cup L_0, \\ -n & \text{on } (K_n \cup L_n) \setminus (K_{n-1} \cup L_{n-1}) \text{ for } n \geq 1 \end{cases}$$

and

$$f_1 = \begin{cases} 0 & \text{on } L_0, \\ -1 & \text{on } (K_0 \cup L_1) \setminus L_0, \\ -n & \text{on } (K_{n-1} \cup L_n) \setminus (K_{n-2} \cup L_{n-1}) \text{ for } n \geq 2. \end{cases}$$

Then f_0 and f_1 are usc functions with $\chi_A = f_0 - f_1$.

Now we complete the remark by showing that $\alpha(f) \leq 2$ for every usc function f. For $p < q$ let $A = \{f \leq p\}$ and $B = \{f \geq q\}$. Then B is closed, so $D_{A,B}(X) = \overline{X \cap A} \cap \overline{X \cap B} = \overline{X \cap A} \cap B \subseteq B$. Hence $D^2_{A,B}(X) \subseteq D_{A,B}(B) = \overline{A \cap B} \cap B = \emptyset \cap B = \emptyset$.

Remark 3.31. One can easily deduce from Theorem 3.29 that $\beta(f \cdot g) \lesssim \max\{\beta(f), \beta(g)\}$ whenever f and g are *bounded* Baire class 1 functions, and similarly for γ. However, we do not know if this holds for arbitrary Baire class 1 functions.

Question 3.32. *Are the ranks β and γ* essentially multiplicative *on the Baire class 1 functions, that is, does $\beta(f \cdot g) \lesssim \max\{\beta(f), \beta(g)\}$ and $\gamma(f \cdot g) \lesssim \max\{\gamma(f), \gamma(g)\}$ hold whenever f and g are Baire class 1 functions?*

Proposition 3.33. *If the sequence of Baire class 1 functions f_n converges uniformly to f then $\beta(f) \leq \sup_n \beta(f_n)$.*

Proof. If $|f - f_n| < \varepsilon/3$ then $|\omega(f, x, F) - \omega(f_n, x, F)| \leq \frac{2}{3}\varepsilon$ for every x and F. Therefore $D_{f,\varepsilon}(F) \subseteq D_{f_n,\varepsilon/3}(F)$ for every F, which in turn implies $\beta(f, \varepsilon) \leq \beta(f_n, \varepsilon/3)$, from which the proposition easily follows. □

Proposition 3.34. *If the sequence of Baire class 1 functions f_n converges uniformly to f then $\gamma(f) \lesssim \sup\limits_n \gamma(f_n)$.*

Proof. By taking a subsequence we can suppose that $|f_n(x) - f(x)| \leq \frac{1}{2^n}$ for every $n \in \mathbb{N}$ and every $x \in X$. With $g_n(x) = f_n(x) - f_{n-1}(x)$ we have $|g(x)| \leq \frac{3}{2^n}$, hence $\sum_{n=1}^{\infty} g_n(x)$ is uniformly convergent, and $f(x) = f_0(x) + \sum_{n=1}^{\infty} g_n(x)$. Using Theorem 3.29 we have $\gamma(g_n) \lesssim \max\{\gamma(f_n), \gamma(f_{n-1})\}$, hence $\sup_n \gamma(g_n) \lesssim \sup_n \gamma(f_n)$. It is enough to prove that for $g = \sum_{n=1}^{\infty} g_n$ we have $\gamma(g) \lesssim \sup_n \gamma(g_n)$, since Theorem 3.29 yields $\gamma(f) \lesssim \max\{\gamma(f_0), \gamma(g)\}$.

Now for every $n \in \mathbb{N}$ let $(\varphi_n^k)_{k\in\mathbb{N}}$ be a sequence of continuous functions converging pointwise to g_n with $\sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon) = \gamma(g_n)$. It is easy to see that we can suppose $|\varphi_n^k(x)| \le \frac{3}{2^n}$ for every $n \in \mathbb{N}$ and $k \in \mathbb{N}$, since by replacing $(\varphi_n^k)_{k\in\mathbb{N}}$ with $\left(\max\left(\min\left(\varphi_n^k, \frac{3}{2^n}\right), -\frac{3}{2^n}\right)\right)_{k\in\mathbb{N}}$ we have a sequence of continuous functions satisfying this, and the sequence is still converging pointwise to g_n, while $\gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon)$ is not increased.

Let $\phi_k = \sum_{n=0}^{k} \varphi_n^k$. We show that $(\phi_k)_{k\in\mathbb{N}}$ converges pointwise to g and also that $\gamma(g) \le \sup_{\varepsilon>0} \gamma((\phi_k)_{k\in\mathbb{N}}, \varepsilon) \lesssim \sup_n \sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon) = \sup_n \gamma(g_n)$, which finishes the proof. To prove pointwise convergence, let $\varepsilon > 0$ be arbitrary and fix $K \in \mathbb{N}$ with $\frac{6}{2^K} < \varepsilon$. For $k > K$ we have

$$|\phi_k(x) - g(x)| = \left|\sum_{n=0}^{k} \varphi_n^k(x) - g(x)\right| \le \left|\sum_{n=0}^{K} \varphi_n^k(x) - g(x)\right| + \left|\sum_{n=K+1}^{k} \varphi_n^k(x)\right|,$$

where the first term of the last expression tends to $\left|\sum_{n=0}^{K} g_n(x) - g(x)\right| \le \frac{3}{2^K}$, while the second is at most $\frac{3}{2^K}$. Hence $\limsup_{k\to\infty} |\phi_k(x) - g(x)| \le 2\frac{3}{2^K} < \varepsilon$ for every $\varepsilon > 0$, showing that $\phi_k(x) \to g(x)$.

Now fix an $\varepsilon > 0$ and $K \in \mathbb{N}$ as before, it is enough to show that $\gamma((\phi_k)_{k\in\mathbb{N}}, 3\varepsilon) \lesssim \sup_n \sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon)$.

For any $x \in X$ and $k, l > K$ we have

$$\begin{aligned} |\phi_k(x) - \phi_l(x)| &= \left|\sum_{n=0}^{k} \varphi_n^k(x) - \sum_{n=0}^{l} \varphi_n^l(x)\right| \\ &\le \sum_{n=0}^{K} \left|\varphi_n^k(x) - \varphi_n^l(x)\right| + \left|\sum_{n=K+1}^{k} \varphi_n^k(x)\right| + \left|\sum_{n=K+1}^{l} \varphi_n^l(x)\right|. \end{aligned} \tag{3.16}$$

As before, the sum of the last two terms is at most ε. We want to use Proposition 3.6 for the derivatives $D = D_{(\phi_k)_{k\in\mathbb{N}}, 3\varepsilon}$ and $D_n = D_{(\varphi_n^k)_{k\in\mathbb{N}}, \frac{\varepsilon}{K+1}}$ for $n \le K$. To check condition (3.1), let $x \in F \setminus \bigcup_{n=0}^{K} D_{(\varphi_n^k)_{k\in\mathbb{N}}, \frac{\varepsilon}{K+1}}(F)$. Then we have a neighborhood U of x and an $N \in \mathbb{N}$ such that $\left|\varphi_n^k(y) - \varphi_n^l(y)\right| < \frac{\varepsilon}{K+1}$ for every $n \le K$, every $y \in U \cap F$ and every $k, l \ge N$. This observation and (3.16) yields that $|\phi_k(y) - \phi_l(y)| \le 2\varepsilon$ for every $y \in U \cap F$ and $k, l \ge N$ showing that $x \notin D_{(\phi_k)_{k\in\mathbb{N}}, 3\varepsilon}(F)$.

Condition (3.2) is similar, and it can be seen as in the proof of Theorem 3.29. Now Proposition 3.6 gives

$$\gamma((\phi_k)_{k\in\mathbb{N}}, 3\varepsilon) \lesssim \max_{n\le K} \gamma\left((\varphi_n^k)_{k\in\mathbb{N}}, \frac{\varepsilon}{K+1}\right) \le \sup_n \sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon),$$

completing the proof. □

Theorem 3.35. *If f is a bounded Baire class 1 function then $\alpha(f) \approx \beta(f) \approx \gamma(f)$.*

Proof. Using Theorem 3.24, it is enough to prove that $\gamma(f) \lesssim \alpha(f)$. First, we prove the theorem for characteristic functions.

Lemma 3.36. *Suppose that $A \in \mathbf{\Delta}_2^0$. Then $\gamma(\chi_A) \lesssim \alpha(\chi_A)$.*

Proof. In order to prove this, first we have to produce a sequence of continuous functions converging pointwise to χ_A.

For this let $(F_\eta)_{\eta<\lambda}$ be a continuous transfinite decreasing sequence of closed sets, so that

$$A = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1})$$

and $\lambda \approx \alpha(\chi_A)$ given by Corollary 3.14. We can assume that the last element of the sequence $(F_\eta)_{\eta<\lambda}$ is $\emptyset$, hence every $x \in X$ is contained in a unique set of the form $F_\eta \setminus F_{\eta+1}$.

For each $k \in \omega$ and $\eta < \lambda$ let $f^k_\eta : X \to [0,1]$ be a continuous function so that $f^k_\eta | F_\eta \equiv 1$, and whenever $x \in X$ and $d(x, F_\eta) \geq \frac{1}{k}$ then $f^k_\eta(x) = 0$. Such a function exists by Urysohn's lemma, since the sets F_η and $\{x \in X : d(x, F_\eta) \geq \frac{1}{k}\}$ are disjoint closed sets.

Now let (η_n) be an enumeration of λ in type $\leq \omega$. Let us define

$$f_k = \sum_{\substack{n\leq k \\ \eta_n \text{ even}}} f^k_{\eta_n} - f^k_{\eta_n+1}.$$

Since the functions f_k are finite sums of continuous functions, they are continuous. We claim that $f_k \to \chi_A$ as $k \to \infty$.

To see this, first let $x \in X$ be arbitrary. Then there exists a unique m so that $x \in F_{\eta_m} \setminus F_{\eta_m+1}$. Choose $k \in \omega$ so that $k \geq m$ and $d(x, F_{\eta_m+1}) \geq \frac{1}{k}$.

Then if $x \in A$ then η_m even and

$$f_k(x) = \sum_{\substack{n\leq k \\ \eta_n \text{ even}}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x) =$$

$$= \left(\sum_{\substack{n\leq k \\ \eta_n \text{ even} \\ \eta_n<\eta_m}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x) \right) + \left(\sum_{\substack{n\leq k \\ \eta_n \text{ even} \\ \eta_n>\eta_m}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x) \right) + f^k_{\eta_m}(x) - f^k_{\eta_m+1}(x).$$

The first sum is clearly 0 since $f^k_{\eta_n} \equiv 1$ on F_{η_m} if $\eta_m > \eta_n$. This is also true for the second one, since if $d(x, F_{\eta_n}) \geq \frac{1}{k}$ then $f^k_{\eta_n}(x) = 0$. Finally, $f_{\eta_m}(x) = 1$ and $f_{\eta_m+1}(x) = 0$, so $f_k(x) = 1$.

