

THE STRUCTURE OF RANDOM HOMEOMORPHISMS

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ABSTRACT. In order to understand the structure of the “typical” element of a homeomorphism group, one has to study how large the conjugacy classes of the group are. When typical means generic in the sense of Baire category, this is well understood, see e.g. the works of Glasner and Weiss, and Kechris and Rosendal. Following Dougherty and Mycielski we investigate the measure theoretic dual of this problem, using Christensen’s notion of Haar null sets. When typical means random, that is, almost every with respect to this notion of Haar null sets, the behaviour of the homeomorphisms is entirely different from the generic case. For $\text{Homeo}^+([0, 1])$ we describe the non-Haar null conjugacy classes and also show that their union is co-Haar null, for $\text{Homeo}^+(\mathbb{S}^1)$ we describe the non-Haar null conjugacy classes, and for $\mathcal{U}(\ell^2)$ we show that, apart from the classes of the multishifts, all conjugacy classes are Haar null. As an application we affirmatively answer the question whether these groups can be written as the union of a meagre and a Haar null set.

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1. INTRODUCTION

The study of generic elements of Polish groups is a flourishing field with a large number of applications, see the works of Kechris and Rosendal [18], Truss [24], and Glasner and Weiss [13] among others. It is natural to ask whether there exist measure theoretic analogues of these results. Unfortunately, on non-locally compact

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groups there is no natural invariant σ -finite measure. However, a generalisation of the ideal of measure zero sets can be defined in every Polish group as follows:

Definition 1.1 (Christensen, [4]). Let G be a Polish group and $B \subset G$ be Borel. We say that B is *Haar null* if there exists a Borel probability measure μ on G such that for every $g, h \in G$ we have $\mu(gBh) = 0$. An arbitrary set S is called Haar null if $S \subset B$ for some Borel Haar null set B .

It is known that the collection of Haar null sets forms a σ -ideal in every Polish group and it coincides with the ideal of measure zero sets in locally compact groups with respect to every left (or equivalently right) Haar measure. Using this definition, it makes sense to talk about the properties of random elements of a Polish group. A property P of elements of a Polish group G is said to *hold almost surely* or *almost every element of G has property P* if the set $\{g \in G : g \text{ has property } P\}$ is co-Haar null.

Since we are primarily interested in homeomorphism and automorphism groups, and in such groups conjugate elements can be considered isomorphic, we are only interested in the conjugacy invariant properties of the elements of our Polish groups. Hence, in order to describe the random element, one must give a complete description of the size of the conjugacy classes with respect to the Haar null ideal. The investigation of this question has been started by Dougherty and Mycielski [9] in the permutation group of a countably infinite set, S_∞ . If $f \in S_\infty$ and a is an element of the underlying set then the set $\{f^k(a) : k \in \mathbb{Z}\}$ is called the *orbit of a (under f)*, while the cardinality of this set is called *orbit length*. Thus, each $f \in S_\infty$ has a collection of orbits (associated to the elements of the underlying set). It is easy to show that two elements of S_∞ are conjugate if and only if they have the same (possibly infinite) number of orbits for each possible orbit length.

Theorem 1.2 (Dougherty, Mycielski, [9]). *Almost every element of S_∞ has infinitely many infinite orbits and only finitely many finite ones.*

Therefore, almost all permutations belong to the union of a countable set of conjugacy classes.

Theorem 1.3 (Dougherty, Mycielski, [9]). *All of these countably many conjugacy classes are non-Haar null.*

Thus, the above theorems give a complete description of the non-Haar null conjugacy classes and the (conjugacy invariant) properties of a random element. In a previous paper [7], we studied the conjugacy classes of automorphism groups of countable structures. We generalised the construction of Dougherty and Mycielski to show that if a closed subgroup G of S_∞ has the *finite algebraic closure property* (that is, for every finite $S \subset \mathbb{N}$, the set $\{b \in \mathbb{N} : |G_{(S)}(b)| < \infty\}$ is finite, where $G_{(S)}(b)$ is the orbit of $b \in \mathbb{N}$ under the stabilizer $G_{(S)}$ of S) then almost all elements of G have infinitely many infinite and finitely many finite orbits. We gave descriptions of the non-Haar null conjugacy classes in $\text{Aut}(\mathbb{Q}, <)$, the group of order-preserving automorphisms of the rationals and in $\text{Aut}(\mathcal{R})$, the automorphism group of the random graph. A surprising feature of these descriptions is that in both cases there are continuum many non-Haar null conjugacy classes, hence in what follows the results stating that in certain groups there are only countably many non-Haar null conjugacy classes is non-trivial.

After studying these automorphism groups, the next natural step is to consider homeomorphism groups. Indeed, the division line between these two types of groups is not at all strict, since e.g. it is well known that the automorphism group of the countable atomless Boolean algebra is actually isomorphic to the homeomorphism group of the Cantor set. Our second motivation is that the structure of the generic elements of homeomorphism groups already has a huge literature (see e.g. the monograph by Akin, Hurley and Kennedy [1], or the survey of Glasner and Weiss [14]) partly because these homeomorphism groups are the central objects in topological dynamics. It is also worth noting that the existence of a generic conjugacy class in the homeomorphism group of the Cantor set was among the main motivations for the seminal paper of Kechris and Rosendal. Therefore it is natural to look at the measure theoretic dual of these questions. The aim of the current paper is to initiate this project by characterising the non-Haar null conjugacy classes in certain important homeomorphism groups.

Let us also mention that there have already been several alternative attempts at investigating random homeomorphisms (with respect to certain natural probability measures), see e.g. Graf, Mauldin and Williams [15], and Downarowicz, Mauldin and Warnock [10]. Our results also imply two theorems of Shi [22] concerning the set of fixed-points of the random homeomorphism of $[0, 1]$.

We now describe our main results.

Let $\text{Homeo}^+([0, 1])$ denote the space of increasing homeomorphisms of the unit interval $[0, 1]$ with the topology of uniform convergence. For a homeomorphism $f \in \text{Homeo}^+([0, 1])$ we let $\text{Fix}(f) = \{x \in [0, 1] : f(x) = x\}$. Our theorem concerning $\text{Homeo}^+([0, 1])$ is the following, which we prove in Section 3.

Theorem 3.2. *The conjugacy class of $f \in \text{Homeo}^+([0, 1])$ is non-Haar null if and only if $\text{Fix}(f)$ has no limit point in the open interval $(0, 1)$, and if $x_0 \in \text{Fix}(f) \cap (0, 1)$ then $f(x) - x$ does not have a local extremum point at x_0 . Moreover, the union of the non-Haar null conjugacy classes is co-Haar null.*

Corollary 1.4. *There are only countably many non-Haar null conjugacy classes in $\text{Homeo}^+([0, 1])$.*

As the definition of Haar null sets suggests, to obtain the "if" part of Theorem 3.2 one has to construct a probability measure with certain properties. Our measure is somewhat similar to some of the measures constructed by Graf, Mauldin and Williams [15]. However, a substantial difference between the two approaches is that proving that a property P holds almost surely (in the sense defined in the Introduction) requires to prove not only that the set $\{f \in \text{Homeo}^+([0, 1]) : f \text{ has property } P\}$ has measure one (with respect to a certain probability measure) but that all of its translates have this property as well. This typically requires much more work.

As an application of the above characterisation, we answer the following question of the first author for the group $\text{Homeo}^+([0, 1])$.

Question 1.5. *Suppose that G is an uncountable Polish group. Can it be written as the union of a meagre and a Haar null set?*

Cohen and Kallman [5] investigated the above question and developed a technique to find Haar null-meagre decompositions. However, their method does not work for the group $\text{Homeo}^+([0, 1])$ as they have also noted. With the help of Theorem 3.2, we have the following corollary.

Corollary 6.1. *The group $\text{Homeo}^+([0, 1])$ can be partitioned into a Haar null and a meagre set.*

Remark 1.6. In [7] we showed that the automorphism group of the countable atomless Boolean algebra can be partitioned into a Haar null and a meagre set, but, as mentioned before, this group is isomorphic to the group of homeomorphisms of the Cantor set, hence we have another important homeomorphism group that can be partitioned into a Haar null and a meagre set.

We now state two results of Shi [22]. Both follow easily from our characterisation of the non-Haar null conjugacy classes of $\text{Homeo}^+([0, 1])$, Theorem 3.2.

Corollary 1.7. *For any function $q \in \text{Homeo}^+([0, 1])$, the set of functions in $\text{Homeo}^+([0, 1])$ which cross the curve $y = q(x)$ in $(0, 1)$ infinitely many times is neither Haar null, nor co-Haar null.*

Proof. Let $\mathcal{F} = \{f \in \text{Homeo}^+([0, 1]) : f(x) = x \text{ infinitely many times}\}$. Then the set in question is $q\mathcal{F}$, hence it is enough to prove that \mathcal{F} is neither Haar null, nor co-Haar null. And the latter statement follows easily from Theorem 3.2. \square

We remark here that in the paper [22], Shi used a slightly modified definition of Haar nullness. Also, instead of proving that the set in the previous corollary is not Haar null, the author proved that it is not *left* or *right Haar null*. We do not define these notions here, only note that since the set is a translate of a conjugacy invariant set, these notions coincide by Lemma 2.1.

The following corollary was also proved by Shi [22].

Corollary 1.8. *For any function $q \in \text{Homeo}^+([0, 1])$, the set of functions in $\text{Homeo}^+([0, 1])$ that are equal to q on some non-degenerate subinterval of $[0, 1]$ is Haar null.*

Proof. Notice that the set in question is a translate of $\{f \in \text{Homeo}^+([0, 1]) : f(x) = x \text{ on some non-degenerate interval}\}$, which is a countable union of Haar null sets using Theorem 3.2. \square

In Section 4 we turn to the group $\text{Homeo}^+(\mathbb{S}^1)$ of order-preserving homeomorphisms of the circle. In what follows, $\tau(f)$ denotes the rotation number of a homeomorphism $f \in \text{Homeo}^+(\mathbb{S}^1)$. If $f \in \text{Homeo}^+(\mathbb{S}^1)$ has finitely many periodic points then we call one of its periodic points x_0 of period q *crossing*, if for a lift F of f^q with $x_0 \in \text{Fix}(F)$, x_0 is not a local extremum point of $F(x) - x$. For the formal definition of these notions, see Section 4.

Theorem 4.5. *The conjugacy class of $f \in \text{Homeo}^+(\mathbb{S}^1)$ is non-Haar null if and only if $\tau(f) \in \mathbb{Q}$, f has finitely many periodic points and all of them are crossing. Every such conjugacy class necessarily contains an even number of periodic orbits, and for every rational number $0 \leq r < 1$ and positive integer k there is a unique non-Haar null conjugacy class with rotation number r containing $2k$ periodic orbits. Moreover, the union of the non-Haar null conjugacy classes and $\{f \in \text{Homeo}^+(\mathbb{S}^1) : \tau(f) \notin \mathbb{Q}\}$ is co-Haar null.*

Corollary 1.9. *There are only countably many non-Haar null conjugacy classes in $\text{Homeo}^+(\mathbb{S}^1)$.*

We could not settle the question whether the union of the non-Haar null conjugacy classes is co-Haar null, since the following question remains open.

Question 1.10. *Is the set $\{f \in \text{Homeo}^+(\mathbb{S}^1) : \tau(f) \notin \mathbb{Q}\}$ Haar null?*

Again, we could use the characterisation to answer Question 1.5 for $\text{Homeo}^+(\mathbb{S}^1)$.

