A covering theorem and the random-indestructibility of the density zero ideal

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Abstract

The main goal of this note is to prove the following theorem. If A_n is a sequence of measurable sets in a σ-finite measure space $(X, \mathcal{A}, \mu)$ that covers μ-a.e. $x \in X$ infinitely many times, then there exists a sequence of integers n_i of density zero so that A_{n_i} still covers μ-a.e. $x \in X$ infinitely many times. The proof is a probabilistic construction.

As an application we give a simple direct proof of the known theorem that the ideal of density zero subsets of the natural numbers is random-indestructible, that is, random forcing does not add a co-infinite set of naturals that almost contains every ground model density zero set. This answers a question of B. Farkas.

1 Introduction

Maximal almost disjoint (MAD) families of subsets of the naturals play a central role in set theory. (Two sets are *almost disjoint* if there intersection is finite.)

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A fundamental question is whether MAD families remain maximal in forcing extensions. This is often studied in a little more generality as follows. For a MAD family $\mathcal{M}$ let $\mathcal{I}_{\mathcal{M}}$ be the ideal of sets that can be almost contained in a finite union of members of $\mathcal{M}$. (*Almost contained* means that only finitely many elements are not contained.) Then it is easy to see that $\mathcal{M}$ remains MAD in a forcing extension if and only if there is no co-infinite set of naturals in the extension that almost contains every (ground model) member of $\mathcal{I}_{\mathcal{M}}$. Hence the following definition is natural.

Definition 1.1 An ideal $\mathcal{I}$ of subsets of the naturals is called *tall* if there is no co-infinite set that almost contains every member of $\mathcal{I}$. Let $\mathcal{I}$ be a tall ideal and $\mathbb{P}$ be a forcing notion. We say that $\mathcal{I}$ is $\mathbb{P}$-*indestructible* if $\mathcal{I}$ remains tall after forcing with $\mathbb{P}$.

This notion is thoroughly investigated for various well-known ideals and forcing notions, for instance Hernández-Hernández and Hrušák proved that the ideal of density zero subsets (see. Definition 2.1) of the natural numbers is random-indestructible. (Indeed, just combine [3, Thm 3.14], which is a result of Brendle and Yatabe, and [3, Thm 3.4].) B. Farkas asked if there is a simple and direct proof of this fact. In this note we provide such a proof.

This proof actually led us to a covering theorem (Thm. 2.5) which we find very interesting in its own right from the measure theory point of view. First we prove this theorem in Section 2 by a probabilistic argument, then we apply it in Section 3 to reprove that the density zero ideal is random-indestructible (Corollary 3.3), and finally we pose some problems in Section 4.

2 A covering theorem

Cardinality of a set A is denoted by $|A|$.

Definition 2.1 A set $A \subset \mathbb{N}$ is of *density zero* if $\lim_{n\to\infty} \frac{|A\cap\{0,\dots,n-1\}|}{n} = 0$. The ideal of density zero sets is denoted by $\mathcal{Z}$.

$A \subset^* B$ means that B *almost contains* A, that is, $A \setminus B$ is finite. The following is well-known.

Fact 2.2 *$\mathcal{Z}$ is a P-ideal, that is, for every sequence $Z_n \in \mathcal{Z}$ there exists $Z \in \mathcal{Z}$ so that $Z_n \subset^* Z$ for every $n \in \mathbb{N}$.*

Lemma 2.3 *Let $(X, \mathcal{A}, \mu)$ be a measure space of σ-finite measure, and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of measurable sets. Suppose that there exists $0 = N_0 < N_1 < N_2 < \dots$ so that $A_{N_{k-1}}, \dots, A_{N_k-1}$ is a cover of X for every $k \in \mathbb{N}^+$, and also that k divides $N_k - N_{k-1}$ for every $k \in \mathbb{N}^+$. Then there exists a set $Z \in \mathcal{Z}$ so that $\{A_n\}_{n\in Z}$ covers μ-a.e. every $x \in X$ infinitely many times.*