If $x \notin A$ then η_m is odd and

$$f_k(x) = \sum_{\substack{n\leq k \\ \eta_n \text{ even}}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x) =$$

$$= \sum_{\substack{n\leq k \\ \eta_n \text{ even} \\ \eta_n<\eta_m}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x) + \sum_{\substack{n\leq k \\ \eta_n \text{ even} \\ \eta_n>\eta_m}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x).$$

Now the previous argument gives $f_k(x) = 0$.

So $f_k \to \chi_A$ holds. Next we prove by induction on η that for every $\eta < \lambda$ and every $\varepsilon > 0$ we have

$$D^{\eta}_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X) \subset F_{\eta}.$$

This will clearly complete the proof.

For $\eta = 0$ we have

$$D^{0}_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X) = X = F_0.$$

If η is a limit ordinal, the statement is clear, since the sequence of derivatives as well as $(F_\eta)_{\eta<\lambda}$ are continuous.

Now let $\eta = \theta + 1$ and $D^{\theta}_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X) \subset F_\theta$. For some m we have $\theta = \eta_m$. Let $x \in F_{\eta_m} \setminus F_{\eta_m+1}$. Then it is enough to prove that $x \notin D^{\eta}_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X)$. Let k be so that $d(x, F_{\eta_m+1}) \geq \frac{2}{k}$.

Whenever $d(x,y) < \frac{1}{k}$ and $y \in D^{\theta}_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X)$ then $y \in F_{\eta_m} \setminus F_{\eta_m+1}$. From this, if $l_1, l_2 \geq k$ we have that $f^{l_1}_{\eta}(y) = f^{l_2}_{\eta}(y) = 1$ if $\eta \leq \eta_m$ and $f^{l_1}_{\eta}(y) = f^{l_2}_{\eta}(y) = 0$ if $\eta > \eta_m$. Hence $f_{l_1}(y) - f_{l_2}(y) = 0$.

So we have that the sequence f_k is eventually constant on a relative neighborhood of x in F_{η_m}, therefore $x \notin D^{\eta}_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X)$, which finishes the proof. □

Next we prove that $\gamma(f) \lesssim \alpha(f)$ for every step function f. We still need the following lemma.

Lemma 3.37. *If A and B are ambiguous sets then*

$$\alpha\left(\chi_{A\cap B}\right) \lesssim \max\left\{\alpha\left(\chi_A\right), \alpha\left(\chi_B\right)\right\}.$$

Proof. It is enough to prove this for β since the previous lemma and Theorem 3.24 yields that the ranks essentially agree on characteristic functions. Theorem 3.29 gives $\beta(\chi_A + \chi_B) \lesssim \max\{\beta(\chi_A), \beta(\chi_B)\}$, hence it suffices to prove that $\beta\left(\chi_{A\cap B}\right) \leq \beta(\chi_A + \chi_B)$. But this easily follows, since one can readily check that for every $\varepsilon < 1$ and F we have $D_{\chi_{A\cap B},\varepsilon}(F) \subseteq D_{\chi_A+\chi_B,\varepsilon}(F)$, finishing the proof. □

Now let f be a step function, so $f = \sum_{i=1}^{n} c_i \chi_{A_i}$, where the A_i's are disjoint ambiguous sets covering X, and we can also suppose that the c_i's form a strictly increasing sequence of real numbers.

Lemma 3.38. $\max_i\{\alpha(\chi_{A_i})\} \lesssim \alpha(f)$.

Proof. Let $H_i = \bigcup_{j=1}^{i} A_j$. By the definition of the rank α, for every i we have

$$\alpha(H_i, H_i^c) \leq \alpha(f). \tag{3.17}$$

This shows that $\alpha(\chi_{A_1}) \lesssim \alpha(f)$, and together with the previous lemma, for $i > 1$

$$\begin{aligned}\alpha(\chi_{A_i}) = \alpha(\chi_{H_i\setminus H_{i-1}}) = \alpha(\chi_{H_i\cap H_{i-1}^c}) &\lesssim \max\{\alpha(\chi_{H_i}), \alpha(\chi_{H_{i-1}^c})\}\\ &= \max\{\alpha(H_i, H_i^c), \alpha(H_{i-1}, H_{i-1}^c)\} \leq \alpha(f),\end{aligned}$$

where the last but one inequality follows from the above lemma and the last inequality from (3.17). □

Now we have

$$\gamma(f) \lesssim \max_i \{\gamma(\chi_{A_i})\} \approx \max_i \{\alpha(\chi_{A_i})\} \lesssim \alpha(f),$$

where we used Theorem 3.29, this theorem for characteristic functions and Lemma 3.38, proving the theorem for step functions.

In particular, $\alpha(f) \leq \beta(f) \leq \gamma(f)$ (Theorem 3.24) gives the following corollary.

Corollary 3.39. *If $f = \sum_{i=1}^{n} c_i \chi_{A_i}$, where the A_i's are disjoint ambiguous sets covering X and the c_i's are distinct then*

$$\alpha(f) \approx \max_i \{\alpha(\chi_{A_i})\}$$

and similarly for β and γ.

Now let f be an arbitrary bounded Baire class 1 function.

Lemma 3.40. *There is a sequence f_n of step functions converging uniformly to f, satisfying $\sup_n \alpha(f_n) \lesssim \alpha(f)$.*

Proof. Let $p_{n,k} = k/2^n$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. The level sets $\{f \leq p_{n,k}\}$ and $\{f \geq p_{n,k+1}\}$ are disjoint $\mathbf{\Pi}^0_2$ sets, hence they can be separated by a $H_{n,k} \in \mathbf{\Delta}^0_2(X)$ (see e.g. [7, 22.16]). We can choose $H_{n,k}$ to satisfy $\alpha_1(H_{n,k}, H^c_{n,k}) \leq 2\alpha(f)$ using Proposition 3.12.

Since f is bounded, for fixed n there are only finitely many $k \in \mathbb{Z}$ for which $H_{n,k+1} \setminus H_{n,k} \neq \emptyset$. Set

$$f_n = \sum_{k \in \mathbb{Z}} p_{n,k} \cdot \chi_{H_{n,k+1} \setminus H_{n,k}}.$$

Now for each n, f_n is a step function with $|f - f_n| \leq 2^{n-1}$. Hence $f_n \to f$ uniformly. Since the level sets of a function f_n are of the form $H_{n,k}$ or $H^c_{n,k}$ for some $k \in \mathbb{Z}$, we have $\alpha(f_n) \leq 2\alpha(f)$, proving the lemma. □

Let f_n be a sequence of step functions given by this lemma. Using Proposition 3.34 and this theorem for step functions, we have $\gamma(f) \lesssim \sup_n \gamma(f_n) \lesssim \sup_n \alpha_n(f_n) \lesssim \alpha(f)$, completing the proof. □

We have seen above that α is not essentially additive on the Baire class 1 functions but β and γ are, therefore α cannot essentially coincide with β or γ. However, in view of the above theorem the following question arises.

Question 3.41. *Does $\beta \approx \gamma$ hold for arbitrary Baire class 1 functions?*

Proposition 3.42. *If the sequence of Baire class 1 functions f_n converges uniformly to f then $\alpha(f) \lesssim \sup\limits_n \alpha(f_n)$.*

Proof. If f is bounded (hence without loss of generality the f_n are also bounded) this is an easy consequence of Theorem 3.35 and Proposition 3.33.

For an arbitrary function g let $g' = \arctan \circ g$. It is easy to show that $\alpha(g') = \alpha(g)$ using Remark 3.9.

If the functions f and f_n are given such that $f_n \to f$ uniformly then $f'_n \to f'$ uniformly, and these are bounded functions, so we have $\alpha(f) = \alpha(f') \lesssim \sup_n \alpha(f'_n) = \sup_n \alpha(f_n)$. □

4. Ranks on the Baire class ξ functions exhibiting strange phenomena

4.1. **The separation rank and the linearized separation rank.** The only rank out of the ones discussed above that has straightforward generalization to the Baire class ξ case is the rank α_1. However, this generalization does not answer Question 1.2, since, similarly to the original α_1, it is not linear. After discussing this, we will propose a very natural modification that transforms an arbitrary rank into a linear one, but we well see that this modified rank will be bounded in ω_1 for characteristic functions!

Definition 4.1. Let A and B be disjoint $\mathbf{\Pi}^0_{\xi+1}$ sets. Then they can be separated by a $\mathbf{\Delta}^0_{\xi+1}$ set (see e.g. [7, 22.16]). Since every $\mathbf{\Delta}^0_{\xi+1}$ set is the transfinite difference of $\mathbf{\Pi}^0_{\xi}$ sets, A and B can be separated by the transfinite difference of such a sequence. Let $\alpha_\xi(A, B)$ denote the length of the shortest such sequence.

Definition 4.2. Let f be a Baire class ξ function, and $p < q \in \mathbb{Q}$. Then $\{f \le p\}$ and $\{f \ge q\}$ are disjoint $\mathbf{\Pi}^0_{\xi+1}$ sets. Let the *separation rank* of f be

$$\alpha_\xi(f) = \sup_{\substack{p<q \\ p,q\in\mathbb{Q}}} \alpha_\xi(\{f \le p\}, \{f \ge q\}).$$

Note that this really extends the definition of α_1.

Theorem 4.3. *For every $1 \le \xi < \omega_1$ the rank α_ξ is unbounded in ω_1 on the characteristic Baire class ξ functions.*

Proof. Let $\mathcal{U} \in \mathbf{\Pi}^0_\xi(2^\omega \times X)$ be a universal set for $\mathbf{\Pi}^0_\xi(X)$ sets, that is, for every $F \subseteq X$, $F \in \mathbf{\Pi}^0_\xi(X)$ there exists a $y \in 2^\omega$ such that $\mathcal{U}^y = F$. For the existence of such a set see [7, 22.3]. Let us use the notation $\Gamma_\zeta(X)$ for the the family of sets $H \subseteq X$ satisfying $\alpha_\xi(H, H^c) < \zeta$. From [7, 22.27] we have $\Gamma_\zeta(X) \subseteq \mathbf{\Delta}^0_{\xi+1}(X)$. We will show that there exists a $\mathbf{\Delta}^0_{\xi+1}$ set for every $\zeta < \omega_1$ which is universal for the family of Γ_ζ sets. Since X is uncountable, there is a continuous embedding of 2^ω into X ([7, 6.5]), hence no universal set exists in $2^\omega \times X$ for the family of $\mathbf{\Delta}^0_{\xi+1}(X)$ sets (easy corollary of [7, 22.7]). This implies for every $\zeta < \omega_1$ that $\Gamma_\zeta \ne \mathbf{\Delta}^0_{\xi+1}$, hence the rank is really unbounded.