Corollary 6.2. *The group $\text{Homeo}^+(\mathbb{S}^1)$ can be partitioned into a Haar null and a meagre set.*

In Section 5 we introduce basic results of spectral theory to prove the following theorem about the group $\mathcal{U}(\ell^2)$ of unitary transformations of the separable, infinite dimensional Hilbert space.

Theorem 5.3. *Let $U \in \mathcal{U}(\ell^2)$ be given with spectral measure μ_U and multiplicity function n_U . If the conjugacy class of U is non-Haar null then $\mu_U \simeq \lambda$ and n_U is constant λ -a. e., where λ denotes Lebesgue measure.*

Corollary 5.4. *In the unitary group every conjugacy class is Haar null possibly except for a countable set of classes, namely the conjugacy classes of the multishifts.*

For the group $\mathcal{U}(\ell^2)$ the fact that it can be written as the union of a meagre and a Haar null set has already been proved, see [5].

2. PRELIMINARIES AND NOTATION

Throughout the paper \mathbb{N} denotes the set of non-negative integers. Let G be an arbitrary Polish group. We call a set $H \subset G$ *conjugacy invariant* if it is the union of some conjugacy classes of G . A map φ with domain G is called *conjugacy invariant* if $\varphi(f) = \varphi(g)$ for conjugate elements $f, g \in G$. For an arbitrary function f , let $\text{Fix}(f) = \{x : f(x) = x\}$ denote the set of fixed points of f .

We will sometimes use the following well-known fact.

Lemma 2.1. *Let G be a Polish group and $B \subset G$ be a Borel set that is conjugacy invariant. If there exists a Borel probability measure μ on G such that for every $g \in G$, $\mu(gB) = 0$ then B is Haar null.*

Proof. It is enough to show that μ is a witness measure for B , that is, for every $g, h \in G$, $\mu(gBh) = 0$. But we have $gBh = gh h^{-1}Bh \subset ghB$, using at the last step that B is conjugacy invariant, hence $\mu(gBh) \leq \mu(ghB) = 0$. \square

3. HOMEOMORPHISMS OF THE INTERVAL

In this section we investigate $\text{Homeo}^+([0, 1])$, the group of increasing homeomorphisms of the unit interval $[0, 1]$ and prove Theorem 3.2. The topology on $\text{Homeo}^+([0, 1])$ is that of uniform convergence, that is generated by the metric $d(f, g) = \|f - g\| = \sup_{x \in [0, 1]} |f(x) - g(x)|$ for $f, g \in \text{Homeo}^+([0, 1])$.

With respect to this metric, we denote the open ball centred at the homeomorphism $f \in \text{Homeo}^+([0, 1])$ having radius $\delta > 0$ by $B(f, \delta)$. Note that if $f_n \rightarrow f$ uniformly for the homeomorphisms $f_n, f \in \text{Homeo}^+([0, 1])$, $n = 0, 1, \dots$, then $f_n^{-1} \rightarrow f^{-1}$ uniformly, hence a compatible metric for $\text{Homeo}^+([0, 1])$ is $d'(f, g) = \|f - g\| + \|f^{-1} - g^{-1}\|$.

Let $\text{sign} : \mathbb{R} \rightarrow \{-1, 0, 1\}$ be the sign function, i.e.,

$$\text{sign}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

For the characterisation of non-Haar null conjugacy classes of $\text{Homeo}^+([0, 1])$, we use the following well-known fact.

Lemma 3.1. *Two homeomorphisms $f, g \in \text{Homeo}^+([0, 1])$ are conjugate if and only if there exists a homeomorphism $h \in \text{Homeo}^+([0, 1])$ such that $\text{sign}(f(x) - x) = \text{sign}(g(h(x)) - h(x))$ for every $x \in [0, 1]$.*

A similar result about homeomorphisms of \mathbb{R} is shown in [20, Lemma 2.3], from which the lemma can be proved easily. Now we are ready to prove our theorem about the non-Haar null conjugacy classes of $\text{Homeo}^+([0, 1])$.

Theorem 3.2. *The conjugacy class of $f \in \text{Homeo}^+([0, 1])$ is non-Haar null if and only if $\text{Fix}(f)$ has no limit point in the open interval $(0, 1)$, and if $x_0 \in \text{Fix}(f) \cap (0, 1)$ then $f(x) - x$ does not have a local extremum point at x_0 . Moreover, the union of the non-Haar null conjugacy classes is co-Haar null.*

Proof. Let

$$\begin{aligned} \mathcal{L} &= \{f \in \text{Homeo}^+([0, 1]) : \text{Fix}(f) \text{ has no limit point in } (0, 1)\} \text{ and} \\ \mathcal{H} &= \{f \in \text{Homeo}^+([0, 1]) : \\ &\quad f(x) - x \text{ has no local extremum point at any } x_0 \in \text{Fix}(f) \cap (0, 1)\}. \end{aligned}$$

We first show that \mathcal{L} and \mathcal{H} are co-Haar null sets, which will prove the ‘‘only if’’ part of the theorem.

Claim 3.3. *\mathcal{L} is co-Haar null.*

Proof. We first prove that $\mathcal{L} \subset \text{Homeo}^+([0, 1])$ is Borel. Let

$$\begin{aligned} \mathcal{L}_{k,l,m} &= \{f \in \text{Homeo}^+([0, 1]) : \\ &\quad \exists x_1, \dots, x_l \in [1/k, 1 - 1/k] \cap \text{Fix}(f) \ (i \neq j \Rightarrow |x_i - x_j| \geq 1/m)\}. \end{aligned}$$

Now $\mathcal{L}_{k,l,m}$ is closed for every $k, l, m > 0$, since if $(f_n) \subset \mathcal{L}_{k,l,m}$ is a sequence converging uniformly to f , and $x_i^n \in [1/k, 1 - 1/k] \cap \text{Fix}(f_n)$ for $i = 1, \dots, l$ satisfies $|x_i^n - x_j^n| \geq 1/m$ for $i \neq j$, then, for an appropriate subsequence, the numbers x_i^n converge to some x_i for all i . It is easy to see that $x_i \in [1/k, 1 - 1/k] \cap \text{Fix}(f)$ and $|x_i - x_j| \geq 1/m$ for $i \neq j$. Hence $f \in \mathcal{L}_{k,l,m}$, showing that $\mathcal{L}_{k,l,m}$ is closed.

Since $f \notin \mathcal{L}$ if and only if f has infinitely many fixed points in some interval $[1/k, 1 - 1/k]$, one can easily check that

$$\mathcal{L}^c = \bigcup_{k>0} \bigcap_{l>0} \bigcup_{m>0} \mathcal{L}_{k,l,m}.$$

This shows that \mathcal{L} is Borel.

Now as \mathcal{L} is Borel and clearly conjugacy invariant, using Lemma 2.1 it is enough to find a Borel probability measure μ with $\mu(\mathcal{L}g) = 1$ for every $g \in \text{Homeo}^+([0, 1])$.

For $1/4 \leq a \leq 3/4$ let $f_a \in \text{Homeo}^+([0, 1])$ be the piecewise linear function defined by

$$(3.1) \quad f_a(x) = \begin{cases} 2xa & \text{if } 0 \leq x < \frac{1}{2}, \\ 2(1-a)x + 2a - 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

The map $\Phi : a \mapsto f_a$ is continuous from $[1/4, 3/4]$ to $\text{Homeo}^+([0, 1])$, hence the pushforward of the normalized Lebesgue measure on $[1/4, 3/4]$ is a Borel probability measure on $\text{Homeo}^+([0, 1])$. Let the pushforward be μ , i.e.,

$$(3.2) \quad \mu(\mathcal{B}) = 2\lambda(\Phi^{-1}(\mathcal{B})) = 2\lambda(\{a : f_a \in \mathcal{B}\})$$

for every Borel subset $\mathcal{B} \subset \text{Homeo}^+([0, 1])$, where λ is the Lebesgue measure.

Let $g \in \text{Homeo}^+([0, 1])$ be arbitrary, it is enough to show that $\mu(\mathcal{L}g) = 1$. It is easy to see that

$$\mu(\mathcal{L}g) = 2\lambda(\{a : \forall k(f_a(x) = g(x) \text{ for only finite many } x \text{ in } [1/k, 1 - 1/k])\}).$$

Now we proceed indirectly. Suppose that $\mu(\mathcal{L}g) < 1$, that is, the set of those a such that g intersects the graph of f_a infinitely many times in some interval $[c_a, 1 - c_a]$ is of positive (outer) measure. Hence, we can find a common $0 < c < 1/2$ such that the set of those a such that g intersects the graph of f_a infinitely many times in the interval $[c, 1 - c]$ is still of positive outer measure.

Having found this common c , we make some modifications to the functions. For a homeomorphism $h \in \text{Homeo}^+([0, 1])$, let $h^* : [c, 1 - c] \rightarrow \mathbb{R}$ be the function defined by

$$h^*(x) = \begin{cases} \frac{h(x)}{2x} & \text{if } c \leq x < \frac{1}{2}, \\ \frac{h(x)}{2(1-x)} + \frac{1-2x}{2(1-x)} & \text{if } \frac{1}{2} \leq x \leq 1 - c. \end{cases}$$

It is easy to see that $f_a^*(x) = a$ for every $a \in [1/4, 3/4]$ and $x \in [c, 1 - c]$.

Note that the maps $(x, y) \mapsto (x, \frac{y}{2x})$ and $(x, y) \mapsto (x, \frac{y}{2(1-x)} + \frac{1-2x}{2(1-x)})$ are Lipschitz maps from the region bounded by the graphs of $f_{1/4}|_{[c, 1-c]}$ and $f_{3/4}|_{[c, 1-c]}$ to $[c, 1 - c] \times [1/4, 3/4]$. This can be shown by checking that the derivatives are bounded. Moreover, the two maps agree on $\{1/2\} \times [1/4, 3/4]$, hence the map

$$F(x, y) = \begin{cases} (x, \frac{y}{2x}) & \text{if } c \leq x < \frac{1}{2}, \\ (x, \frac{y}{2(1-x)} + \frac{1-2x}{2(1-x)}) & \text{if } \frac{1}{2} \leq x \leq 1 - c \end{cases}$$

is Lipschitz on the same region. This implies that the image of a rectifiable curve under F is rectifiable, hence g^* is of bounded variation, as its graph is the image of the graph of $g|_{[c, 1-c]}$ under F .

Now we finish the proof of the theorem by deducing a contradiction. We derived from our indirect assumption that the set of those a such that the graph of g intersects the graph of f_a infinitely many times over the interval $[c, 1 - c]$ is of positive outer measure. This implies that the set of those a such that the graph of g^* intersects the graph of the constant function f_a^* infinitely many times is of positive outer measure. Hence, the set of those y such that $(g^*)^{-1}(y)$ is infinite is of positive outer measure. But g^* is of bounded variation, hence according to a result of Banach (see [2, Corollary 1]), the measure of this set should be zero, a contradiction. \square

Claim 3.4. \mathcal{H} is co-Haar null.