Proof. Write $\{N_{k-1}, \dots, N_k - 1\} = W_0^k \cup \dots \cup W_{k-1}^k$, where the W_i^k's are the k disjoint arithmetic progressions of difference k. Let $\{\xi_k\}_{k \in \mathbb{N}^+}$ be a sequence of independent random variables so that ξ_k is uniformly distributed on $\{0, \dots, k-1\}$. Define

$$Z = \cup_{k \in \mathbb{N}^+} W_{\xi_k}^k.$$

It is easy to see that $Z \in \mathcal{Z}$. Hence it suffices to show that with probability 1 μ-a.e. $x \in X$ is covered infinitely many times by $\{A_n\}_{n \in Z}$.

Let us now fix an $x \in X$. Let E_k be the event $\{x \in \cup_{n \in W_{\xi_k}^k} A_n\}$, that is, x is covered by the set chosen in the k^{th} block. As the k^{th} block is a cover of X, $Pr(E_k) \geq \frac{1}{k}$, so $\sum_{k \in \mathbb{N}^+} Pr(E_k) = \infty$. Moreover, the events $\{E_k\}_{k \in \mathbb{N}^+}$ are independent. Hence by the second Borel-Cantelli Lemma $Pr(\text{Infinitely many of the } E_k\text{'s occur}) = 1$. So every fixed x is covered infinitely many times with probability 1, but then by the Fubini theorem with probability 1 μ-a.e. x is covered infinitely many times, and we are done. (To be more precise, let $(\Omega, \mathcal{S}, Pr)$ be the probability measure space, then $Z(\omega) = \cup_{k \in \mathbb{N}} W_{\xi_k(\omega)}^k$. Since the sets $\{(x, \omega) : x \in A_n\}$ and $\{(x, \omega) : \xi_k(\omega) = n\}$ are clearly $\mathcal{A} \times \mathcal{S}$-measurable, it is straightforward to show that

$$\{(x, \omega) : x \text{ is covered infinitely many times by } \{A_n\}_{n \in Z(\omega)}\} \subset X \times \Omega$$

is $\mathcal{A} \times \mathcal{S}$-measurable, and hence Fubini applies.) $\square$

Lemma 2.4 *Let $(X, \mathcal{A}, \mu)$ be a measure space of finite measure, and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of measurable sets that covers μ-a.e. every $x \in X$ infinitely many times. Then there exists a set $Z \in \mathcal{Z}$ so that $\{A_n\}_{n \in Z}$ still covers μ-a.e. every $x \in X$ infinitely many times.*

Proof. Let $\varepsilon > 0$ be arbitrary and set $N_0 = 0$. By the continuity of measures, there exists N_1 so that $\mu(X \setminus (A_{N_0} \cup \dots \cup A_{N_1 - 1})) \leq \frac{\varepsilon}{2}$. Since $\{A_n\}_{n \geq N_1}$ still covers μ-a.e. $x \in X$ infinitely many times, we can continue this procedure, and recursively define $0 = N_0 < N_1 < N_2 < \dots$ so that $\mu(X \setminus (A_{N_{k-1}} \cup \dots \cup A_{N_k - 1})) \leq \frac{\varepsilon}{2^k}$ for every $k \in \mathbb{N}^+$. We can also assume (by choosing larger N_k's at each step) that k divides $N_k - N_{k-1}$ for every $k \in \mathbb{N}^+$.

Let $X_\varepsilon = \cap_{k \in \mathbb{N}^+} (A_{N_{k-1}} \cup \dots \cup A_{N_k - 1})$, then $\mu(X \setminus X_\varepsilon) \leq \varepsilon$. Let us restrict $\mathcal{A}$, the A_n's and μ to X_ε, and apply the previous lemma with this setup to obtain Z_ε.