Let $p : \zeta \times \mathbb{N} \to \mathbb{N}$ be a bijection. For $\eta < \zeta$ and $y \in 2^\omega$ we define a $\phi(y)_\eta \in 2^\omega$ by $\phi(y)_\eta(n) = y(p(\eta, n))$. First we check that for a fixed $\eta < \zeta$ the map $y \mapsto \phi(y)_\eta$ is continuous. Let $U = \{x \in 2^\omega : x(0) = i_0, \dots, x(n) = i_n\}$ be a set from the usual basis of 2^ω. The preimage of U is the set $\{y \in 2^\omega : \forall k \le n\ \phi(y)_\eta(k) = i_k\} = \{y \in 2^\omega : \forall k \le n\ y(p(\eta, k)) = i_k\}$, which is a basic open set, too. Now $\mathcal{U}_\eta = \{(y, x) : (\phi(y)_\eta, x) \in \mathcal{U}\}$ is a continuous preimage of a $\mathbf{\Pi}^0_\xi$ set, hence $\mathcal{U}_\eta \in \mathbf{\Pi}^0_\xi(2^\omega \times X)$ (see

[7, 22.1]). Let

$$\mathcal{U}' = \{(y,x) \in 2^\omega \times X : \text{the smallest ordinal } \eta \text{ such that } (y,x) \notin \mathcal{U}_\eta \text{ is odd,}$$
$$\text{if such an } \eta \text{ exists, or no such } \eta \text{ exists and } \zeta \text{ is odd}\}.$$

Now we check that $\mathcal{U}' \in \mathbf{\Delta}^0_{\xi+1}(2^\omega \times X)$. Let $\mathcal{V}_\eta = \bigcap_{\theta<\eta} \mathcal{U}_\theta$, then these sets form a continuous decreasing sequence of $\mathbf{\Pi}^0_\xi$ sets and it is easy to see that $\mathcal{U}'^c$ is the transfinite difference of the sequence $(\mathcal{V}_\eta)_{\eta<\zeta+1}$, hence $\mathcal{U}'^c \in \mathbf{\Delta}^0_{\xi+1}$, proving that $\mathcal{U}' \in \mathbf{\Delta}^0_{\xi+1}$, since the family of $\mathbf{\Delta}^0_{\xi+1}$ sets is closed under complements (see [7, 22.1]).

Now we show that $\mathcal{U}'$ is universal. For a set $H \in \Gamma_\zeta(X)$ there is a sequence $(z_\eta)_{\eta<\zeta}$ in 2^ω, such that H is the transfinite difference of the sets $\mathcal{U}^{z_\eta}$. For every sequence $(z_\eta)_{\eta<\zeta}$ we can find a $y \in 2^\omega$ such that $\phi(y)_\eta = z_\eta$, namely $y : p(\eta, n) \mapsto z_\eta(n)$ makes sense (since p is a bijection), and works. Consequently, for H there is a $y \in 2^\omega$, such that H is the transfinite difference of the sets $\mathcal{U}^{z_\eta} = \mathcal{U}^{\phi(y)_\eta} = (\mathcal{U}_\eta)^y$. It is easy to see that if H is the transfinite difference of the sequence $((\mathcal{U}_\eta)^y)_{\eta<\zeta}$ then

$$H = \{x \in X : \text{the smallest ordinal } \eta \text{ such that } x \notin (\mathcal{U}_\eta)^y \text{ is odd,}$$
$$\text{if such an } \eta \text{ exists, or no such } \eta \text{ exists and } \zeta \text{ is odd}\},$$

hence $H = \mathcal{U}'^y$. □

Corollary 4.4. *For every $1 \le \xi < \omega_1$, every non-empty perfect set $P \subseteq X$ and every ordinal $\zeta < \omega_1$ there is a characteristic function $\chi_A \in \mathcal{B}_\xi(X)$ with $A \subseteq P$ and $\alpha_\xi(\chi_A) \ge \zeta$.*

Proof. Since P is perfect, it is an uncountable Polish space with the subspace topology, hence the rank α_ξ is unbounded on the characteristic Baire class ξ functions defined on P by the previous theorem. Hence we can take a characteristic function $f' \in \mathcal{B}_\xi(P)$ with $\alpha_\xi(f') \ge \zeta$, and set

$$f(x) = \begin{cases} f'(x) & \text{if } x \in P \\ 0 & \text{if } x \in X \setminus P. \end{cases}$$

It is easy to see that $f \in \mathcal{B}_\xi(X)$, hence it is enough to prove that $\alpha_\xi(f) \ge \zeta$.

For this, it is enough to prove that $\alpha_\xi(\{f' \le p\}, \{f' \ge q\}) \le \alpha_\xi(\{f \le p\}, \{f \ge q\})$ for every pair of rational numbers $p < q$. For this, let $H \in \mathbf{\Delta}^0_{\xi+1}(X)$ where $\{f \le p\} \subseteq H \subseteq \{f \ge q\}^c$ and H is the transfinite difference of the sets $(F_\eta)_{\eta<\lambda}$ with $\lambda = \alpha_\xi(\{f \le p\}, \{f \ge q\})$ and $F_\eta \in \mathbf{\Pi}^0_\xi(X)$ for every $\eta < \lambda$.

Let $H' = P \cap H$ and for every $\eta < \lambda$ let $F'_\eta = P \cap F_\eta$. It is easy to see that H' separates the level sets $\{f' \le p\}$ and $\{f' \ge q\}$ and H' is the transfinite difference of the sets $(F'_\eta)_{\eta<\lambda}$. And since $H' \in \mathbf{\Delta}^0_{\xi+1}(P)$ and $F'_\eta \in \mathbf{\Pi}^0_\xi(P)$ for every $\eta < \lambda$ ([7, 22.A]), we have the desired inequality $\alpha_\xi(\{f' \le p\}, \{f' \ge q\}) \le \alpha_\xi(\{f \le p\}, \{f \ge q\})$. Thus the proof is complete. □

The main disadvantage of this rank is that the construction of Remark 3.30 easily yields that the rank does not behave nicely under linear operations. We leave the easy proof of the next statement to the reader.

Proposition 4.5. *Let $1 \leq \xi < \omega_1$. Then α_ξ is not essentially linear, actually not even essentially additive.*

However, there is a natural way to make a rank linear.

Definition 4.6. For an $f \in \mathcal{B}_\xi$, let

$$\alpha'_\xi(f) = \min\{\max\{\alpha_\xi(f_1), \dots, \alpha_\xi(f_n)\} : n \in \omega, f_1, \dots, f_n \in \mathcal{B}_\xi, \\ f = f_1 + \dots + f_n\}.$$

It can be easily seen that α'_ξ is now linear, but we do not know whether it is still unbounded in ω_1.

Question 4.7. *Let $1 \leq \xi < \omega_1$. Is α'_ξ unbounded in ω_1?*

We have the following partial result, which is a very strong indication that the answer to this question is in the negative, since in every single case when we can show that a rank is unbounded it is actually unbounded on the characteristic functions.

Theorem 4.8. *If $1 \leq \xi < \omega_1$ and f is a* characteristic *Baire class ξ function then $\alpha'_\xi(f) \leq 2$.*

Proof. Let us call a function f a semi-Borel class ξ function if the level sets $\{f < c\}$ are in $\mathbf{\Sigma}^0_\xi$ for every $c \in \mathbb{R}$. Note that then the level sets $\{f > c\}$ are in $\mathbf{\Sigma}^0_{\xi+1}$, hence $f \in \mathcal{B}_\xi$.

We first show that a semi-Borel class ξ function has α_ξ rank at most 2. Let $p < q$ be a pair of rational numbers. The level set $\{f \geq q\} \in \mathbf{\Pi}^0_\xi(X)$, hence the transfinite difference of the sequence $F_0 = X, F_1 = \{f \geq q\}$ separates the level sets $\{f \leq p\}$ and $\{f \geq q\}$.

Now using the same idea as in Remark 3.30, it is clear that every characteristic Baire class ξ function can be written as the difference of two semi-Borel class ξ functions, completing the proof of this theorem. $\square$

The following question is very closely related to Question 4.7.

Question 4.9. *Let $1 \leq \xi < \omega_1$ and let f_n and f be Baire class ξ functions such that $f_n \to f$ uniformly. Does this imply that $\alpha'_\xi(f) \lesssim \sup_n \alpha'_\xi(f_n)$?*

Remark 4.10. An affirmative answer to this question would provide a negative answer to Question 4.7. Indeed, it is not hard to show that α'_ξ is bounded for step functions, and hence, by taking uniform limit, for every bounded function. Then one can check that the rank of an arbitrary function f equals to the rank of the bounded function $\arctan \circ f$, hence α'_ξ is bounded.

4.2. **Limit ranks.** In this section we apply an even more natural approach to define ranks on the Baire class ξ functions starting from an arbitrary rank on the Baire class 1 functions. Surprisingly, they will all turn out to be bounded in ω_1.

Definition 4.11. Let ρ be a rank on the Baire class 1 functions. We inductively define a rank $\overline{\rho}_\xi$ on the Baire class ξ functions. First, let $\overline{\rho}_1 = \rho$. For a successor ordinal $\xi+1$ and a Baire class $\xi+1$ function f let

$$\overline{\rho}_{\xi+1}(f)=\min\left\{\sup_n \overline{\rho}_{\xi}(f_n) : f_n \to f,\ f_n \text{ is of Baire class } \xi\right\}.$$

Finally, for a limit ordinal ξ and a Baire class ξ function f let

$$\overline{\rho}_{\xi}(f)=\min\Big\{\sup_n \overline{\rho}_{\xi_n}(f_n) : f_n \to f,\ f_n \text{ is of Baire class } \xi_n,\ \xi_n<\xi,$$
$$f_n \text{ is not of Baire class } \zeta \text{ if } \zeta<\xi_n\Big\}.$$

Surprisingly, the ranks $\overline{\alpha}_\xi$, $\overline{\beta}_\xi$ and $\overline{\gamma}_\xi$ will all be bounded for $\xi \geq 2$.

Theorem 4.12. *If $2 \leq \xi < \omega_1$ then $\overline{\alpha}_\xi \leq \overline{\beta}_\xi \leq \overline{\gamma}_\xi \leq \omega$.*

Proof. It is enough to prove the theorem for $\xi = 2$. Let Φ be a class of real valued functions on X. As in [5], we say that Φ is *ordinary* if it contains the constant functions and if $f, g \in \Phi$ then $\max(f, g)$, $\min(f, g)$, $f+g$, $f-g$, fg and f/g (if g is nowhere zero) are all in Φ. An ordinary class of functions is called *complete* if it is closed under uniform limits.