Proof. First we show that $\mathcal{H} \subset \text{Homeo}^+([0, 1])$ is Borel. It is easy too see that the complement of \mathcal{H} is $\bigcup_{m=1}^{\infty} \mathcal{H}_m$, where

$$\mathcal{H}_m = \left\{ f \in \text{Homeo}^+([0, 1]) : \exists x_0 \in [1/m, 1 - 1/m] \text{ such that } x_0 \text{ is the} \right. \\ \left. \text{minimum or the maximum of } f - \text{id}|_{(x_0 - \frac{1}{m}, x_0 + \frac{1}{m})} \right\}.$$

We show that each \mathcal{H}_m is closed, from which it follows that \mathcal{H} is G_δ , thus Borel. Let $(f_n)_{n \in \mathbb{N}}$ be a convergent sequence of functions from \mathcal{H}_m . Let $x_n \in [1/m, 1 - 1/m]$ be a point with the property that it is the minimum or the maximum of $f_n|_{(x_n - 1/m, x_n + 1/m)}$. Then there exists a subsequence n_k such that x_{n_k} converges to some $x' \in [1/m, 1 - 1/m]$ with x_{n_k} always either a local minimum or a local maximum of f_{n_k} . It is easy to see that if f is the limit of the sequence $(f_n)_{n \in \mathbb{N}}$ then

x' is the minimum or maximum of $f|_{(x'-1/m, x'+1/m)}$, showing that \mathcal{H}_m is closed. This finishes the proof that \mathcal{H} is Borel.

Now we construct a Borel probability measure on $\text{Homeo}^+([0, 1])$, such that for every $g \in \text{Homeo}^+([0, 1])$, $\mu(g\mathcal{H}) = 1$. By Lemma 2.1, this indeed implies that \mathcal{H} is co-Haar null, as it can easily be seen using Lemma 3.1 that \mathcal{H} is conjugacy invariant.

This measure is the same as in the proof of Claim 3.3, for $1/4 \leq a \leq 3/4$ let $f_a \in \text{Homeo}^+([0, 1])$ be the piecewise linear function as in (3.1). The measure μ is defined as in (3.2), $\mu(\mathcal{B}) = 2\lambda(\{a : f_a \in \mathcal{B}\})$ for a Borel set \mathcal{B} , where λ is the Lebesgue measure. As before, this defines a Borel probability measure on $\text{Homeo}^+([0, 1])$. Let $g \in \text{Homeo}^+([0, 1])$, to finish the proof that \mathcal{H} is co-Haar null, it is enough to show that the set $\{a : f_a \notin g\mathcal{H}\} = \{a : g^{-1}f_a \notin \mathcal{H}\}$ is countable.

If for some $a \in [1/4, 3/4]$, the function $g^{-1}f_a(x) - x$ has a local maximum at $x_a \in \text{Fix}(g^{-1}f) \cap (0, 1)$, then we assign an open interval (c_a, d_a) to a , where $c_a, d_a \in \mathbb{Q} \cap (0, 1)$, $x_a \in (c_a, d_a)$ and $g^{-1}(f_a(x)) - x \leq 0$ if $x \in (c_a, d_a)$. Such an interval exists, as $x_a \in \text{Fix}(g^{-1}f)$, hence $g^{-1}(f(x_a)) - x_a = 0$, and x_a is the local maximum of $g^{-1}f_a(x) - x$. Now we show that we cannot assign the same interval to two distinct points. Suppose that we assign (c_a, d_a) to a and a' , with $a < a'$. Then for every $x \in (c_a, d_a)$, we have $g^{-1}(f_{a'}(x)) - x \leq 0$, but for $x_a \in (c_a, d_a)$, $0 = g^{-1}(f_a(x_a)) - x_a < g^{-1}(f_{a'}(x_a)) - x_a$, a contradiction, using the fact that g^{-1} is a strictly increasing function, and $f_a(x) < f_{a'}(x)$ for every $x \in (0, 1)$.

This shows that the set of those a 's, such that $g^{-1}f_a \notin \mathcal{H}$ because there is a fixed point in $(0, 1)$ which is a local maximum of $g^{-1}(f_a(x)) - x$, is countable. The other reason excluding $g^{-1}f_a$ from being in \mathcal{H} is the existence of a fixed point that is a local minimum, and one can similarly prove that this happens for only countably many a 's. This shows that $\{a : f_a \notin g\mathcal{H}\}$ is countable, hence \mathcal{H} is co-Haar null. \square

Thus the proof of the ‘‘only if’’ part of the theorem is complete.

Now we prove that every conjugacy class described in the theorem is indeed non-Haar null. Let \mathcal{C} be such a conjugacy class, then for those fixed points of a function $f \in \mathcal{C}$ that fall into the open interval $(0, 1)$ we have the following possibilities:

- (i) $\text{Fix}(f) \cap (0, 1)$ is a countably infinite set with limit points 0 and 1,
- (ii) $\text{Fix}(f) \cap (0, 1)$ is a countably infinite set with the unique limit point 0,
- (iii) $\text{Fix}(f) \cap (0, 1)$ is a countably infinite set with the unique limit point 1,
- (iv) $|\text{Fix}(f) \cap (0, 1)| = n$ for some $n \in \mathbb{N}$.

The set $(0, 1) \setminus \text{Fix}(f)$ is the union countably many open intervals, and since every fixed point $x_0 \in \text{Fix}(f) \cap (0, 1)$ is isolated, each of these intervals has a neighbouring interval on each side. In each of these intervals, the function $f(x) - x$ is either positive or negative, but as $f(x) - x$ has no local extremum point at any fixed point, the sign of $f(x) - x$ is distinct in neighbouring intervals.

Hence, each of the last three cases in the list describes two conjugacy classes (the fourth case describes two conjugacy classes for each n): there is a first or last open interval of $(0, 1) \setminus \text{Fix}(f)$, in which the sign of $f(x) - x$ can be chosen to be either positive or negative, but this choice determines the sign on every other interval. Whereas the first case describes only one possible conjugacy class. One can easily see using Lemma 3.1 that these possibilities indeed describe conjugacy classes.

Now we prove that in case (ii), (iii) and (iv), in order to show that each of these classes are non-Haar null, it is enough to show that the union of the two described

conjugacy classes is non-Haar null. The proof is the same in each case, so let \mathcal{C}_1 and \mathcal{C}_2 be the two conjugacy classes in either case and suppose that \mathcal{C}_1 is Haar null. We show that then \mathcal{C}_2 is Haar null, hence so is the union, contradicting our assumption. We use the following claim for the proof.

Claim 3.5. *Let G be a Polish topological group and $H \subset G$ a Haar null set. Then the set $\{h^{-1} : h \in H\}$ is also Haar null.*

Proof. Let μ be a witness measure for H , and let μ' be the measure with $\mu'(B) = \mu(\{b^{-1} : b \in B\})$. Now μ' is a witness measure for the set $H^{-1} = \{h^{-1} : h \in H\}$, since

$$\mu'(aH^{-1}b) = \mu'(\{ah^{-1}b : h \in H\}) = \mu(\{b^{-1}ha^{-1} : h \in H\}) = 0.$$

□

Using the claim, the image of \mathcal{C}_1 under the homeomorphism $f \mapsto f^{-1}$ is also Haar null. But it is easy to see that the image is \mathcal{C}_2 (regardless of which case we are working with), hence \mathcal{C}_2 is also Haar null, and so is $\mathcal{C}_1 \cup \mathcal{C}_2$.

Now we state three lemmas that will be essential for the rest of the proof.

Lemma 3.6. *Let $\mathcal{K} \subset \text{Homeo}^+([0, 1])$ be a compact set. Then there exists $\delta : (0, \infty) \rightarrow (0, \infty)$ such that for every $\varepsilon > 0$, $f \in \mathcal{K}$ and $x, y \in [0, 1]$ with $|x - y| \leq \delta(\varepsilon)$, we have $|f(x) - f(y)| < \varepsilon$ and $|f^{-1}(x) - f^{-1}(y)| < \varepsilon$.*

Proof. Let $\mathcal{K}' = \mathcal{K} \cup \{f^{-1} : f \in \mathcal{K}\}$. Then \mathcal{K}' is compact, since the map $f \mapsto f^{-1}$ is a homeomorphism. As $\text{Homeo}^+([0, 1])$ is a subspace of $C([0, 1])$, the space of continuous function defined on $[0, 1]$ with the supremum norm, we can apply the Arzelà–Ascoli theorem for \mathcal{K}' . Hence, there exists a function $\delta : (0, \infty) \rightarrow (0, \infty)$ such that for all $f \in \mathcal{K}'$ and $x, y \in [0, 1]$ with $|x - y| \leq \delta(\varepsilon)$, we have $|f(x) - f(y)| < \varepsilon$. As \mathcal{K}' contains f and f^{-1} for every $f \in \mathcal{K}$, we are done. □

Lemma 3.7. *Let $\mathcal{K} \subset \text{Homeo}^+([0, 1])$ be compact and $x_0, y_0 \in (0, 1)$, then there exists $g \in \text{Homeo}^+([0, 1])$ with $g(x_0) = y_0$ and such that both 0 and 1 are among the limit points of $\text{Fix}(g^{-1}f)$ for every $f \in \mathcal{K}$.*

Proof. Let $\delta : (0, \infty) \rightarrow (0, \infty)$ be the function given by the previous lemma for \mathcal{K} . Let $x_1 = x_0 + (1 - x_0)/2 < 1$, we show that there exists $y_1 < 1$ such that $y_1 > y_0$ and $y_1 > f(x_1)$ for every $f \in \mathcal{K}$. Indeed, any y_1 with $\max\{y_0, 1 - \delta(1 - x_1)\} < y_1 < 1$ works, since if $|1 - y_1| < \delta(1 - x_1)$ then $|f^{-1}(1) - f^{-1}(y_1)| < 1 - x_1$, showing that $f^{-1}(y_1) > x_1$, hence $f(x_1) < y_1$ for every $f \in \mathcal{K}$. Then let $y_2 = y_1 + (1 - y_1)/2$ and $x_2 < 1$ be such that $x_2 > x_1$ and $f(x_2) > y_2$ for every $f \in \mathcal{K}$. One can easily get an appropriate x_2 by a similar argument, choosing x_2 such that $\max\{x_1, 1 - \delta(1 - y_2)\} < x_2 < 1$.

By iterating the argument, one can easily get two strictly increasing sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ such that they are converging to 1, and $f(x_{2n}) > y_{2n}$, $f(x_{2n+1}) < y_{2n+1}$ for every $f \in \mathcal{K}$ and $n \in \mathbb{N} \setminus \{0\}$. Moreover, similarly one can get two decreasing sequences $(z_n)_{n \in \mathbb{N}}$ and $(u_n)_{n \in \mathbb{N}}$ with $z_0 < x_0$ and $u_0 < y_0$ such that they are converging to 0, and $f(z_{2n}) < u_{2n}$, $f(z_{2n+1}) > u_{2n+1}$ for every $f \in \mathcal{K}$ and $n \in \mathbb{N}$.