Let us now consider $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \dots$, then for every $m \in \mathbb{N}^+$ every $x \in X_{\frac{1}{m}}$ is covered infinitely many times by $\{A_n\}_{n \in Z_{\frac{1}{m}}}$. Since $\mathcal{Z}$ is a P-ideal, there exists a $Z \in \mathcal{Z}$ such that $Z_{\frac{1}{m}} \subset^* Z$ for every m. Hence for every $m \in \mathbb{N}^+$ every $x \in X_{\frac{1}{m}}$ is covered infinitely many times by $\{A_n\}_{n \in Z}$. But then we are done, since μ-a.e. $x \in X$ is in $\cup_m X_{\frac{1}{m}}$. $\square$

Theorem 2.5 *Let $(X, \mathcal{A}, \mu)$ be a measure space of σ-finite measure, and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of measurable sets that covers μ-a.e. every $x \in X$ infinitely many times. Then there exists a set $Z \subset \mathbb{N}$ of density zero so that $\{A_n\}_{n \in Z}$ still covers μ-a.e. every $x \in X$ infinitely many times.*

Proof. Write $X = \cup X_m$, where each X_m is of finite measure. For each X_m obtain Z_m by the previous lemma. Then a $Z \in \mathcal{Z}$ such that $Z_m \subset^* Z$ for every m clearly works. $\square$

The following example shows that the purely topological analogue of Theorem 2.5 is false.

Example 2.6 *There exists a sequence U_n of clopen sets covering every point of the Cantor space infinitely many times so that for every $Z \in \mathcal{Z}$ there exists a point covered only finitely many times by $\{U_n : n \in Z\}$.*

Proof. By an easy recursion we can define a sequence U_n of clopen subsets of the Cantor set C and a sequence of naturals $0 = N_0 < N_1 < \dots$ with the following properties.

1. $U_{N_{k-1}}, \dots, U_{N_k - 1}$ (called a 'block') is a disjoint cover of C,
2. every block is a refinement of the previous one,
3. if U_n is in the k^{th} block and is partitioned into $U_t, \dots, U_s$ in the $k+1^{st}$ block (called the 'immediate successors of U_n') then $s \geq 2t$.

Let $Z \in \mathcal{Z}$ be given, and let n_0 be so that $\frac{|Z \cap \{0,\dots,n-1\}|}{n} < \frac{1}{2}$ for every $n \geq n_0$. By 3. $\{U_n : n \in Z\}$ cannot contain all immediate successors of any U_m above n_0. Therefore, starting at a far enough block, we can recursively pick a U_{n_i} from each block so that $n_i \notin Z$ for every i, and $\{U_{n_i}\}_{i \in \mathbb{N}}$ is a nested sequence of clopen sets. But then the intersection of this sequence is only covered finitely many times by $\{U_n : n \in Z\}$. $\square$

Remark 2.7 We can 'embed' this example to any topological space containing a copy of the Cantor set by just adding the complement of the Cantor set to all U_n's. Of course, the new U_n's will only be open, not clopen.

3 An application: The density zero ideal is random-indestructible

In this section we give a simple and direct proof of the random-indestructibility of $\mathcal{Z}$, which was first proved in [3].

$[\mathbb{N}]^\omega$ denotes the set of infinite subsets of $\mathbb{N}$. It carries a natural Polish space topology where the sub-basic open sets are the sets of the form $[n] = \{A \in [\mathbb{N}]^\omega : n \in A\}$ and their complements. Let λ denote Lebesgue measure.