For a class of functions Φ, we denote by Φ^p the set of functions that are pointwise limits of functions from Φ. We denote the pair of families of level sets of functions in Φ by $\mathcal{P}(\Phi)$, that is,

$$\mathcal{P}(\Phi) = \left(\{\{f>c\} : f\in\Phi, c\in\mathbb{R}\}, \{\{f\geq c\} : f\in\Phi, c\in\mathbb{R}\}\right).$$

If $\mathcal{P} = (\mathcal{M}, \mathcal{N})$ is a pair of systems of sets then we denote the class of functions whose levels sets are in $\mathcal{P}$ by $\Phi(\mathcal{P})$, that is,

$$\Phi(\mathcal{P}) = \{f : X \to \mathbb{R} \mid \forall c\in\mathbb{R}\ \{f>c\}\in\mathcal{M}, \{f\geq c\}\in\mathcal{N}\}.$$

Now we state three theorems based on results in [5].

Theorem 4.13. *If a class of functions Φ is ordinary then Φ^p is ordinary and complete.*

Theorem 4.14. *If a class of functions Φ is ordinary and $\mathcal{P}(\Phi) = (\mathcal{M}, \mathcal{N})$ then $\mathcal{P}(\Phi^p) = (\mathcal{N}_{\delta\sigma}, \mathcal{M}_{\sigma\delta})$.*

Theorem 4.15. *If a class of functions Φ is complete and ordinary then $\Phi = \Phi(\mathcal{P}(\Phi))$.*

Theorem 4.13 is shown in [5, §41. IV.], Theorem 4.14 is an easy corollary of [5, §41. V., VI.] and Theorem 4.15 is shown in [5, §41. VIII.].

Now let Φ consist of the Baire class 1 functions of the form

$$\sum_{i=1}^{n} c_i \chi_{H_i},$$

where H_i is in the algebra $\mathcal{A}$ generated by the open sets (an algebra is a family closed under finite unions and complements). It is easy to check that $\mathcal{A}$ contains exactly the sets that can be written as the finite disjoint union of sets of the form

$F \cap G$, where F is closed and G is open. Indeed, the intersection of two such set is of the same form, and the complement of such a set is

$$\left(\bigcup_{i=0}^{n-1}(F_i \cap G_i)\right)^c = \bigcap_{i=0}^{n-1}(F_i \cap G_i)^c = \bigcap_{i=0}^{n-1}(F_i^c \cup G_i^c) =$$

$$\bigcup\left\{\bigcap_{i=0}^{n-1} F_i^{a(i)} \cap \bigcap_{i=0}^{n-1} G_i^{b(i)} : a, b \in 2^n,\ \forall i < n \text{ at least one of } a(i) \text{ and } b(i) \text{ is } 1\right\},$$

where for a set H, $H^0 = H$ and $H^1 = H^c$, and the last equality holds, since a point x is contained in either of the two sets in question iff for every $i < n$ it is contained in at least one of F_i^c and G_i^c. Now we check that the sets in the union are disjoint. Without loss of generality we have two terms with distinct a's, so $a(i) = 0$ and $a'(i) = 1$ for a suitable i. But then the term belonging to a is a subset of F_i and the other one is a subset of F_i^c, proving disjointness.

An easy consequence of these observations is that Φ is ordinary.

Lemma 4.16. *$\gamma(f) \leq \omega$ for every $f \in \Phi$.*

Proof. First we prove that $\gamma(\chi_F) \leq 2$ for every closed set F. Let F be a closed set, and define $f_n(x) = 1 - \min\{1, n \cdot d(x, F)\}$. It is easy to check that $f_n \to \chi_F$ pointwise. We now show that $\gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) \leq 2$ for every $\varepsilon > 0$, which will imply $\gamma(\chi_F) \leq 2$. Fix $\varepsilon > 0$. If $x \notin F$ then x has a neighborhood U such that $d(U, F) > 0$ and then if we fix an $N > \frac{1}{d(U,F)}$ then $f_n(y) = 0$ for every $y \in U$ and $n \geq N$, therefore $\omega((f_n)_{n\in\mathbb{N}}, x, X) = 0$. This implies $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(X) \subseteq F$. But $f_n|_F \equiv 1$ for every n, hence if $x \in F$ then $\omega((f_n)_{n\in\mathbb{N}}, x, F) = 0$, therefore $D^2_{(f_n)_{n\in\mathbb{N}},\varepsilon}(X) \subseteq D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) = \emptyset$, proving $\gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) \leq 2$.

It is easy to check that $\gamma(f) = \gamma(1 - f)$ for every $f \in \mathcal{B}_1$. This implies that $\gamma(\chi_G) \leq 2$ for every open set G, since $\chi_G = 1 - \chi_{X\setminus G}$.

Now, let $H = F \cap G$, where F is closed and G is open. We show that $\gamma(\chi_H) \leq \omega$. By Theorem 3.29 there exists a sequence f_n of continuous functions with $f_n \to \chi_F + \chi_G$ and $\gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) \leq \omega$ for every $\varepsilon > 0$. Define $f'_n = \max\{0, f_n - 1\}$. Then it is easy to check that $f'_n \to \chi_H$ and $\gamma((f'_n)_{n\in\mathbb{N}}, \varepsilon) \leq \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) \leq \omega$ for every $\varepsilon > 0$.

Since any $H \in \mathcal{A}$ is a finite disjoint union of sets of the form $F \cap G$, the above paragraph shows that $\chi_H = \chi_{H_0} + \cdots + \chi_{H_n}$, where $\gamma(\chi_{H_i}) \leq \omega$. But then Theorem 3.29 yields that $\gamma(\chi_H) \leq \omega$. Then applying Theorem 3.29 once again we obtain that $\gamma(f) \leq \omega$ for every $f \in \Phi$. $\square$

Now we turn to the proof of the theorem. By Theorem 3.24 and the previous lemma, it is enough to show that Φ^p equals the family of Baire class 2 functions. Since every $f \in \Phi$ is of Baire class 1, we have that Φ^p is a subclass of the Baire class 2 functions.

For the converse, let us define $\mathcal{M}$ and $\mathcal{N}$ by $\mathcal{P}(\Phi) = (\mathcal{M}, \mathcal{N})$. By the definition of Φ, $\mathcal{M}$ and $\mathcal{N}$ both contain the open and closed sets. By Theorem 4.14 $\mathcal{P}(\Phi^p) = (\mathcal{N}_{\delta\sigma}, \mathcal{M}_{\sigma\delta})$, hence $\mathbf{\Sigma}^0_3 \subseteq \mathcal{N}_{\delta\sigma}$ and $\mathbf{\Pi}^0_3 \subseteq \mathcal{M}_{\sigma\delta}$. And by Theorem 4.13 and Theorem 4.15 $\Phi^p = \Phi(\mathcal{P}(\Phi^p)) = \Phi(\mathcal{N}_{\delta\sigma}, \mathcal{M}_{\sigma\delta}) \supseteq \Phi(\mathbf{\Sigma}^0_3, \mathbf{\Pi}^0_3) = \mathcal{B}_2$, finishing the proof. $\square$

4.3. **Partition ranks.** The following well known fact also gives rise to a very natural rank on the Baire class ξ functions. However, this also turns out to be bounded.

Proposition 4.17. *A function f is of Baire class ξ if and only if for every $\varepsilon > 0$ there exists a function g of the form $g = \sum_{n\in\omega} c_n \cdot \chi_{H_n}$, where $H_n \in \mathbf{\Delta}^0_{\xi+1}(X)$, the H_n's form a partition of X and $|f(x) - g(x)| \leq \varepsilon$ for every $x \in X$. Moreover, if f is bounded then each set H_n can be chosen to be empty for all but finitely many $n \in \omega$.*

Proof. If f is of Baire class ξ then for a fixed $\varepsilon > 0$ let the numbers p_n be defined by $p_n = n \cdot \frac{\varepsilon}{2}$ for every $n \in \mathbb{Z}$. The sets $\{f \leq p_n\}$ and $\{f \geq p_{n+1}\}$ are disjoint $\mathbf{\Pi}^0_{\xi+1}$ sets, hence they can be separated by a set $A_n \in \mathbf{\Delta}^0_{\xi+1}$. Now let $H_n = A_n \setminus A_{n-1}$. Note that if f is bounded then $H_n = \emptyset$ for all but finitely many $n \in \omega$. These sets form a partition, and with $g = \sum_{n\in\mathbb{Z}} p_n \cdot \chi_{H_n}$ the proof of the first direction is complete.

For the other one, note that the function g is of Baire class ξ, hence f is the uniform limit of Baire class ξ functions, implying that f is of Baire class ξ (see e.g. [7, 24.4]). $\square$

Definition 4.18. Let f be a Baire class ξ function and let the *partition rank* of f be

$$\delta(f) = \sup_{\varepsilon>0} \min \Bigg\{ \sup_{n\in\omega} \alpha_\xi(H_n, H_n^c) : \ H_n \in \mathbf{\Delta}^0_{\xi+1}, \ \bigcup_{n\in\omega} H_n = X, \\ H_n \cap H_m = \emptyset \ (n \neq m), \ \exists (c_n)_{n\in\omega} \left| f - \sum_{n\in\omega} c_n \cdot \chi_{H_n} \right| \leq \varepsilon \Bigg\}.$$

Proposition 4.19. $\delta(f) \leq 4$ *for every Baire class ξ function f.*

Proof. Fix $\varepsilon > 0$. Obtain a function of the form $\sum_{n\in\omega} c_n \cdot \chi_{H_n}$ as in the above proposition. It is enough to prove that every H_n has a further partition into a sequence of sets $H_{n,k} \in \mathbf{\Delta}^0_{\xi+1}$ with $\alpha_\xi(H_{n,k}, H^c_{n,k}) \leq 4$.

But this is easy, since H_n can be written as the transfinite difference of $\mathbf{\Pi}^0_\xi$ sets, so H_n is obtained as the countable disjoint union of sets of the form $F_\eta \setminus F_{\eta+1}$ with $F_\eta, F_{\eta+1} \in \mathbf{\Pi}^0_\xi$, and the α_ξ rank of $F_\eta \setminus F_{\eta+1}$ at most 4, as the sequence $(X, X, F_\eta, F_{\eta+1})$ shows. $\square$

Now we focus our attention on finite partitions and investigate the resulting rank, which we can only define for bounded functions.