One can easily see that $(0, 1) = (z_0, x_0) \cup \bigcup_{n \in \mathbb{N}} (z_{n+1}, z_n] \cup [x_n, x_{n+1})$. Moreover, by defining the function $g : [0, 1] \rightarrow [0, 1]$ such that $g(x_n) = y_n$ and $g(z_n) = u_n$, for every $n \in \mathbb{N}$ and g is linear on intervals of the form $[x_n, x_{n+1}]$, $[z_{n+1}, z_n]$ and on

$[z_0, x_0]$ with $g(0) = 0$ and $g(1) = 1$, one gets an increasing homeomorphism of $[0, 1]$ such that the graph of g intersects the graph of every $f \in \mathcal{K}$ infinitely many times in every neighbourhood of 0 and 1. Thus the proof of the lemma is complete. \square

Lemma 3.8. *Let $0 < x_0, y_0 < 1$ with $f(x_0) > y_0$ for every $f \in \mathcal{K}$. Then there exists a function $g \in \text{Homeo}^+([0, 1])$ such that $g(x_0) = y_0$ and $f(x) > g(x)$ for every $x \in (0, 1)$ and $f \in \mathcal{K}$.*

Proof. For $x \in [0, 1]$, let $h(x) = \min_{f \in \mathcal{K}} f(x)$. The minimum exists, since the image of \mathcal{K} is compact in $[0, 1]$ under the continuous map $f \mapsto f(x)$. Using [23, Theorem 2.2], $h \in \text{Homeo}^+([0, 1])$. Let $T : [0, 1] \rightarrow [0, 1]$ be the function with $T(0) = T(x_0) = \frac{y_0}{h(x_0)}$, $T(1) = 1$ and T is linear in the intervals $[0, x_0]$ and $[x_0, 1]$. Let

$$g(x) = T(x)h(x),$$

we show that g satisfies the conditions of the lemma. It is easy to see that $g(0) = 0$, $g(1) = 1$ and $g(x_0) = y_0$, and also that it is continuous. Since T is a positive, increasing function, for $0 \leq x < y \leq 1$ we also have

$$g(y) - g(x) = T(y)h(y) - T(x)h(x) \geq T(x)(h(y) - h(x)) > 0,$$

showing that g is strictly increasing, hence $g \in \text{Homeo}^+([0, 1])$. For $x \in (0, 1)$, $T(x) < 1$, hence $g(x) < h(x) \leq f(x)$ for every $f \in \mathcal{K}$. This finishes the proof of the lemma. \square

Now we prove that the conjugacy classes described by (i), (ii), (iii) and (iv) are indeed non-Haar null. We first deal with the cases (i), (ii) and (iii), as they are easy consequences of the above facts. As already discussed above, cases (ii) and (iii) each describe two conjugacy classes, and in order to prove that both are non-Haar null, it is enough to show that their union is non-Haar null.

So let

$$\mathcal{F}_{(i)} = \{f \in \text{Homeo}^+([0, 1]) : 0 \text{ and } 1 \text{ is among the limit points of } \text{Fix}(f)\},$$

$$\mathcal{F}_{(ii)} = \{f \in \text{Homeo}^+([0, 1]) : 0 \text{ is a limit point of } \text{Fix}(f), \text{ but } 1 \text{ is not}\},$$

$$\mathcal{F}_{(iii)} = \{f \in \text{Homeo}^+([0, 1]) : 1 \text{ is a limit point of } \text{Fix}(f), \text{ but } 0 \text{ is not}\}.$$

Since the conjugacy class described by (i) is $\mathcal{F}_{(i)} \cap \mathcal{H} \cap \mathcal{L}$, and \mathcal{H} and \mathcal{L} are co-Haar null, it is enough to prove that $\mathcal{F}_{(i)}$ is non-Haar null to finish this case. Similarly, in cases (ii) and (iii), the union of the two conjugacy classes are $\mathcal{F}_{(ii)} \cap \mathcal{H} \cap \mathcal{L}$ and $\mathcal{F}_{(iii)} \cap \mathcal{H} \cap \mathcal{L}$, hence for these cases it is enough to show that $\mathcal{F}_{(ii)}$ and $\mathcal{F}_{(iii)}$ are non-Haar null.

To show that these three sets are non-Haar null, it is enough to prove that every compact set can be translated into them, as above. So let $\mathcal{K} \subset \text{Homeo}^+([0, 1])$ be compact. Let $x_0 \in (0, 1)$ be such that $f(x_0) > 1/2$ for every $f \in \mathcal{K}$, for example $x_0 = 1 - \delta(1/2)$ works, where δ is the function provided by Lemma 3.6. To show that \mathcal{K} can be translated into $\mathcal{F}_{(i)}$, let g be the function provided by Lemma 3.7 with x_0 and $y_0 = 1/2$. It is clear that $g^{-1}\mathcal{K} \subset \mathcal{F}_{(i)}$.

Now let h be the function provided by Lemma 3.8 with x_0 as above and $y_0 = 1/2$, and let $g_{(ii)}$ agree with g on $[0, x_0]$, and agree with h on $[x_0, 1]$. Then $g_{(ii)}$ intersects the graph of every $f \in \mathcal{K}$ infinitely many times in every neighbourhood of 0, but is disjoint from the graph of every $f \in \mathcal{K}$ in a small deleted neighbourhood of 1. From this one can easily see that $g_{(ii)}^{-1}\mathcal{K} \subset \mathcal{F}_{(ii)}$. Similarly, if $g_{(iii)}$ agrees with g on $[x_0, 1]$

and agrees with h on $[0, x_0]$, then $g_{(iii)}^{-1}\mathcal{K} \subset \mathcal{F}_{(iii)}$, showing that each of these three sets are non-Haar null. This finishes the proof that the conjugacy classes described in (i), (ii) and (iii) are non-Haar null.

It remains to be proven that for each n , the two conjugacy classes described by (iv) are non-Haar null.

Let

$$\mathcal{F}_n = \{f \in \text{Homeo}^+([0, 1]) : |\text{Fix}(f)| = n + 2\}.$$

We claim that it is enough to prove that for each $n \in \mathbb{N}$, \mathcal{F}_n is non-Haar null. For a fixed n , if \mathcal{F}_n is non-Haar null, then the union of the two conjugacy classes described by (iv) is also non-Haar null, since the union equals $\mathcal{F}_n \cap \mathcal{H} \cap \mathcal{L}$, the intersection of a non-Haar null and two co-Haar null sets. And as noted before, this implies that both of these classes are non-Haar null.

First we show that for each $n \in \mathbb{N}$, the set

$$\mathcal{F}_{\geq n}^{fin} = \{f \in \text{Homeo}^+([0, 1]) : n + 2 \leq |\text{Fix}(f)| < \aleph_0\}$$

is non-Haar null. In order to show this, it is enough to prove that the set

$$\mathcal{F}'_{\geq n} = \{f \in \text{Homeo}^+([0, 1]) : n + 2 \leq |\text{Fix}(f)| \text{ and neither } 0 \text{ nor } 1 \\ \text{is among the limit points of } \text{Fix}(f)\}$$

is non-Haar null, since $\mathcal{F}_{\geq n}^{fin} = \mathcal{F}'_{\geq n} \cap \mathcal{L}$, thus then $\mathcal{F}_{\geq n}^{fin}$ is the intersection of a non-Haar null and a co-Haar null set.

We prove that $\mathcal{F}'_{\geq n}$ is non-Haar null by showing that every compact subset of $\text{Homeo}^+([0, 1])$ can be translated into it. So let $\mathcal{K} \subset \text{Homeo}^+([0, 1])$ be compact. Similarly as before, there exist $x_0, y_0 \in (0, 1)$ with $f(x_0) > y_0$ for every $f \in \mathcal{K}$. Then, again with the same ideas as in the proof of Lemma 3.7, one can find two strictly increasing finite sequences in $(0, 1)$, $(x_k)_{k=0}^{2n}$ and $(y_k)_{k=0}^{2n}$ such that $f(x_{2m}) > y_{2m}$, $f(x_{2m+1}) < y_{2m+1}$ for every $f \in \mathcal{K}$ and every suitable m . Then let g be defined on the interval $[x_0, x_{2n}]$ by $g(x_k) = y_k$ for every $k \leq 2n$ and g is linear on each interval $[x_k, x_{k+1}]$ for every $k < 2n$. Then g is a strictly increasing continuous function on $[x_0, x_{2n}]$. Define g on the remaining intervals of $[0, 1]$ with the help of Lemma 3.8 such that $g(x) < f(x)$ for every $x \in (0, x_0) \cup (x_{2n}, 1)$ and $f \in \mathcal{K}$. Then $g \in \text{Homeo}^+([0, 1])$, and clearly $g^{-1}\mathcal{K} \subset \mathcal{F}'_{\geq n}$, since g intersects every $f \in \mathcal{K}$ at least n times (actually at least $2n$ times), but there is no intersection in the intervals $(0, x_0)$ and $(x_{2n}, 1)$. This shows that $\mathcal{F}'_{\geq n}$ and thus also that $\mathcal{F}_{\geq n}^{fin}$ is non-Haar null.

Using that $\mathcal{F}_{\geq n}^{fin} = \bigcup_{m \geq n} \mathcal{F}_m$, and the fact that the countable union of Haar null sets is Haar null, we conclude that at least one of the \mathcal{F}_m 's must be non-Haar null, hence for each $n \in \mathbb{N}$ there exists $m \geq n$ such that \mathcal{F}_m is non-Haar null. It follows that to show that each \mathcal{F}_n is non-Haar null, it is enough to prove the following lemma:

Lemma 3.9. *For a fixed n if \mathcal{F}_{n+1} is non-Haar null then \mathcal{F}_n is non-Haar null.*

Proof. First we prove that \mathcal{F}_k is Borel for every $k \in \mathbb{N}$. Let

$$\mathcal{F}_{\geq n} = \{f \in \text{Homeo}^+([0, 1]) : n + 2 \leq |\text{Fix}(f)|\},$$

then $\mathcal{F}_k = \mathcal{F}_{\geq k} \setminus \mathcal{F}_{\geq k+1}$, hence it is enough to show that $\mathcal{F}_{\geq k}$ is Borel for every k . Let

$$\mathcal{F}_{\geq k}^\delta = \{f : \exists x_1, \dots, x_{k+2} \in [0, 1] \forall i \neq j (f(x_i) = x_i \wedge |x_i - x_j| \geq \delta)\}.$$

It is easy to see that $\mathcal{F}_{\geq k} = \bigcup_{m \in \mathbb{N}} \mathcal{F}_{\geq k}^{1/m}$, and also that $\mathcal{F}_{\geq k}^\delta$ is closed, hence $\mathcal{F}_{\geq k}$ is F_σ , thus Borel.

Now, suppose towards a contradiction that \mathcal{F}_n is Haar null. Then there exists a Borel probability measure μ such that every two-sided translate of \mathcal{F}_n is of measure 0 with respect to μ . Using [17, 17.11], there is a compact set \mathcal{K} with $\mu(\mathcal{K}) > 0$, hence by restricting μ to \mathcal{K} and multiplying accordingly to get a probability measure, we again obtain a measure with the property that every translate of \mathcal{F}_n is of measure 0. Hence, we can suppose that μ has compact support, which we do from now on. Since \mathcal{F}_{n+1} is not Haar null, μ cannot be a witness for it, hence there exists $g \in \text{Homeo}^+([0, 1])$ such that $\mu(g\mathcal{F}_{n+1}) > 0$, where we used Lemma 2.1 for the fact that it is enough to consider one sided translates of \mathcal{F}_{n+1} .