Lemma 3.1 *For every Borel function $f : \mathbb{R} \to [\mathbb{N}]^\omega$ there exists a set $Z \in \mathcal{Z}$ such that $f(x) \cap Z$ is infinite for λ-a.e. $x \in \mathbb{R}$.*

Proof. Let $A_n = f^{-1}([n])$, then A_n is clearly Borel, hence Lebesgue measurable. For every $x \in \mathbb{R}$

$$x \in A_n \iff x \in f^{-1}([n]) \iff f(x) \in [n] \iff n \in f(x). \tag{3.1}$$

Since every $f(x)$ is infinite, (3.1) yields that every $x \in \mathbb{R}$ is covered by infinitely many A_n's. By Theorem 2.5 there exists a $Z \in \mathcal{Z}$ such that for λ-a.e. $x \in \mathbb{R}$ we have $x \in A_n$ for infinitely many $n \in Z$. But then by (3.1) for λ-a.e. $x \in \mathbb{R}$ we have $n \in f(x)$ for infinitely many $n \in Z$, so $f(x) \cap Z$ is infinite. □

Recall that *random forcing* is $\mathbb{B} = \{p \subset \mathbb{R} : p \text{ is Borel}, \lambda(p) > 0\}$ ordered by inclusion. The *random real* r is defined by $\{r\} = \cap_{p \in G}\, p$, where G is the generic filter. For the terminology and basic facts concerning random forcing consult e.g. [5], [4], [1], or [6]. In particular, we will assume familiarity with coding of Borel sets and functions, and will freely use the same symbol for all versions of a Borel set or function. The following fact is well-known and easy to prove.

Fact 3.2 *Let $B \subset \mathbb{R}$ be Borel. Then $p \Vdash$ "$r \in B$" iffy $\lambda(p \setminus B) = 0$.*

Corollary 3.3 *The ideal of density zero subsets of the natural numbers is random-indestructible, that is, random forcing does not add a co-infinite set of naturals that almost contains every ground model density zero set.*

Proof. For a Borel function $f : \mathbb{R} \to [\mathbb{N}]^\omega$ and a set $Z \in \mathcal{Z}$ let

$$B_{f,Z} = \{x \in \mathbb{R} : f(x) \cap Z \text{ is infinite}\},$$

then by the previous lemma for every f there is a Z so that $B_{f,Z}$ is of full measure. By Fact 3.2 for every f there is a Z so that $1_\mathbb{B} \Vdash$ "$f(r) \cap Z$ is infinite". Hence for every f $1_\mathbb{B} \Vdash$ "$\exists Z \in \mathcal{Z} \cap V$ so that $f(r) \cap Z$ is infinite". But every $y \in [\mathbb{N}]^\omega \cap V[r]$ is of the form $f(r)$ for some ground model (coded) Borel function $f : \mathbb{R} \to [\mathbb{N}]^\omega$, so we obtain that for every $y \in [\mathbb{N}]^\omega \cap V[r]$ $1_\mathbb{B} \Vdash$ "$\exists Z \in \mathcal{Z} \cap V$ so that $y \cap Z$ is infinite". Therefore $1_\mathbb{B} \Vdash$ "$\forall y \in [\mathbb{N}]^\omega \exists Z \in \mathcal{Z} \cap V$ so that $y \cap Z$ is infinite", and setting $x = \mathbb{N} \setminus y$ yields $1_\mathbb{B} \Vdash$ "$\forall x \subset \omega$ co-infinite $\exists Z \in \mathcal{Z} \cap V$ so that $Z \not\subset^* x$", so we are done. □

Remark 3.4 Clearly, $\mathcal{Z}$ is also $\mathbb{B}(\kappa)$-indestructible, since every new real is already added by sub-poset isomorphic to $\mathbb{B}$. ($\mathbb{B}(\kappa)$ is the usual poset for adding κ many random reals by the measure algebra on 2^κ.)

4 Problems

There are numerous natural directions in which one can ask questions in light of Corollary 3.3 and Theorem 2.5. As for the former one, one can consult e.g. [2] and the references therein. As for the latter one, it would be interesting to investigate what happens if we replace the density zero ideal by another well-known one, or if we replace the measure setup by the Baire category analogue, or if we consider non-negative functions (summing up to infinity a.e.) instead of sets, or even if we consider κ-fold covers and ideals on κ.

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