Definition 4.20. Let f be a bounded Baire class ξ function and let the *finite partition rank* of f be

$$\delta_{fin}(f) = \sup_{\varepsilon>0} \min \Bigg\{ \sup_{n\leq N} \alpha_\xi(H_n, H_n^c) : N \in \omega, H_n \in \mathbf{\Delta}^0_{\xi+1}(n \leq N), \bigcup_{n\leq N} H_n = X, \\ H_n \cap H_m = \emptyset \ (n, m \leq N, \ n \neq m), \ \exists (c_n)_{n\leq N} \left| f - \sum_{n\leq N} c_n \cdot \chi_{H_n} \right| \leq \varepsilon \Bigg\}.$$

Theorem 4.21. $\delta_{fin}(f) \approx \alpha_\xi(f)$ *for every bounded Baire class ξ function f.*

Proof. Let f be an arbitrary bounded Baire class ξ function. First we prove that $\delta_{fin} \lesssim \alpha_\xi(f)$. For a fixed $\varepsilon > 0$ let the numbers p_n be defined by $p_n = n \cdot \frac{\varepsilon}{2}$ for every $n \in \mathbb{Z}$. The sets $\{f \leq p_n\}$ and $\{f \geq p_{n+1}\}$ are disjoint $\mathbf{\Pi}^0_{\xi+1}$ sets, hence they can be separated by a set $A_n \in \mathbf{\Delta}^0_{\xi+1}$ with $\alpha_\xi(A_n, A_n^c) \leq \alpha_\xi(f)$. Now let $H_n = A_n \setminus A_{n-1}$. Since f is bounded, $H_n = \emptyset$ for all but finitely many $n \in \omega$. Clearly, these sets form a partition, and $g = \sum_{n\in\mathbb{Z}} p_n \cdot \chi_{H_n}$ is ε-close to f.

We will prove in Corollary 5.18 below that α_ξ is essentially linear for bounded functions. Therefore we obtain $\alpha_\xi(H_n, H_n^c) = \alpha_\xi(\chi_{H_n}) = \alpha_\xi(\chi_{A_n} - \chi_{A_{n-1}}) \lesssim \max\{\alpha_\xi(\chi_{A_n}), \alpha_\xi(\chi_{A_{n-1}})\} = \max\{\alpha_\xi(A_n, A_n^c), \alpha_\xi(A_{n-1}, A_{n-1}^c)\} \leq \alpha_\xi(f)$, proving $\delta_{fin} \lesssim \alpha_\xi(f)$.

Now we prove the other direction. Let $p < q$ be arbitrary rational numbers, it is enough to prove that there is a set $H \in \mathbf{\Delta}^0_{\xi+1}$ separating the level sets $\{f \leq p\}$ and $\{f \geq q\}$ with $\alpha_\xi(H, H^c) \leq \delta_{fin}(f)$. Now set $\varepsilon = \frac{q-p}{2}$. From the definition of δ_{fin}, we can find a finite partition $X = H_0 \cup \dots \cup H_N$ into disjoint $\mathbf{\Delta}^0_{\xi+1}$ sets and $c_n \in \mathbb{R}$ with $g = \sum_{n=0}^N c_n \cdot \chi_{H_n}$ satisfying $|f - g| < \varepsilon$ and $\alpha_\xi(H_n, H_n^c) \leq \delta_{fin}(f)$ for $n \leq N$.

Let $A = \{n \leq N : H_n \cap \{f \leq p\} \neq \emptyset\}$ and $H = \bigcup_{n\in A} H_n$. Clearly, $\{f \leq p\} \subseteq H$. Moreover, no H_n can intersect both $\{f \leq p\}$ and $\{f \geq q\}$, since g is constant on H_n and $|f - g| < \varepsilon = \frac{q-p}{2}$. Therefore $H \cap \{f \geq q\} = \emptyset$. Using the essential linearity of α_ξ for bounded functions again we obtain $\alpha_\xi(H, H^c) = \alpha_\xi(\chi_H) \lesssim \max\{\alpha_\xi(\chi_{H_n}) : n \in A\} = \max\{\alpha_\xi(H_n, H_n^c) : n \in A\} \leq \delta_{fin}(f)$, completing the proof. □

5. Ranks answering Question 1.1 and Question 1.2

In this section we finally show that there actually exist ranks with very nice properties. Throughout the section, let $1 \leq \xi < \omega_1$ be fixed.

Let f be of Baire class ξ. Let

$$T_{f,\xi} = \{\tau' : \tau' \supseteq \tau \text{ Polish}, \tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau), f \in \mathcal{B}_1(\tau')\}.$$

So $T_{f,\xi}$ is the set of those Polish refinements of the original topology that are subsets of the $\mathbf{\Sigma}^0_\xi$ sets turning f to a Baire class 1 function.

Remark 5.1. Clearly, $T_{f,1} = \{\tau\}$ for every Baire class 1 function f.

In order to show that the ranks we are about to construct are well-defined, we need the following proposition.

Proposition 5.2. *$T_{f,\xi} \neq \emptyset$ for every Baire class ξ function f.*

Proof. By the previous remark we may assume $\xi \geq 2$. For every rational p the level sets $\{f \leq p\}$ and $\{f \geq p\}$ are $\mathbf{\Pi}^0_{\xi+1}$ sets, hence they are countable intersections of $\mathbf{\Sigma}^0_\xi$ sets. In turn, these $\mathbf{\Sigma}^0_\xi$ sets are countable unions of sets from $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(\tau)$. Clearly, $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(\tau) \subseteq \mathbf{\Delta}^0_\xi$ for $\xi \geq 2$. By Kuratowski's theorem [7, 22.18], there exists a Polish refinement $\tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau)$ of τ for which all these countable many $\mathbf{\Delta}^0_\xi$ sets are in $\mathbf{\Delta}^0_1(\tau')$. Then for every rational p the level sets are now $\mathbf{\Pi}^0_2(\tau')$ sets, and the same holds for irrational numbers too, since these level sets can be written as countable intersection of rational level sets, proving $T_{f,\xi} \neq \emptyset$. □

Now similarly to the limit ranks, we define a rank on the Baire class ξ functions starting from an arbitrary rank on the Baire class 1 functions.

Definition 5.3. Let ρ be a rank on the Baire class 1 functions. Then for a Baire class ξ function f let

$$\rho^*_\xi(f) = \min_{\tau' \in T_{f,\xi}} \rho_{\tau'}(f), \tag{5.1}$$

where $\rho_{\tau'}(f)$ is just the ρ rank of f in the τ' topology.

Remark 5.4. From Remark 5.1 it is clear that $\rho^*_1 = \rho$ for every ρ.

Proposition 5.5. *Let ρ and η be ranks on the Baire class 1 functions. If $\rho = \eta$, or $\rho \le \eta$, or $\rho \approx \eta$, or $\rho \lesssim \eta$ then $\rho^*_\xi = \eta^*_\xi$, or $\rho^*_\xi \le \eta^*_\xi$, or $\rho^*_\xi \approx \eta^*_\xi$, or $\rho^*_\xi \lesssim \eta^*_\xi$, respectively. Moreover, the same implications hold relative to the class of bounded functions.*

Proof. The statement for $=$ and $\le$ is immediate from the definitions, and the case of $\approx$ obviously follows from the case $\lesssim$, so it suffices to prove this latter case only. So assume $\rho \lesssim \eta$ (or $\rho \lesssim \eta$ on the bounded Baire class 1 functions). Choose an optimal $\tau' \in T_{f,\xi}$ for η, that is, $\eta^*_\xi(f) = \eta_{\tau'}(f)$. Then $\rho^*_\xi(f) \le \rho_{\tau'}(f) \lesssim \eta_{\tau'}(f) = \eta^*_\xi(f)$, completing the proof. □

Then the following two corollaries are immediate from Theorem 3.24, and Theorem 3.35.

Corollary 5.6. $\alpha^*_\xi \le \beta^*_\xi \le \gamma^*_\xi$.

Corollary 5.7. $\alpha^*_\xi(f) \approx \beta^*_\xi(f) \approx \gamma^*_\xi(f)$ *for every bounded Baire class ξ function f.*

Similarly to Question 3.41 (the case of Baire class 1 functions), we do not know whether $\beta^*_\xi(f) \approx \gamma^*_\xi(f)$ holds for arbitrary Baire class ξ functions.

Question 5.8. *Does $\beta^*_\xi(f) \approx \gamma^*_\xi(f)$ hold for every Baire class ξ function?*

Note that by repeating the argument of Remark 3.30 one can show that α^*_ξ differs from β^*_ξ and γ^*_ξ. It is easy to see that an affirmative answer to Question 3.41 would imply an affirmative answer to the last question, however, the other direction is not clear.

Theorem 5.9. *If X is a Polish group then the ranks α^*_ξ, β^*_ξ and γ^*_ξ are translation invariant.*

Proof. Note first that for a Baire class ξ function f and $x_0 \in X$ the functions $f \circ L_{x_0}$ and $f \circ R_{x_0}$ are also of Baire class ξ. We prove the statement only for the rank α^*_ξ, because an analogous argument works for the ranks β^*_ξ and γ^*_ξ.

Let f be a Baire class ξ function and $x_0 \in X$, first we prove that $\alpha^*_\xi(f) \ge \alpha^*_\xi(f \circ R_{x_0})$. Let $\tau' \in T_{f,\xi}$ be arbitrary and consider the topology $\tau'' = \{U \cdot x_0^{-1} : U \in \tau'\}$. The map $\phi : x \mapsto x \cdot x_0^{-1}$ is a homeomorphism between the spaces (X, τ') and (X, τ''), satisfying $f(x) = (f \circ R_{x_0})(\phi(x))$. From this it is clear that $\tau'' \in T_{f \circ R_{x_0},\xi}$ and since the definition of the rank α depends only on the topology of the space, we have $\alpha_{\tau'}(f) = \alpha_{\tau''}(f \circ R_{x_0})$. Since $\tau' \in T_{f,\xi}$ was arbitrary, the fact that $\alpha^*_\xi(f) \geq \alpha^*_\xi(f \circ R_{x_0})$ easily follows.

Repeating the argument with the function $f \circ R_{x_0}$ and element x_0^{-1}, we have $\alpha^*_\xi(f \circ R_{x_0}) \geq \alpha^*_\xi(f \circ R_{x_0} \circ R_{x_0^{-1}}) = \alpha^*_\xi(f)$, hence $\alpha^*_\xi(f) = \alpha^*_\xi(f \circ R_{x_0})$. For the function $f \circ L_{x_0}$ we can do same using the topology $\tau'' = \{x_0^{-1} \cdot U : U \in \tau'\}$ and homeomorphism $\phi : x \mapsto x_0^{-1} \cdot x$ yielding $\alpha^*_\xi(f) = \alpha^*_\xi(f \circ L_{x_0})$, finishing the proof. $\square$

Theorem 5.10. *If f is a Baire class ξ function and $F \subseteq X$ is a closed set then $f \cdot \chi_F$ is of Baire class ξ and $\alpha^*_\xi(f \cdot \chi_F) \leq 1 + \alpha^*_\xi(f)$, $\beta^*_\xi(f \cdot \chi_F) \leq 1 + \beta^*_\xi(f)$ and $\gamma^*_\xi(f \cdot \chi_F) \leq 1 + \gamma^*_\xi(f)$.*

Proof. Examining the level sets of the function $f \cdot \chi_F$, it is easy to check that it is of Baire class ξ.