Let us fix three rational numbers p, q and r with $0 < p < q < r < 1$. For a function $f \in \mathcal{F}_{n+1}$, the first, i.e., the smallest fixed point of f is 0. Let $\mathcal{F}_{n+1}^{p,q,r}$ contain those f 's, such that the second fixed point of f is in (p, q) , while the other $n + 1$ fixed points are in $(r, 1]$, that is

$$\mathcal{F}_{n+1}^{p,q,r} = \{f \in \mathcal{F}_{n+1} : |\text{Fix}(f) \cap (p, q)| = 1, |\text{Fix}(f) \cap (r, 1]| = n + 1\}.$$

It is easy to see that

$$\mathcal{F}_{n+1} = \bigcup_{\substack{p,q,r \in \mathbb{Q} \cap (0,1) \\ p < q < r}} \mathcal{F}_{n+1}^{p,q,r},$$

hence for some $0 < p, q, r < 1$, $\mu(g\mathcal{F}_{n+1}^{p,q,r}) > 0$. A function $f \in \mathcal{F}_{n+1}^{p,q,r}$ has no fixed points in $[q, r]$, hence either $f(x) < x$ or $f(x) > x$ for every $x \in [q, r]$. Let $\mathcal{F}_{n+1}^{p,q,r'}$ contain those $f \in \mathcal{F}_{n+1}^{p,q,r}$ with $f(x) > x$ for every $x \in [q, r]$, and let $\mathcal{F}_{n+1}^{p,q,r''}$ contain those with $f(x) < x$ for every $x \in [q, r]$. Then either $\mu(g\mathcal{F}_{n+1}^{p,q,r'}) > 0$ or $\mu(g\mathcal{F}_{n+1}^{p,q,r''}) > 0$. The rest of the proof of the lemma is similar in these two cases, we only show the proof for the first case.

So suppose that $\mu(g\mathcal{F}_{n+1}^{p,q,r'}) > 0$. This implies $\mu(g\mathcal{F}_{n+1}^{p,q,r'} \cap \mathcal{K}) > 0$ where \mathcal{K} is the support of μ . Now we define a function $h \in \text{Homeo}^+([0, 1])$ in the following way: let $h(x) = x$ for $x \in [r, 1]$, let $h(\frac{q+r}{2}) = \frac{q+r}{4}$ and let h be linear on the interval $[\frac{q+r}{2}, r]$. Now $h(\frac{q+r}{2}) < f(\frac{q+r}{2})$ for every f in the closure of $\mathcal{F}_{n+1}^{p,q,r'} \cap g^{-1}\mathcal{K}$. The closure is a compact set, hence we can use Lemma 3.8 to extend h to the interval $[0, \frac{q+r}{2}]$ to be in $\text{Homeo}^+([0, 1])$ with the property that $h(x) < f(x)$ for every $x \in (0, \frac{q+r}{2})$ and every f in the closure of $\mathcal{F}_{n+1}^{p,q,r'} \cap g^{-1}\mathcal{K}$.

One can easily see that for every $f \in h^{-1}(\mathcal{F}_{n+1}^{p,q,r'} \cap g^{-1}\mathcal{K})$, $f \in \mathcal{F}_n$, since h was constructed so that the translate of $\mathcal{F}_{n+1}^{p,q,r'} \cap g^{-1}\mathcal{K}$ by h^{-1} preserves every fixed point in $[r, 1]$ but the fixed point in (p, q) vanishes. Consequently,

$$g\mathcal{F}_{n+1}^{p,q,r'} \cap \mathcal{K} = g(\mathcal{F}_{n+1}^{p,q,r'} \cap g^{-1}\mathcal{K}) = gh h^{-1}(\mathcal{F}_{n+1}^{p,q,r'} \cap g^{-1}\mathcal{K}) \subset gh\mathcal{F}_n,$$

hence $\mu(gh\mathcal{F}_n) \geq \mu(g\mathcal{F}_{n+1}^{p,q,r'} \cap \mathcal{K}) > 0$, contradicting our assumption that μ is a witness measure for \mathcal{F}_n . This finishes the proof of the lemma. \square

The moreover part of Theorem 3.2 follows from the first part of the theorem and from the fact that the set $\{f \in \text{Homeo}^+([0, 1]) : \text{Fix}(f) \text{ has no limit point in } (0, 1) \text{ and } f(x) - x \text{ has no local extremum point at any } x_0 \in \text{Fix}(f) \cap (0, 1)\}$ is co-Haar null using Claim 3.3 and Claim 3.4. Therefore, the proof of the theorem is also complete. \square

4. HOMEOMORPHISMS OF THE CIRCLE

In this section we investigate $\text{Homeo}^+(\mathbb{S}^1)$, the group of orientation preserving homeomorphisms of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, with the uniform topology. The purpose of this section is to prove our main theorem about the group $\text{Homeo}^+(\mathbb{S}^1)$, Theorem 4.5. Before turning to the proof, we first introduce the basic definitions and state some classical results about circle homeomorphisms. All of these results can be found in [16].

For $f \in \text{Homeo}^+(\mathbb{S}^1)$, an increasing homeomorphism $F \in \text{Homeo}^+(\mathbb{R})$ is a *lift* of f , if $f(x) \equiv F(x) \pmod{1}$ for each $x \in [0, 1)$ and $F(x+1) = F(x) + 1$ for each $x \in \mathbb{R}$. It is easy to see that if F_1 and F_2 are two lifts of $f \in \text{Homeo}^+(\mathbb{S}^1)$ then there exists a $k \in \mathbb{Z}$ such that $F_1(x) - F_2(x) = k$ for each $x \in \mathbb{R}$. For $\alpha \in \mathbb{R}$ let $R_\alpha \in \text{Homeo}^+(\mathbb{S}^1)$ denote the function $x \mapsto x + \alpha \pmod{1}$. Then let $\overline{R}_\alpha \in \text{Homeo}^+(\mathbb{R})$ be the homeomorphism $\overline{R}_\alpha(x) = x + \alpha$. Note that $R_0 \equiv R_1$, although $\overline{R}_0 \not\equiv \overline{R}_1$.

If $f \in \text{Homeo}^+(\mathbb{S}^1)$ and F is a lift of f then the quantity

$$\tau(F) = \lim_{n \rightarrow \infty} \frac{1}{n} (F^n(x) - x)$$

is independent of x , and is well defined up to an integer, that is, $\tau(F) - \tau(F') \in \mathbb{Z}$ if F' is another lift of f . Then it makes sense to define the *rotation number* of f as $\tau(f) \equiv \tau(F) \pmod{1}$, hence $\tau(f) \in \mathbb{S}^1$. It is well-known that τ is continuous and conjugacy invariant. We will use it to classify the conjugacy classes in $\text{Homeo}^+(\mathbb{S}^1)$ and also to prove some properties of them.

An *orbit* of a point $x \in \mathbb{S}^1$ under $f \in \text{Homeo}^+(\mathbb{S}^1)$ is $\{f^n(x) : n \in \mathbb{Z}\}$. A point $x \in \mathbb{S}^1$ is a *periodic point* of $f \in \text{Homeo}^+(\mathbb{S}^1)$ if $f^n(x) = x$ for some $n \in \mathbb{N}$, $n \geq 1$. An orbit is *periodic* if it is the orbit of a periodic point. The *period* of a periodic point x is the least $n \geq 1$ such that $f^n(x) = x$. It is a well known fact that $\tau(f) \in \mathbb{Q}$ if and only if f has periodic points, and if $\tau(f) \in \mathbb{Q}$ then all periodic points have the same period. Moreover, if $\tau(f) = p/q$ with p, q relatively prime, $0 \leq p < q$ then every periodic point of f has period q .

If $x \in \mathbb{S}^1$ is a fixed point of $f \in \text{Homeo}^+(\mathbb{S}^1)$ and F is a lift of f then $F(x') = x' + k$ for some $k \in \mathbb{Z}$, where $x' \in \mathbb{R}$ is any real number with $x \equiv x' \pmod{1}$. We say that x is a *crossing fixed point* if for every neighbourhood $U \subset \mathbb{R}$ of x' there is a point $y_1 \in U$ with $F(y_1) < y_1 + k$, and there is another point $y_2 \in U$ with $F(y_2) > y_2 + k$. If x is a periodic point of f and n is the smallest positive integer with $f^n(x) = x$ then we say that x is a *crossing periodic point* of f if x is a crossing fixed point of f^n .

We introduce the relation $x < y < z$ for three elements $x, y, z \in \mathbb{S}^1$, by saying that $x < y < z$ holds if and only if $x' < y' < z'$ or $z' < x' < y'$ or $y' < z' < x'$, where x', y' and z' are the unique representatives of x, y and z , respectively, in $[0, 1)$. We also use the notation $x_1 < x_2 < \dots < x_n$ for $x_1, \dots, x_n \in \mathbb{S}^1$ if and only if $x_i < x_j < x_k$ for $i < j < k$. Also, for $x, y \in \mathbb{S}^1$ let $(x, y) = \{z \in \mathbb{S}^1 : x < z < y\}$, and analogously for $[x, y]$, $[x, y)$ and $(x, y]$.

A fixed point $x \in \mathbb{S}^1$ of f is *attractive* if there is an interval (u, v) such that $x \in (u, v)$ and for every $y \in (u, v)$, $f^n(y) \rightarrow x$. A fixed point x is *repulsive* if there is an interval $(u, v) \ni x$ such that for every $y \in (u, v) \setminus \{x\}$ and for every large enough $n \in \mathbb{N}$, $f^n(y) \notin (u, v)$. The proofs of the following lemmas are standard and are left to the reader.

Lemma 4.1. *Suppose that $f \in \text{Homeo}^+(\mathbb{S}^1)$ has some, but only finitely many fixed points. Let x be a fixed point of f , and let y and z be two fixed points (not necessarily different from x) such that $y < x < z$ and there are no fixed points in the intervals (y, x) and (x, z) . Then x is attractive if and only if $f(p) \in (p, x)$ for every $p \in (y, x)$ and $f(p) \in (x, p)$ for every $p \in (x, z)$. Similarly, x is repulsive if and only if $f(p) \notin (p, x)$ for every $p \in (y, x)$ and $f(p) \notin (x, p)$ for every $p \in (x, z)$.*

Lemma 4.2. *Let $f \in \text{Homeo}^+(\mathbb{S}^1)$ be a homeomorphism with finitely many fixed points, all of whom are crossing. Then every fixed point of f is either repulsive or attractive, and the two types are alternating.*

Now we describe the conjugacy classes of $\text{Homeo}^+(\mathbb{S}^1)$ with rational rotation number using results from [12]. Suppose that $f \in \text{Homeo}^+(\mathbb{S}^1)$ has a fixed point. Then for some $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, $F(x) = x + k$, where F is a lift of f . The *signature* of f is then defined as $\Delta_f(x) = \text{sign}(F(x) - x - k)$. Note that value $F(x) - x - k$ only depends on the fractional part of x , hence it is legitimate to interpret Δ_f as a function from \mathbb{S}^1 to $\{-1, 0, 1\}$, which we will do. Also notice that if $F(x') = x' + k'$ for another $x' \in \mathbb{R}$ and $k' \in \mathbb{Z}$ then $k = k'$, hence Δ_f is well-defined. Now we state the facts we need about conjugation.

Lemma 4.3 ([12]). *If the two homeomorphisms $f, g \in \text{Homeo}^+(\mathbb{S}^1)$ have fixed points then they are conjugate if and only if there exists a homeomorphism $h \in \text{Homeo}^+(\mathbb{S}^1)$ such that $\Delta_g = \Delta_f \circ h$. If $\tau(f) = \tau(g) = \frac{p}{q}$ with p and q relatively prime and $0 \leq p < q$ then f and g are conjugate if and only if f^q and g^q are conjugate.*

Now we prove a proposition about the random homeomorphism.

Proposition 4.4. *The set of $f \in \text{Homeo}^+(\mathbb{S}^1)$ with infinitely many periodic points and the set of $f \in \text{Homeo}^+(\mathbb{S}^1)$ with finitely many periodic points at least one of which is non-crossing are both Haar null.*

Proof. Since all the periodic points have the same period, if f has infinitely many periodic points then for some n , f^n has infinitely many fixed points. Hence, it is enough to prove that for each n , for co-Haar null many $f \in \text{Homeo}^+(\mathbb{S}^1)$, f^n only has finitely many fixed points and each of them are crossing. Let n be a fixed positive integer and let $\mathcal{F}_n = \{f \in \text{Homeo}^+(\mathbb{S}^1) : f^n \text{ has finitely many fixed points and all of them are crossing}\}$. It is enough to show that \mathcal{F}_n is co-Haar null.

As the number of fixed points is the same for conjugate homeomorphisms, and $(h^{-1}fh)^n = h^{-1}f^n h$, the number of periodic points with period n is also the same for conjugate automorphisms. The fact that a homeomorphism has a non-crossing periodic point also holds simultaneously for conjugate homeomorphisms. It follows that \mathcal{F}_n is conjugacy invariant.

We now claim that \mathcal{F}_n is a Borel set. Let $\mathcal{F} = \{f \in \text{Homeo}^+(\mathbb{S}^1) : f \text{ has finitely many fixed points and all of them are crossing}\}$. It is enough to show that \mathcal{F} is a Borel set, since \mathcal{F}_n is the inverse image of \mathcal{F} under the continuous map $f \mapsto f^n$.

Let

$$\mathcal{F}_{k,l} = \{f \in \text{Homeo}^+(\mathbb{S}^1) : \exists x_1, \dots, x_l \in \mathbb{S}^1 \cap \text{Fix}(f) (i \neq j \Rightarrow |x_i - x_j| \geq 1/k)\},$$

where the distance $|x-y|$ of two elements $x, y \in \mathbb{S}^1$ are interpreted as $\min(y'-x', x'+1-y')$, where $x', y' \in \mathbb{R}$ with $x \equiv x' \pmod{1}$, $y \equiv y' \pmod{1}$ and $x' \leq y' < x'+1$.

Also let

$$\mathcal{C}_m = \left\{ f \in \text{Homeo}^+(\mathbb{S}^1) : \text{for a lift } F \text{ of } f \text{ there exists } x \in [0, 1] \text{ and } k \in \mathbb{Z} \right. \\ \left. \begin{aligned} &\text{with } F(x) = x + k \text{ and either } F(y) \leq y + k \text{ for all } y \in \left(x - \frac{1}{m}, x + \frac{1}{m}\right) \\ &\text{or } F(y) \geq y + k \text{ for all } y \in \left(x - \frac{1}{m}, x + \frac{1}{m}\right) \end{aligned} \right\}.$$

Clearly $f \notin \mathcal{F}$ if and only if f has infinitely many fixed points or it has a non-crossing fixed point, thus

$$\mathcal{F}^c = \bigcap_{l \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \mathcal{F}_{k,l} \cup \bigcup_{m \in \mathbb{N}} \mathcal{C}_m.$$

Hence, in order to show that \mathcal{F} is Borel, it is enough to prove that $\mathcal{F}_{k,l}$ and \mathcal{C}_m are Borel sets for every $k, l, m \in \mathbb{N}$.

First we show that $\mathcal{F}_{k,l}$ is closed for every $k, l \in \mathbb{N}$. Let $f_n \in \mathcal{F}_{k,l}$ be a sequence converging uniformly to f , and $x_i^n \in \mathbb{S}^1 \cap \text{Fix}(f_n)$ for $i = 1, \dots, l$ satisfy $|x_i^n - x_j^n| \geq 1/k$ for $i \neq j$. Then, for an appropriate subsequence, the numbers x_i^n converge to some x_i for all i . It is easy to see that $x_i \in \text{Fix}(f)$ and $|x_i - x_j| \geq 1/k$ for $i \neq j$. Hence $f \in \mathcal{F}_{k,l}$, showing that $\mathcal{F}_{k,l}$ is closed, thus Borel.

Now we prove that each \mathcal{C}_m is Borel by showing that it is also closed. Let $(f_k) \subset \mathcal{C}_m$ be a uniformly convergent sequence with $f_k \rightarrow f$. For each k , let $x_k \in [0, 1]$ and F_k a suitable lift of f_k with $F_k(x_k) = x_k$ and either $F_k(y) \leq y$ for every $y \in (x_k - 1/m, x_k + 1/m)$, or $F_k(y) \geq y$ for every $y \in (x_k - 1/m, x_k + 1/m)$. Then there exists a subsequence k_l such that x_{k_l} converges to some $x \in [0, 1]$ and we also suppose that for each l , $F_{k_l}(y) \leq y$ for every $y \in (x_{k_l} - 1/m, x_{k_l} + 1/m)$. The case where for each l , $F_{k_l}(y) \geq y$ for every $y \in (x_{k_l} - 1/m, x_{k_l} + 1/m)$, is analogous. Now x is clearly a fixed point of f . Hence, for a suitable lift F of f , $F(x) = x$, and it is also easy to see that for every $y \in (x - 1/m, x + 1/m)$, $F(y) \leq y$, thus $f \in \mathcal{C}_m$. This shows that \mathcal{C}_m , and thus \mathcal{F}_n is a Borel set.

Now we prove that \mathcal{F}_n is co-Haar null by constructing a Borel probability measure μ on $\text{Homeo}^+(\mathbb{S}^1)$ such that for every $f \in \text{Homeo}^+(\mathbb{S}^1)$, $\mu(\mathcal{F}_n f^{-1}) = 1$. Then Lemma 2.1 implies that \mathcal{F}_n is co-Haar null, since \mathcal{F}_n is a conjugacy invariant Borel set.

Let

$$(4.1) \quad \mu(B) = \lambda(\{\alpha \in [0, 1] : R_\alpha \in B\}),$$

where λ is the Lebesgue measure. It is easy to see that μ is a Borel probability measure. Let $f \in \text{Homeo}^+(\mathbb{S}^1)$ be arbitrary, we need to show that $\mu(\mathcal{F}_n f^{-1}) = 1$, or equivalently, $\lambda(\{\alpha \in [0, 1] : R_\alpha f \in \mathcal{F}_n\}) = 1$.

Let F be the lift of f with $F(0) \in [0, 1]$ and $H = \{(x, \alpha) \in \mathbb{R} \times \mathbb{R} : (\overline{R}_\alpha F)^n(x) - x \in \mathbb{Z}\}$. First of all, it is easy to see that H is closed, since the map $\phi(x, \alpha) = (\overline{R}_\alpha F)^n(x) - x$ is continuous, and $H = \phi^{-1}(\mathbb{Z})$. Let $k \in \mathbb{Z}$ be fixed. The map $\alpha \mapsto (\overline{R}_\alpha F)^n(x) - x$ is strictly increasing and continuous, and has limits $-\infty$ at $-\infty$ and ∞ at ∞ . Hence, $\phi^{-1}(k)$ is the graph of the function assigning to $x \in \mathbb{R}$ the unique $\alpha \in \mathbb{R}$ with $(\overline{R}_\alpha F)^n(x) - x = k$. Let $\psi_k : \mathbb{R} \rightarrow \mathbb{R}$ denote the corresponding function. It is easy to see that $\phi(x + 1, \alpha) = \phi(x, \alpha)$, hence ψ_k is periodic for each k with period 1. Since $F(0) \in [0, 1]$, clearly $F(x) \in [x - 1, x + 1]$ for every $x \in \mathbb{R}$. It follows that for $x \in [0, 1)$, $(\overline{R}_\alpha F)^n(x) - x \in [n\alpha - n - 1, n\alpha + n + 1]$. Hence,

$k \in [n\psi_k(x) - n - 1, n\psi_k(x) + n + 1]$, meaning that

$$(4.2) \quad \psi_k(x) \in [k/n - 1 - 1/n, k/n + 1 + 1/n] \text{ for } x \in [0, 1]$$

and actually for every $x \in \mathbb{R}$ using the periodicity of ψ_k .

Thus ψ_k is bounded and its graph is closed, hence it is continuous. It also satisfies

$$(4.3) \quad \psi_k(y) - \psi_k(x) \leq y - x \text{ if } x < y,$$

since either $\psi_k(y) \leq \psi_k(x)$, and it is automatic, or $\psi_k(y) > \psi_k(x)$, in which case it follows from

$$\begin{aligned} k &= (\overline{R}_{\psi_k(x)}F)^n(x) - x \leq (\overline{R}_{\psi_k(x)}F)^n(y) - y + y - x \\ &\leq \psi_k(x) - \psi_k(y) + (\overline{R}_{\psi_k(y)}F)^n(y) - y + y - x \\ &= \psi_k(x) - \psi_k(y) + k + y - x. \end{aligned}$$

Then (4.3) implies that $\psi_k|_{[0,1]}$ is of bounded variation, since $\psi_k(0) = \psi_k(1)$, hence in any subdivision $0 = a_0 < a_1 < \dots < a_l = 1$, the sums

$$\sum_{\substack{i \\ \psi_k(a_i) < \psi_k(a_{i+1})}} \psi_k(a_{i+1}) - \psi_k(a_i) \quad \text{and} \quad \sum_{\substack{i \\ \psi_k(a_i) > \psi_k(a_{i+1})}} \psi_k(a_{i+1}) - \psi_k(a_i)$$

have the same absolute value, and the former is at most 1. Using the fact that $\psi_k|_{[0,1]}$ is of bounded variation, a result of Banach [2] then implies that

$$(4.4) \quad \text{for almost all } \alpha \in [0, 1], \psi_k|_{[0,1]}^{-1}(\alpha) \text{ is finite for every } k \in \mathbb{Z}.$$

To finish the proof of the proposition, we need to show that $\lambda(\{\alpha \in [0, 1] : R_\alpha f \notin \mathcal{F}_n\}) = 0$. Now we have $\{\alpha \in [0, 1] : R_\alpha f \notin \mathcal{F}_n\} = \{\alpha \in [0, 1] : (R_\alpha f)^n(x) = x \text{ for infinitely many } x \in \mathbb{S}^1\} \cup \{\alpha \in [0, 1] : (R_\alpha f)^n \text{ has finitely many fixed points and at least one of them is non-crossing}\}$. Now we show that both sets are of measure zero, starting with the former.

It is easy to see that $\{\alpha \in [0, 1] : (R_\alpha f)^n(x) = x \text{ for infinitely many } x \in \mathbb{S}^1\} = \{\alpha \in [0, 1] : (\overline{R}_\alpha F)^n(x) - x \in \mathbb{Z} \text{ for infinitely many } x \in [0, 1]\} = \{\alpha \in [0, 1] : \bigcup_k \psi_k|_{[0,1]}^{-1}(\alpha) \text{ is infinite}\}$. Using (4.2), ψ_k has values in $[0, 1]$ for only finitely many k , hence the union can be changed to a finite union, meaning that the set above equals $\{\alpha \in [0, 1] : \psi_k|_{[0,1]}^{-1}(\alpha) \text{ is infinite for some } k\}$. And this is clearly of measure zero using (4.4).