Now let $\tau' \in T_{f,\xi}$ be arbitrary. Clearly, $f \cdot \chi_F$ is of Baire class 1 with respect to τ', and by Proposition 3.28 we have $\alpha_{\tau'}(f \cdot \chi_F) \leq 1 + \alpha_{\tau'}(f)$ for every $\tau' \in T_{f,\xi}$, hence $\alpha^*_\xi(f \cdot \chi_F) \leq 1 + \alpha^*_\xi(f)$. The other two inequalities follow similarly. $\square$

Proposition 5.11. *If f is a Baire class ζ function with $\zeta < \xi$ then $\alpha^*_\xi(f) = \beta^*_\xi(f) = \gamma^*_\xi(f) = 1$.*

Proof. Using Proposition 3.27, it is enough to show that there exists a topology $\tau' \in T_{f,\xi}$ such that $f : (X, \tau') \to \mathbb{R}$ is continuous, and this is clear from [7, 24.5]. $\square$

Next we prove a useful lemma, and then investigate further properties of the ranks α^*_ξ, β^*_ξ and γ^*_ξ.

Lemma 5.12. *For every n let τ_n be a Polish refinement of τ with $\tau_n \subseteq \mathbf{\Sigma}^0_\xi(\tau)$. Then there exists a common Polish refinement τ' of the τ_n's also satisfying $\tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau)$.*

Proof. The case $\xi = 1$ is again trivial, so we may assume $\xi \geq 2$. Take a base $\{G^k_n : k \in \mathbb{N}\}$ for τ_n. Since these sets are in $\mathbf{\Sigma}^0_\xi(\tau)$, they can be written as the countable unions of sets from $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(\tau)$. Clearly, $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(\tau) \subseteq \mathbf{\Delta}^0_\xi$ for $\xi \geq 2$. As above, by Kuratowski's theorem [7, 22.18], we have a Polish topology τ', for which these countably many $\mathbf{\Delta}^0_\xi(\tau)$ sets are in $\mathbf{\Delta}^0_1(\tau')$ satisfying $\tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau)$. This τ' works. $\square$

Lemma 5.13. *If $\tau' \subseteq \tau''$ are two Polish topologies with $f \in \mathcal{B}_1(\tau')$ then $f \in \mathcal{B}_1(\tau'')$, moreover, $\beta_{\tau'}(f) \geq \beta_{\tau''}(f)$ and $\gamma_{\tau'}(f) \geq \gamma_{\tau''}(f)$.*

Proof. To prove that $f \in \mathcal{B}_1(\tau'')$ note that the level sets $\{f < c\}, \{f > c\} \in \mathbf{\Sigma}^0_2(\tau')$, hence $\{f < c\}, \{f > c\} \in \mathbf{\Sigma}^0_2(\tau'')$, so $f \in \mathcal{B}_1(\tau'')$.

Now recall the definition of the derivative defining β:

$$\omega(f, x, F) = \inf \left\{ \sup_{x_1, x_2 \in U \cap F} |f(x_1) - f(x_2)| : U \text{ open}, x \in U \right\},$$

$$D_{f,\epsilon}(F) = \{x \in F : \omega(f, x, F) \geq \epsilon\}.$$

Let us now fix f and $\varepsilon > 0$ and let us denote the derivative $D_{f,\epsilon}$ with respect to the topology τ' by $D_{\tau'}$, and with respect to the topology τ'' by $D_{\tau''}$. By Proposition 3.5 it is enough to prove that $D_{\tau''}(F) \subseteq D_{\tau'}(F)$ for every closed set $F \subseteq X$.

For this it is enough to show that $\omega_{\tau''}(f,x,F) \leq \omega_{\tau'}(f,x,F)$ for every $x \in F$ where $\omega_{\tau'}(f,x,F)$ is the oscillation with respect to the topology τ'. And this is clear, since in the case of τ'', the infimum in the definition goes through more open set containing x, hence the resulting oscillation will be less.

For the rank γ, we proceed similarly. First we recall the definition of γ:

$$\omega((f_n)_{n\in\mathbb{N}},x,F) = \inf_{\substack{x\in U\\ U \text{ open}}} \inf_{N\in\mathbb{N}} \sup\left\{|f_m(y)-f_n(y)| : n,m \geq N,\ y \in U \cap F\right\},$$

$$D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) = \{x \in F : \omega((f_n)_{n\in\mathbb{N}},x,F) \geq \varepsilon\},$$

$$\gamma(f) = \min\left\{\sup_{\varepsilon>0} \gamma((f_n)_{n\in\mathbb{N}},\varepsilon) : \forall n\ f_n \text{ is continuous and } f_n \to f \text{ pointwise}\right\}.$$

Let us fix a sequence $(f_n)_{n\in\mathbb{N}}$ of τ'-continuous (hence also τ''-continuous) functions converging pointwise to f, and also fix $\varepsilon > 0$. Let us denote the derivative $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}$ with respect to τ' by $D_{\tau'}$ and with respect to τ'' by $D_{\tau''}$. Again, by Proposition 3.5 it is enough to prove that $D_{\tau''}(F) \subseteq D_{\tau'}(F)$ for every closed set $F \subseteq X$. And similarly to the previous case it is enough to prove that the oscillation $\omega((f_n)_{n\in\mathbb{N}},x,F)$ with respect to the topology τ'' is at most the oscillation with respect to τ', but this is clear, since, as before, the infimum goes through more open set in the case of τ''. □

Theorem 5.14. *The ranks β^*_ξ and γ^*_ξ are essentially linear.*

Proof. We only consider β^*_ξ, since the proof for the rank γ^*_ξ is completely analogous.

It is easy to see that $\beta^*_\xi(cf) = \beta^*_\xi(f)$ for every $c \in \mathbb{R}\setminus\{0\}$, hence it suffices to show that β^*_ξ is essentially additive.

For f and g let τ_f and τ_g be such that $\beta_{\tau_f}(f) = \beta^*_\xi(f)$ and $\beta_{\tau_g}(g) = \beta^*_\xi(g)$. Using Lemma 5.12 we have a common refinement τ' of τ_f and τ_g with $\tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau)$. Now $f,g \in \mathcal{B}_1(\tau')$, so $f+g \in \mathcal{B}_1(\tau')$, hence $\tau' \in T_{f+g,\xi}$. Therefore $\beta^*_\xi(f+g) \leq \beta_{\tau'}(f+g)$. By Lemma 5.13 we have that $\beta_{\tau'}(f) \leq \beta_{\tau_f}(f)$ (in fact equality holds), and similarly for g. But $\beta_{\tau'}$ is additive by Theorem 3.29, so

$$\beta^*_\xi(f+g) \leq \beta_{\tau'}(f+g) \lesssim \max\{\beta_{\tau'}(f),\beta_{\tau'}(g)\} \leq \max\{\beta_{\tau_f}(f),\beta_{\tau_g}(g)\} =$$

$$\max\{\beta^*_\xi(f),\beta^*_\xi(g)\}.$$

□

Remark 5.15. One can easily deduce from Theorem 5.14 that $\beta^*_\xi(f\cdot g) \lesssim \max\{\beta^*_\xi(f),\beta^*_\xi(g)\}$ for every $\xi < \omega_1$ whenever f and g are bounded Baire class ξ functions, and similarly for γ^*_ξ. Again, as in the case of β and γ, the situation is unclear for unbounded functions.

Question 5.16. *Let $1 \leq \xi < \omega_1$. Are the ranks β^*_ξ and γ^*_ξ essentially multiplicative?*

Theorem 5.17. *If f is a Baire class ξ function then*

$$\alpha^*_\xi(f) \le \alpha_\xi(f) \le 2\alpha^*_\xi(f), \textit{ hence } \alpha^*_\xi(f) \approx \alpha_\xi(f).$$

Proof. For $\xi = 1$ the claim is an easy consequence of the definition of the two ranks and Corollary 3.14. From now on, we suppose that $\xi \ge 2$.

For the first inequality, for every pair of rationals $p < q$ pick a sequence $(F^\zeta_{p,q})_{\zeta<\alpha_\xi(f)} \subseteq \mathbf{\Pi}^0_\xi(X)$, whose transfinite difference separates the level sets $\{f \le p\}$ and $\{f \ge q\}$.

Every $\mathbf{\Pi}^0_\xi(X)$ set is the intersection of countably many $\mathbf{\Delta}^0_\xi$ sets, hence $F^\zeta_{p,q} = \bigcap_n H^\zeta_{p,q,n}$, with $H^\zeta_{p,q,n} \in \mathbf{\Delta}^0_\xi$. By Kuratowski's theorem [7, 22.18], there is a finer Polish topology $\tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau)$, for which $H^\zeta_{p,q,n} \in \mathbf{\Delta}^0_1(\tau')$ for every p, q, n and $\zeta < \alpha_\xi(f)$, hence $F^\zeta_{p,q} \in \mathbf{\Pi}^0_1(\tau')$.

This means that the level sets of f can be separated by transfinite differences of closed sets with respect to τ', hence they can be separated by sets in $\mathbf{\Delta}^0_2(\tau')$. Then it is easy to see that for every $c \in \mathbb{R}$ the level sets $\{f \le c\}$ and $\{f \ge c\}$ are countable intersections of $\mathbf{\Delta}^0_2(\tau')$ sets, hence they are $\mathbf{\Pi}^0_2(\tau')$ sets, proving that $f \in \mathcal{B}_1(\tau')$. Moreover, $\alpha_{1,\tau'}(f) \le \alpha_\xi(f)$ easily follows from the construction (here $\alpha_{1,\tau'}$ is the rank α_1 with respect to τ'). And by Corollary 3.14 we have $\alpha^*_\xi \le \alpha_{\tau'}(f) \le \alpha_{1,\tau'}(f) \le \alpha_\xi(f)$, proving the first inequality of the theorem.

For the second inequality, take a topology τ' with $\alpha_{\tau'}(f) = \alpha^*_\xi(f)$. Again, by Corollary 3.14, we have $\alpha_{1,\tau'}(f) \le 2\alpha_{\tau'}(f) = 2\alpha^*_\xi(f)$.