Now we turn to the latter set. Suppose that for some $\alpha \in [0, 1]$, $(R_\alpha f)^n \in \text{Homeo}^+(\mathbb{S}^1)$ has finitely many fixed points and one of them is non-crossing. Let $x \in [0, 1]$ represent the non-crossing fixed point, then $(\overline{R}_\alpha F)^n(x) = x + k$ for some $k \in \mathbb{Z}$. Since $(R_\alpha f)^n$ only has finitely many fixed points and x is non-crossing, there is a neighbourhood $(x - \delta, x + \delta)$ of x such that either $(\overline{R}_\alpha F)^n(y) > y + k$ for every $y \in (x - \delta, x + \delta) \setminus \{x\}$ or $(\overline{R}_\alpha F)^n(y) < y + k$ for every $y \in (x - \delta, x + \delta) \setminus \{x\}$. In the first case x is a strict local maximum point of ψ_k , in the second case x is a strict local minimum point of ψ_k . Hence, we get that $\{\alpha \in [0, 1] : (R_\alpha f)^n \text{ has finitely many fixed point and at least one of them is non-crossing}\} \subset \{\alpha \in [0, 1] : \alpha \text{ is a strict local minimum or maximum of } \psi_k \text{ for some } k \in \mathbb{Z}\}$. And the latter is well known to be countable (see e.g. [23, Theorem 7.2]), thus it is of measure zero. This completes the proof of the proposition. \square

Now we are ready to prove the main theorem of this section.

Theorem 4.5. *The conjugacy class of $f \in \text{Homeo}^+(\mathbb{S}^1)$ is non-Haar null if and only if $\tau(f) \in \mathbb{Q}$, f has finitely many periodic points and all of them are crossing. Every such conjugacy class necessarily contains an even number of periodic orbits, and for every rational number $0 \leq r < 1$ and positive integer k there is a unique non-Haar null conjugacy class with rotation number r containing $2k$ periodic orbits. Moreover, the union of the non-Haar null conjugacy classes and $\{f \in \text{Homeo}^+(\mathbb{S}^1) : \tau(f) \notin \mathbb{Q}\}$ is co-Haar null.*

Proof. For each rational number r let

$$(4.5) \quad \mathcal{F}_r = \{f \in \text{Homeo}^+(\mathbb{S}^1) : \tau(f) = r, f \text{ has finitely many periodic points and each of them is crossing}\}.$$

It is clear that each set \mathcal{F}_r is the union of conjugacy classes, since the rotation number is conjugacy invariant.

We only prove the first assertion of the theorem. The second will follow using Claim 4.6 and Claim 4.7 below. The moreover part follows from Proposition 4.4.

Let us prove the “only if” part of the first statement. Using Proposition 4.4, it is enough to show that if $\tau(f) \notin \mathbb{Q}$ then the conjugacy class of f is Haar null. Let $\mathcal{C} = \{g \in \text{Homeo}^+(\mathbb{S}^1) : \tau(g) = \tau(f)\}$. It is enough to show that \mathcal{C} is Haar null. In order to do so, we use the measure μ defined in (4.1). Note that \mathcal{C} is closed, since the map $f \mapsto \tau(f)$ is continuous. Hence, using Lemma 2.1 and the fact that \mathcal{C} is conjugacy invariant, it is enough to show that $\mu(\mathcal{C}g) = 0$ for every $g \in \text{Homeo}^+(\mathbb{S}^1)$. Clearly, $\{\alpha \in [0, 1] : R_\alpha \in \mathcal{C}g\} = \{\alpha \in [0, 1] : \tau(R_\alpha g^{-1}) = \tau(f)\}$. Let G be a lift of g^{-1} . Using [16, Proposition 11.1.9], the map $\alpha \mapsto \tau(\overline{R}_\alpha G)$ is strictly increasing at points where $\tau(\overline{R}_\alpha G) \notin \mathbb{Q}$. Hence the inverse image of $\mathbb{Z} + \tau(f)$ is countable, thus $\{\alpha \in [0, 1] : \tau(R_\alpha g^{-1}) = \tau(f)\}$ is countable, finishing the proof that \mathcal{C} is indeed Haar null.

Now we prove the “if” part. First we describe the conjugacy classes that satisfy the conditions of the theorem.

Claim 4.6. *Let $f, g \in \mathcal{F}_r$ for some $r \in \mathbb{Q}$. Then f and g are conjugate if and only if they have the same number of periodic points.*

Proof. Let $r = p/q$ with p, q relatively prime, $q > 0$. If f and g are conjugate then f^q and g^q are also conjugate, hence f^q and g^q have the same number of fixed points, thus indeed, f and g have the same number of periodic points. Now suppose that they have the same number of periodic points. Then f^q and g^q have the same number of fixed points, moreover, $f^q, g^q \in \mathcal{F}_0$. Using the first statement of Lemma 4.3, it is easy to see that if two homeomorphisms from \mathcal{F}_0 have the same number of fixed points then they are conjugate. Hence f^q and g^q are conjugate, but then, using Lemma 4.3 again, f and g are also conjugate. This proves the claim. \square

Claim 4.7. *Let p, q be relatively prime numbers with $0 \leq p < q$ and let $n > 0$. We claim that there exists a homeomorphism $f \in \mathcal{F}_{p/q}$ with exactly n periodic points if and only if the number of periodic orbits, n/q is even.*

Proof. Let p, q and n be as above with n/q even, we first show that such homeomorphisms exist. Let F , an increasing homeomorphism of \mathbb{R} be defined as follows. For every $k \in \{0, 1, \dots, n-1\}$, F maps the interval $[\frac{k}{n}, \frac{k+1}{n}]$ bijectively (increasingly and continuously) into $[\frac{k+(np)/q}{n}, \frac{k+1+(np)/q}{n}]$. We also require that if k is even then every $y \in (\frac{k}{n}, \frac{k+1}{n})$, is “shifted up”, that is, $F(y) - \frac{k+(np)/q}{n} > y - \frac{k}{n}$. On the other hand, if

k is odd then we require for such points that $F(y) - \frac{k+(np)/q}{n} < y - \frac{k}{n}$. Then we defined F on $[0, 1]$ as a strictly increasing, continuous function with $F(1) = F(0) + 1$, hence we can extend it uniquely to a homeomorphism of \mathbb{R} (which we also denote by F) with $F(x+1) = F(x) + 1$ for every $x \in \mathbb{R}$. Thus, F is a lift of a homeomorphism $f \in \text{Homeo}^+(\mathbb{S}^1)$.

We need to show that $f \in \mathcal{F}_{p/q}$ and it has exactly n periodic points. For every $k, \ell \in \mathbb{N}$, $F^{\ell q}(k/n) = k/n + \ell p$, from which it follows that $\tau(f) = p/q$ and it has at least n periodic points. Let $y \in (\frac{k}{n}, \frac{k+1}{n})$ for some natural number k with $0 \leq k < n$. If k is even then $k + (np)/q$ is also even and so is $k + \ell(np)/q$ for every $\ell \in \mathbb{N}$. Hence $y - \frac{k}{n} < F(y) - \frac{k+(np)/q}{n} < \dots < F^q(y) - \frac{k+(np)}{n} = F^q(y) - \frac{k}{n} + p$. If k is odd then one can similarly obtain $y - \frac{k}{n} > F^q(y) - \frac{k}{n} + p$. These facts imply that in both cases, y is not a fixed point of f^q , hence it is not a periodic point of f , since all periodic points of f have the same period, q . On the other hand, these facts imply that each fixed point of f^q is crossing, showing that f satisfies the required properties.

Now we prove that if such a homeomorphism f exists then n/q is even. Suppose indirectly that n/q is odd. It is clear from the fact that each fixed point of f^q is crossing that the number of fixed points of f^q is even, hence n is even. Thus q is even and p is odd. Let $0 \leq x_0 < x_1 < \dots < x_{n-1} < 1$ represent the periodic points of f . Since they form orbits of size q , there is a $k < q$ such that for every $i < n$, $f(x_i) = x_{j(i)}$, where $j(i) \equiv i + \frac{n}{q}k \pmod{n}$. Here k and q are relatively prime, since otherwise the size of the orbits would be less than q , hence k is odd. Actually, $k = p$, but we do not need this fact. The fixed points of f^q are exactly the points x_i , hence it makes sense to define the map $\phi : \{0, 1, \dots, n-1\} \rightarrow \{-1, 1\}$ so that $\phi(i) = 1$ if and only if the values over the interval (x_i, x_{i+1}) are above the diagonal with respect to the unique lift of f^q where the fixed points of f^q are on the diagonal. Since every fixed point of f^q is crossing, the values of ϕ are alternating.

Let us assume that $\phi(0) = 1$, the case where $\phi(0) = -1$ can be handled similarly. This means that if we choose an arbitrary point $y \in (x_0, x_1)$ then $(f^q)^\ell(y)$ converges to x_1 as ℓ tends to infinity, hence $f((f^q)^\ell(y))$ converges to $x_{j(1)}$. Since k and n/q are odd numbers, $j(0)$ is odd, hence $\phi(j(0)) = -1$. Consequently, for any point in $(x_{j(0)}, x_{j(0)+1})$, for example, for $f(y)$, we have that $(f^q)^\ell(f(y)) \rightarrow x_{j(0)}$ as $\ell \rightarrow \infty$. But $(f^q)^\ell(f(y)) = f((f^q)^\ell(y))$, a contradiction. Hence the proof of the claim is complete. \square

Now let $p, q \in \mathbb{N}$ be relatively prime numbers with $0 \leq p < q$, $k \in \mathbb{N}$ with $k \geq 1$ and let $\mathcal{F}_{p/q}^k = \{f \in \mathcal{F}_{p/q} : |\text{Fix}(f^q)| = 2kq\}$. Using Claim 4.7 and Claim 4.6, $\mathcal{F}_{p/q}^k$ is a conjugacy class and $\bigcup_{k \geq 1} \mathcal{F}_{p/q}^k = \mathcal{F}_{p/q}$. Hence to finish the proof of the theorem, we need to show that each $\mathcal{F}_{p/q}^k$ is non-Haar null. We do so by proving the following two facts. First, we show that $\mathcal{F}_{p/q}^k$ is non-Haar null for infinitely many k , that is, $\bigcup_{k \geq \ell} \mathcal{F}_{p/q}^k$ is non-Haar null for every $\ell \geq 1$. Then we show that if $\mathcal{F}_{p/q}^k$ is Haar null then $\mathcal{F}_{p/q}^{k+1}$ is also Haar null. One can easily see that these two facts imply that $\mathcal{F}_{p/q}^k$ is non-Haar null for every $k \geq 1$. Now we prove these claims one after the other.

Claim 4.8. *For every $\ell \geq 1$, $\bigcup_{k \geq \ell} \mathcal{F}_{p/q}^k$ is non-Haar null.*

Proof. Let $\ell \geq 1$ be fixed and $f \in \mathcal{F}_{p/q}^\ell$. Since every fixed point of f^q is crossing, there is an open neighbourhood \mathcal{U} of f^q such that $|\text{Fix}(g)| \geq 2\ell q$ for every $g \in \mathcal{U}$. Let \mathcal{U}' be the inverse image of \mathcal{U} under the continuous map $g \mapsto g^q$. Now $f \in \mathcal{U}'$ and for every $g \in \mathcal{U}'$, $|\text{Fix}(g^q)| \geq 2\ell q$. Since the map $g \mapsto \tau(g)$ from $\text{Homeo}^+(\mathbb{S}^1)$ to \mathbb{S}^1 is continuous (see e.g. [16, 11.1.6]), we can also suppose by shrinking \mathcal{U}' that if $\tau(g) = r/q$ for some $0 \leq r < q$ and $g \in \mathcal{U}'$ then $r = p$. This observation and the fact that $\text{Fix}(g^q) \neq \emptyset$ for \mathcal{U}' imply that $\tau(g) = p/q$ for every $g \in \mathcal{U}'$.