It remains to prove that $\alpha_\xi(f) \le \alpha_{1,\tau'}(f)$. A τ'-closed set is $\mathbf{\Pi}^0_\xi$ with respect to τ. Therefore, if $(F_\eta)_{\eta<\zeta}$ is a decreasing continuous sequence of τ'-closed sets whose transfinite difference separates $\{f \le p\}$ and $\{f \ge q\}$ then the same sequence is a decreasing continuous sequence of sets from $\mathbf{\Pi}^0_\xi(\tau)$, proving $\alpha_\xi(f) \le \alpha_{1,\tau'}(f)$. □

Corollary 5.18. *α_ξ and α^*_ξ are essentially linear for bounded functions for every ξ.*

Proof. $\alpha_\xi \approx \alpha^*_\xi$ by the previous theorem, $\alpha^*_\xi \approx \beta^*_\xi$ for bounded functions by Corollary 5.7, and β^*_ξ is essentially linear by Theorem 5.14. □

From Corollary 3.39 we can obtain the appropriate statement for the ranks $\alpha^*_\xi, \beta^*_\xi$ and γ^*_ξ.

Proposition 5.19. *If $f = \sum_{i=1}^n c_i \chi_{A_i}$, where the A_i's are disjoint $\mathbf{\Delta}^0_{\xi+1}$ sets covering X and the c_i's are distinct then*

$$\alpha^*_\xi(f) \approx \max_i\{\alpha^*_\xi(\chi_{A_i})\},$$

and similarly for β^*_ξ and γ^*_ξ.

Proof. The additivity of α^*_ξ implies $\alpha^*_\xi(f) \lesssim \max_i\{\alpha^*_\xi(\chi_{A_i})\}$. For the other inequality let τ' be a topology for which f is Baire class 1. Then the characteristic functions χ_{A_i} are also Baire class 1, and hence by Corollary 3.39 we obtain $\alpha_{\tau'}(f) \approx \max_i\{\alpha_{\tau'}(\chi_{A_i})\}$. But by the definition of α^*_ξ for every such topology

$\alpha^*_\xi(\chi_{A_i}) \leq \alpha_{\tau'}(\chi_{A_i})$, therefore $\max_i\{\alpha^*_\xi(\chi_{A_i})\} \leq \max_i\{\alpha_{\tau'}(\chi_{A_i})\} \approx \alpha_{\tau'}(f)$. Then choosing τ' so that $\alpha_{\tau'}(f) = \alpha^*_\xi(f)$ the proof is complete.

□

Theorem 5.20. *The ranks α^*_ξ, β^*_ξ and γ^*_ξ are unbounded in ω_1. Moreover, for every non-empty perfect set $P \subseteq X$ and ordinal $\zeta < \omega_1$ there exists a characteristic function $\chi_A \in \mathcal{B}_\xi(X)$ with $A \subseteq P$ such that $\alpha^*_\xi(\chi_A), \beta^*_\xi(\chi_A)$, $\gamma^*_\xi(\chi_A) \geq \zeta$.*

Proof. In order to prove the theorem, by Corollary 5.6 it suffices to prove the statement for α^*_ξ. Moreover, instead of $\alpha^*_\xi(\chi_A) \geq \zeta$ it suffices to obtain $\alpha^*_\xi(\chi_A) \gtrsim \zeta$. And this is clear from Theorem 5.17 and Corollary 4.4. □

Proposition 5.21. *If f_n, f are Baire class ξ functions and $f_n \to f$ uniformly then $\beta^*_\xi(f) \leq \sup_n \beta^*_\xi(f_n)$.*

Proof. For every n let $\tau_n \in T_{f_n,\xi}$ with $\beta_{\tau_n}(f_n) = \beta^*_\xi(f_n)$. Using Lemma 5.12, let τ' be their common refinement satisfying $\tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau)$, where τ is the original topology. Note that $f_n \in \mathcal{B}_1(\tau')$ for every n, and the Baire class 1 functions are closed under uniform limits [7, 24.4], hence $\tau' \in T_{f,\xi}$. Then by Proposition 3.33 and Lemma 5.13 we have

$$\beta^*_\xi(f) \leq \beta_{\tau'}(f) \leq \sup_n \beta_{\tau'}(f_n) \leq \sup_n \beta_{\tau_n}(f_n) = \sup_n \beta^*_\xi(f_n).$$

□

Proposition 5.22. *If f_n, f are Baire class ξ functions and $f_n \to f$ uniformly then $\alpha^*_\xi(f) \lesssim \sup_n \alpha^*_\xi(f_n)$ and $\gamma^*_\xi(f) \lesssim \sup_n \gamma^*_\xi(f_n)$.*

Proof. Repeat the previous argument but apply Proposition 3.42 and Proposition 3.34 instead of Proposition 3.33. □

6. Uniqueness of the ranks

As we have seen, the natural unbounded ranks defined on the Baire class ξ functions essentially coincide on the bounded functions. Now we will formulate a general theorem which states that if a rank on the bounded functions has certain natural properties then it must agree with the ranks defined above. Because of some not completely clear technical difficulties we only work out the details in the Baire class 1 case.

The main reason why we treat this result separately and did not use it to prove that the ranks considered so far all agree for bounded functions is the following. So far, formally, a rank was simply a map defined on a set of functions. Now we slightly modify this concept: in this section a rank will be a family of maps $\rho = \{\rho^{(X,\tau)}\}_{(X,\tau) \text{ Polish}}$, where $\rho^{(X,\tau)}$ is a rank on the Baire class 1 functions defined on the Polish space (X,τ). However, since there is no danger of confusion, we will abuse notation and will simply continue to use ρ. Notice that the ranks α, β and γ can naturally be viewed this way.

Theorem 6.1. *Let ρ be a rank on the bounded Baire class 1 functions. Suppose that ρ has the following properties for every $A \in \mathbf{\Delta}^0_2$ and Baire class 1 functions f and f_n:*

(1) $\rho(\chi_A) \approx \alpha_1(A, A^c)$
$(\approx \alpha(A, A^c) \approx \alpha(\chi_A) \approx \beta(\chi_A) \approx \gamma(\chi_A)$*, that is, the rank of A is essentially its complexity in the difference hierarchy),*
(2) *ρ is essentially linear,*
(3) *if $f_n \to f$ uniformly then $\rho(f) \lesssim \sup_n \rho(f_n)$,*
(4) *if $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function then $\rho(h \circ f) \lesssim \rho(f)$,*
(5) *if f is defined on the Polish space X and $Y \subset X$ is Polish (or equivalently, $\mathbf{\Pi}^0_2(X)$, see e.g.* [7, 3.11]*) then $\rho(f|_Y) \lesssim \rho(f)$.*

Then $\rho \approx \alpha$ for bounded Baire class 1 functions.

Property (5) is probably the most ad hoc among the conditions, however it is easy to see that it holds for ranks α, β and γ:

Lemma 6.2. *Let X, Y be Polish spaces with $Y \subset X$ and f be a bounded Baire class 1 function on X. Then $\alpha(f|_Y) \lesssim \alpha(f)$, and hence similarly for β and γ.*

Proof. Using Corollary 3.14, it is enough to prove the lemma for α_1. By the definition of the rank α_1, if $p < q$ are rational numbers then there exists a $\mathbf{\Delta}^0_2(X)$ set A so that $\alpha_1(A, A^c) \leq \alpha_1(f)$ and A separates $\{f \leq p\}$ and $\{f \geq q\}$. Clearly, $A \cap Y$ separates the sets $\{f|_Y \leq p\}$ and $\{f|_Y \geq q\}$. So it is enough to show that $\alpha_{1,Y}(A \cap Y, A^c \cap Y) \leq \alpha_1(A, A^c)$.

Now, there exists a sequence of closed sets $(F_\eta)_{\eta<\alpha_1(A,A^c)}$ so that

$$A = \bigcup_{\substack{\eta<\alpha_1(A,A^c)\\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1}).$$

But the sets $(F_\eta \cap Y)_{\eta<\alpha_1(A,A^c)}$ witness that $\alpha_{1,Y}(A \cap Y, A^c \cap Y) \leq \alpha_1(A, A^c)$, so we are done. □

Proof of Theorem 6.1. We split the proof of the theorem into two easy lemmas.

Lemma 6.3. *If $f = \sum_{i=1}^n c_i \chi_{A_i}$ where the A_i's are disjoint $\mathbf{\Delta}^0_2$ sets covering the underlying space X and the c_i's are distinct then $\rho(f) \approx \alpha(f)$.*

Proof. By the essential linearity of ρ clearly

$$\rho(f) \lesssim \max_i \rho(\chi_{A_i}).$$

Now let $0 \leq j \leq n$ be fixed and $h : \mathbb{R} \to \mathbb{R}$ be Lipschitz so that $h(c_i) = 0$ for $i \neq j$ and $h(c_j) = 1$. Then

$$\rho(\chi_{A_j}) = \rho(h \circ f) \lesssim \rho(f)$$

by Property (4), so we have that

$$\rho(f) \approx \max_i \rho(\chi_{A_i}).$$

Using Corollary 3.39 and Property (1) we obtain that α and ρ essentially agree on step functions. □

Now let f be an arbitrary bounded Baire class 1 function. Then by Lemma 3.40 and Proposition 3.42 there exists a sequence of step functions f_n converging uniformly to f so that $\alpha(f) \approx \sup_n \alpha(f_n)$. Hence, by Property (3) and the previous lemma,

$$\rho(f) \lesssim \sup_n \rho(f_n) \approx \sup_n \alpha(f_n) \approx \alpha(f).$$

Hence, interchanging the role of α and ρ in the above argument, in order to prove $\rho(f) \approx \alpha(f)$ it is enough to construct a sequence f_n of step functions converging uniformly to f so that

$$\sup_n \rho(f_n) \lesssim \rho(f). \tag{6.1}$$

The construction goes similarly to that of Lemma 3.40, but we need an additional step.

Lemma 6.4. *Suppose that f is a bounded Baire class 1 function on the Polish space X and $p, q \in \mathbb{R}$ with $p < q$. Then there exists a set $H \in \mathbf{\Delta}^0_2(X)$ so that $\rho(\chi_H) \lesssim \rho(f)$ and H separates the sets $\{f \leq p\}$ and $\{f \geq q\}$.*

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be Lipschitz so that $h|_{(-\infty,p]} \equiv 0$ and $h|_{[q,\infty)} \equiv 1$. Let $f_1 = h \circ f$, Property (4) ensures that

$$\rho(f_1) \lesssim \rho(f). \tag{6.2}$$

Let $Y = \{f \leq p\} \cup \{f \geq q\}$ and $f_2 = f_1|_Y$. Clearly, f_2 is a step function on the Polish space Y (note that Y is $\mathbf{\Pi}^0_2(X)$), hence by the previous lemma and Property (5) we obtain

$$\alpha(f_2) \approx \rho(f_2) \lesssim \rho(f_1). \tag{6.3}$$

In particular, $\{f_2 \leq 0\}$ and $\{f_2 \geq 1\}$ can be separated by a $\mathbf{\Delta}^0_2(Y)$ set H' so that

$$H' = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F'_\eta \setminus F'_{\eta+1})$$

for some $F'_\eta \in \mathbf{\Pi}^0_1(Y)$ and

$$\lambda \lesssim \alpha(f_2), \tag{6.4}$$

using Corollary 3.14.

Now let F_η be the closure of F'_η in X and

$$H = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1}).$$

Then H is a $\mathbf{\Delta}^0_2(X)$ set, and by Property (1), Corollary 3.14, (6.4), (6.3) and (6.2) we obtain

$$\rho(\chi_H) \approx \alpha(\chi_H) \leq \lambda \lesssim \alpha(f_2) \approx \rho(f_2) \lesssim \rho(f_1) \lesssim \rho(f).$$

Moreover,

$$H \cap Y = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \cap F^c_{\eta+1} \cap Y) = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F'_\eta \cap F'^c_{\eta+1} \cap Y) = H' \cap Y.$$

Since H' separates $\{f_2 \leq 0\}$ and $\{f_2 \geq 1\}$, and it is easy to see that $\{f \leq p\} \subset \{f_2 \leq 0\} \subset Y$ and analogously for $\{f \geq q\}$, we obtain that H separates $\{f \leq p\}$ and $\{f \geq q\}$, which completes the proof. $\square$

Now we complete the proof by constructing a sequence f_n converging uniformly to f and satisfying (6.1). We basically repeat the proof of Lemma 3.40. Let $p_{n,k} = k/2^n$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ so that $\inf(f) \leq p_{n,k} \leq \sup(f)$. By the boundedness of f there are just finitely many $p_{n,k}$'s. The level sets $\{f \leq p_{n,k}\}$ and $\{f \geq p_{n,k+1}\}$ are disjoint $\mathbf{\Pi}^0_2$ sets, hence by the previous lemma they can be separated by a $H_{n,k} \in \mathbf{\Delta}^0_2$ so that $\rho(\chi_{H_{n,k}}) \lesssim \rho(f)$. Set

$$f_n = \sum_k p_{n,k} \cdot (\chi_{H_{n,k+1}} - \chi_{H_{n,k}}).$$

Clearly, $f_n \to f$ uniformly. Now, for every n

$$\rho(f_n) = \rho\left(\sum_k p_{n,k} \cdot (\chi_{H_{n,k+1}} - \chi_{H_{n,k}})\right) \lesssim \max_k \rho(\chi_{H_{n,k}}) \lesssim \rho(f)$$

by the essential linearity of ρ, which finishes the proof of the theorem. $\square$

It is not hard to see that if the range of our functions is 2^ω instead of $\mathbb{R}$ (or any other zero dimensional linearly ordered Polish space) then we can drop Property (5) in Theorem 6.1.

Question 6.5. *Does there exist a rank ρ with Properties $(1) - (4)$, so that $\rho \not\approx \alpha$?*

Now we very briefly discuss the Baire class ξ case. It is not hard to check that if the family of ranks is defined not only on functions on the Polish spaces, but also on functions on all subsets (or just Borel or $\mathbf{\Pi}^0_{\xi+1}$ subsets) of Polish spaces, and Property (5) is modified accordingly, then a result analogous to Theorem 6.1 holds. However, the following question, where the ranks are only defined on functions on the Polish spaces is more natural.

Question 6.6. *Let ρ be rank on the bounded Baire class ξ functions (defined on Polish spaces). Suppose that ρ has the following properties:*

1. *if $A \in \mathbf{\Delta}^0_{\xi+1}(X)$ then $\rho(\chi_A) \approx \alpha_\xi(\chi_A)$,*
2. *ρ is essentially linear,*
3. *if $f_n \to f$ uniformly then $\rho(f) \lesssim \sup_n \rho(f_n)$,*
4. *if $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function then $\rho(h \circ f) \lesssim \rho(f)$,*
5. *if $H \in \mathbf{\Pi}^0_2(X)$ then $\rho(f|_H) \lesssim \rho(f)$.*

Does this imply that $\rho \approx \alpha$ for bounded Baire class ξ functions?

7. Conclusion

First we answered Question 1.7 affirmatively by showing that the underlying compact metric space in the theory of Kechris and Louveau can be replaced by an arbitrary Polish space.

Then, after proving that certain very natural attempts surprisingly result in ranks that are bounded in ω_1, we have defined three ranks on the Baire class ξ functions,

$\alpha^*_\xi \le \beta^*_\xi \le \gamma^*_\xi$, corresponding to the three ranks on the Baire class 1 functions investigated by Kechris and Louveau. All the other ranks for which we could prove unboundedness, namely α_ξ and δ_{fin} defined on the bounded Baire class ξ functions, essentially agree with α^*_ξ. (It is unclear whether α'_ξ is unbounded, see the next section of Open problems.)

If we consider the ranks of sets, i.e., the ranks of characteristic functions, or more generally, the ranks of bounded functions, then in addition $\alpha^*_\xi \approx \beta^*_\xi \approx \gamma^*_\xi$ holds, hence all ranks are essentially the same for bounded functions! We also have a general result (only spelt out in the Baire 1 case) that all ranks satisfying certain natural requirements agree on the bounded functions. Moreover, the rank of a step function $\sum_{i=1}^n c_i \chi_{A_i}$ (where the A_i's form a partition and the c_i's are distinct) is the maximum of the ranks of the χ_{A_i}'s.

We were able to prove most of the known properties of the ranks on the Baire class 1 functions for α^*_ξ, β^*_ξ and γ^*_ξ. All three ranks are translation invariant and unbounded in ω_1. The ranks β^*_ξ and γ^*_ξ are essentially linear, while α^*_ξ is not. The ranks $\alpha^*_\xi, \beta^*_\xi$ and γ^*_ξ behave nicely under uniform limits.

This may well be considered as an affirmative answer to the (slightly vague) Question 1.1. Moreover, we have the following.

Corollary 7.1. *The rank β^*_ξ (or γ^*_ξ) provides an affirmative answer to Question 1.2.*

Proof. The proofs of the requirements listed in the question can be found in

- Theorem 5.20,
- Theorem 5.9,
- Theorem 5.14,
- Theorem 5.10 (note that $1 + \eta \lesssim \eta$ for every η),

respectively. □

Then, by considering the proof of [3, Theorem 6.2] and replacing the class of Borel functions by $\mathcal{B}_\xi$, the Borel class by the rank β^*_ξ and the functions χ_{B_α} by functions supported in P_α with β^*_ξ rank at least α we obtain the following.

Corollary 7.2. *For every $2 \le \xi < \omega_1$ the solvability cardinal $sc(\mathcal{B}_\xi) \ge \omega_2$, hence under the Continuum Hypothesis $sc(\mathcal{B}_\xi) = \omega_2 = (2^\omega)^+$.*

8. Open problems

In this last section we collect the open problems of the paper.

Throughout the paper we almost always considered only the relations $\approx$ and $\lesssim$. It would be interesting to know which statements remain true using $=$ and $\le$ instead.

Question 8.1. *Let ρ and ρ' be two of the ranks defined in this paper for which $\rho \lesssim \rho'$ holds. Is it true that $\rho \le \rho'$?*

We have shown in Theorem 4.8 that if $1 \le \xi < \omega_1$ and f is a *characteristic* Baire class ξ function then the linearized separation rank $\alpha'_\xi(f) \le 2$.

Question 8.2. *Is the linearized separation rank α'_ξ unbounded in ω_1 for the Baire class ξ functions?*

Actually, we do not even know the answer in the case of $\xi = 1$.

The following question is very closely related to this.

Question 8.3. *Let $1 \leq \xi < \omega_1$ and let f_n and f be Baire class ξ functions such that $f_n \to f$ uniformly. Does this imply that $\alpha'_\xi(f) \lesssim \sup_n \alpha'_\xi(f_n)$?*

As mentioned above, an affirmative answer to this question would provide a negative answer to the previous one.

Recall that a rank ρ is *essentially multiplicative* if $\rho(f \cdot g) \lesssim \max\{\rho(f), \rho(g)\}$ for every f and g. Remarks 3.31 and 5.15 indicate that the ranks β, γ, β^*_ξ and γ^*_ξ are essentially multiplicative on the *bounded* functions from the appropriate Baire classes.

Question 8.4. *Let $1 \leq \xi < \omega_1$. Are the ranks β, γ, β^*_ξ and γ^*_ξ essentially multiplicative?*

We have shown in Theorem 4.12 that the limit ranks are bounded by ω, but do not know whether this is optimal.

Question 8.5. *Is there an $n \in \omega$ such that $\overline{\gamma}_2 \leq n$? If yes, which is the smallest such n?*

We have seen that for every $1 \leq \xi < \omega_1$ we have $\beta^*_\xi \approx \gamma^*_\xi$ on the bounded Baire class ξ functions (even on non-compact Polish spaces), but $\alpha^*_\xi \not\approx \beta^*_\xi$ for arbitrary Baire class ξ functions. So the following question is natural.

Question 8.6. *Let $1 \leq \xi < \omega_1$. Does $\beta^*_\xi \approx \gamma^*_\xi$ hold for arbitrary Baire class ξ functions?*

We believe that an affirmative answer might help extend Theorem 6.1 to the unbounded case.

Our next questions concern the uniqueness of ranks.

Question 8.7. *Does there exist a rank ρ with Properties* $(1)-(4)$ *of Theorem 6.1 so that $\rho \not\approx \alpha$ on bounded Baire class* 1 *functions?*

Question 8.8. *Let ρ be rank on the bounded Baire class ξ functions (defined on Polish spaces). Suppose that ρ has the following properties:*

(1) *if $A \in \mathbf{\Delta}^0_{\xi+1}(X)$ then $\rho(\chi_A) \approx \alpha_\xi(\chi_A)$,*
(2) *ρ is essentially linear,*
(3) *if $f_n \to f$ uniformly then $\rho(f) \lesssim \sup_n \rho(f_n)$,*
(4) *if $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function then $\rho(h \circ f) \lesssim \rho(f)$,*
(5) *if $H \in \mathbf{\Pi}^0_2(X)$ then $\rho(f|_H) \lesssim \rho(f)$.*

Does this imply that $\rho \approx \alpha$ for bounded Baire class ξ functions?

Question 8.9. *The fourth chapter of* [8] *discusses two more ranks on the bounded Baire class 1 functions that turn out to be essentially equivalent to α, β and γ. Is there a well-behaved generalization of these theories to the Baire class ξ case?*

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