Since \mathcal{U}' is a non-empty open set, it is non-Haar null. Hence to finish the proof of the claim, it is enough to show that the set

$$\mathcal{H} = \mathcal{U}' \setminus \bigcup_{k \geq \ell} \mathcal{F}_{p/q}^k$$

is Haar null. But if $f \in \mathcal{H}$ then either f has infinitely many periodic points, or f has a non-crossing periodic point. Hence Proposition 4.4 implies that \mathcal{H} is indeed Haar null, finishing the proof of the claim. \square

Now we finish the proof of the theorem by showing the following.

Claim 4.9. *If $\mathcal{F}_{p/q}^k$ is Haar null for some $k \geq 1$ then $\mathcal{F}_{p/q}^{k+1}$ is also Haar null.*

Proof. Suppose indirectly that $\mathcal{F}_{p/q}^k$ is Haar null, but $\mathcal{F}_{p/q}^{k+1}$ is not. Then, since $\mathcal{F}_{p/q}^{k+1}$ is conjugacy invariant, there exists a Borel probability measure μ such that for every $f \in \text{Homeo}^+(\mathbb{S}^1)$, $\mu(f\mathcal{F}_{p/q}^k) = 0$, but $\mu(g\mathcal{F}_{p/q}^{k+1}) > 0$ for some $g \in \text{Homeo}^+(\mathbb{S}^1)$. We denote by K the number of periodic points of a homeomorphism $f \in \mathcal{F}_{p/q}^{k+1}$, that is, $K = q(2k + 2)$.

Let $r_0 < s_0 < r_1 < s_1 < \dots < r_{K-1} < s_{K-1}$ be rational elements of \mathbb{S}^1 , ordered cyclically. For such numbers, let $\mathcal{F}_{p/q}^{k+1}(r_0, s_0, \dots, r_{K-1}, s_{K-1}) \subset \mathcal{F}_{p/q}^{k+1}$ contain $f \in \mathcal{F}_{p/q}^{k+1}$ if and only if each interval (r_i, s_i) contains exactly one periodic point of f . It is easy to see that each $f \in \mathcal{F}_{p/q}^{k+1}$ is contained in some $\mathcal{F}_{p/q}^{k+1}(r_0, s_0, \dots, r_{K-1}, s_{K-1})$, hence for at least one of these sets, $\mu(g\mathcal{F}_{p/q}^{k+1}(r_0, s_0, \dots, r_{K-1}, s_{K-1})) > 0$. We now fix $r_0, s_0, \dots, r_{K-1}, s_{K-1}$ in such a way, and let $\mathcal{F}^+ \subset \mathcal{F}_{p/q}^{k+1}(r_0, s_0, \dots, r_{K-1}, s_{K-1})$ contain $f \in \mathcal{F}_{p/q}^{k+1}(r_0, s_0, \dots, r_{K-1}, s_{K-1})$ if and only if the unique fixed point of f^q in the interval (r_0, s_0) is repulsive. Similarly, let $\mathcal{F}^- \subset \mathcal{F}_{p/q}^{k+1}(r_0, s_0, \dots, r_{K-1}, s_{K-1})$ contain f if and only if the fixed point of f^q in (r_0, s_0) is attractive. It is easy to see that $\mathcal{F}_{p/q}^{k+1}(r_0, s_0, \dots, r_{K-1}, s_{K-1}) = \mathcal{F}^+ \cup \mathcal{F}^-$, hence we can suppose without loss of generality that $\mu(g\mathcal{F}^+) > 0$: If $\mu(g\mathcal{F}^+) = 0$ then $\mu(g\mathcal{F}^-) > 0$ and by rearranging the intervals cyclically (e.g. $r'_{K-1} = r_0$, $s'_{K-1} = s_0$, and $r'_i = r_{i+1}$, $s'_i = s_{i+1}$ otherwise) we can achieve that the sets \mathcal{F}^+ and \mathcal{F}^- are swapped, using the fact that attractive and repulsive fixed points of f^q are alternating by Lemma 4.2.

To get a contradiction, we show that $h\mathcal{F}^+ \subset \mathcal{F}_{p/q}^k$ for some $h \in \text{Homeo}^+(\mathbb{S}^1)$, hence $\mathcal{F}^+ \subset h^{-1}\mathcal{F}_{p/q}^k$, thus $\mu(gh^{-1}\mathcal{F}_{p/q}^k) > 0$.

For each $f \in \mathcal{F}^+$, let $p_i^f \in (r_i, s_i)$ be the unique periodic point of f in the interval (r_i, s_i) . It follows easily from Lemma 4.1 that if $x \in (p_0^f, p_1^f)$ then the sequence $(f^{nq}(x))_{n \in \mathbb{N}}$ is monotone and contained in (p_0^f, p_1^f) , thus it converges. The limit is necessarily a fixed point of f^q . Since p_0^f is repulsive and hence p_1^f is

attractive, we have the following:

$$(4.6) \quad \forall f \in \mathcal{F}^+ \quad \forall x \in (p_0^f, p_1^f) \left(f^{nq}(x) \rightarrow p_1^f \right).$$

It is easy to see that f maps a fixed point of f^q to another fixed point in a monotone way, that is, there is an integer ℓ with $0 \leq \ell < K$ such that $f(p_i^f) = p_{j(i)}^f$, where $j(i) \equiv i + \ell \pmod{K}$ for every i . Let us denote by $P_0^f \in [0, 1)$ the real number corresponding to p_0^f and for every $i < K$, let $P_i^f \in \mathbb{R}$ be the real number corresponding to p_i^f with $P_0^f < P_1^f < \dots < P_{K-1}^f < P_0^f + 1$. For $n \in \mathbb{N}$ we also use the notation $P_n^f = P_r^f + m$, where $n = mK + r$ with $0 \leq r < K$. It is easy to see that $P_0^f < P_1^f < \dots < P_{K-1}^f < P_K^f < \dots$. Let F be the lift of f with $F(P_0^f) = P_\ell^f$. It follows that $F^{iK}(P_0^f) = P_{iK\ell}^f = P_0^f + i\ell$ for every $i \in \mathbb{N}$, hence $\tau(F) = \ell/K$. Since $\tau(F) \equiv \tau(f) \pmod{1}$, and $\tau(f) \equiv p/q \pmod{1}$, and both ℓ/K and p/q are numbers from $[0, 1)$, it follows that $\ell/K = p/q$. Hence $\ell = p(2k+2)$, the same for each $f \in \mathcal{F}^+$. To simplify notation, for each $n \in \mathbb{N}$ we denote by p_n^f, r_n and s_n the elements of \mathbb{S}^1 equal to p_r^f, r_r and s_r , respectively, where $0 \leq r < K$ and $n \equiv r \pmod{K}$.

Now we define a homeomorphism $h \in \text{Homeo}^+(\mathbb{S}^1)$ such that the periodic points $p_{i\ell+1}^f$ and $p_{i\ell+2}^f$ ($i = 0, 1, \dots, q-1$) are not periodic for hf . Let \mathcal{P}'_f denote the set of these periodic points, and let \mathcal{P}_f denote the set of all periodic points of f . Note that \mathcal{P}'_f consists of two periodic orbits.

For $i = 0, 1, \dots, q-1$ let the points x_i, y_i, u_i and v_i be chosen so that $s_{i\ell} < x_i < y_i < r_{i\ell+1}$ and $s_{i\ell+2} < u_i < v_i < r_{i\ell+3}$. We use the convention $x_n = x_r, y_n = y_r, u_n = u_r$ and $v_n = v_r$, where $0 \leq r < q$ and $n \equiv r \pmod{q}$. Now we define h the following way. On each interval of the form $[x_i, v_i]$, let h map $[x_i, y_i]$ to $[x_i, u_i]$ linearly, and also map $[y_i, v_i]$ to $[u_i, v_i]$ linearly, so that h is order preserving. Then $h(x_i) = x_i$ and $h(v_i) = v_i$, hence it makes sense to require that $h(x) = x$ if x is outside each of these intervals. It is clear that

$$(4.7) \quad \forall f \in \mathcal{F}^+ \quad \forall n \in \mathbb{N} \quad \forall y \in (p_{n\ell}^f, p_{n\ell+3}^f) \left(h(y) \in [y, p_{n\ell+3}^f) \right).$$

We now claim that $hf \in \mathcal{F}_{p/q}^k$ for every $f \in \mathcal{F}^+$.

Let $f \in \mathcal{F}^+$. If $p \in \mathcal{P}_f \setminus \mathcal{P}'_f$ then it is easy to see that $(hf)^n(p) = f^n(p)$ for every $n \in \mathbb{N}$, hence p is a periodic point of hf with period q , hence hf has at least $2kq$ periodic points, that is, at least $2k$ periodic orbits. It is also an immediate consequence that $\tau(hf) = p/q$, hence

$$(4.8) \quad \text{every periodic point of } hf \text{ has period } q.$$

We now show that no other point is periodic. Let

$$(4.9) \quad x \in (\mathbb{S}^1 \setminus \mathcal{P}_f) \cup \mathcal{P}'_f$$

be such a point, we complete the proof by showing that x is not periodic. We distinguish several cases depending on x .

Case 1: there is no $n \in \mathbb{N}$ with $x \in (p_{n\ell}^f, p_{n\ell+3}^f)$. Then the same holds for $f^m(x)$, that is, for every $m \in \mathbb{N}$ there is no $n \in \mathbb{N}$ with $f^m(x) \in (p_{n\ell}^f, p_{n\ell+3}^f)$. Since h is the identity restricted to the complement of the intervals of the form $(p_{n\ell}^f, p_{n\ell+3}^f)$, one can show by induction that

$$(4.10) \quad f^m(x) = (hf)^m(x) \text{ for every } m \in \mathbb{N}.$$

From the fact that $x \notin (p_{n\ell}^f, p_{n\ell+3}^f)$ for every $n \in \mathbb{N}$, it follows that $x \notin \mathcal{P}_f'$. Using also (4.9), it is clear that x is not a periodic point of f , hence by (4.10), it is not a periodic point of hf .

Case 2: there exists an $n \in \mathbb{N}$ such that

$$(4.11) \quad x \in (p_{n\ell}^f, p_{n\ell+3}^f).$$

In order to show that x is not a periodic point of hf , using (4.8), it is enough to prove that

$$(4.12) \quad p_{n\ell}^f < x < (hf)^q(x) < p_{n\ell+3}^f.$$

We first show that

$$(4.13) \quad p_{n\ell}^f < f^q(x) \leq (hf)^q(x) < p_{n\ell+3}^f.$$

In order to do so, it is enough to prove that

$$(4.14) \quad p_{(n+m)\ell}^f < f^m(x) \leq (hf)^m(x) < p_{(n+m)\ell+3}^f$$

for every $m \in \mathbb{N}$, since applying (4.14) with $m = q$ gives (4.13), using that $\ell q = Kp \equiv 0 \pmod{K}$. For $m = 0$, (4.14) is immediate from (4.11). Now suppose that (4.14) is true for some $m \in \mathbb{N}$. Then, using that f is an orientation preserving homeomorphism, $p_{(n+m+1)\ell}^f < f^{m+1}(x) \leq f((hf)^m(x)) < p_{(n+m+1)\ell+3}^f$. Using (4.7) with $n = n + m + 1$ and $y = f((hf)^m(x))$, we get (4.14) for $m + 1$. Now we distinguish three subcases.

Case 2a: $x \in (p_{n\ell}^f, p_{n\ell+1}^f)$. Then, using the fact that $p_{n\ell}^f$ is a repulsive fixed point of f^q and Lemma 4.1, $p_{n\ell}^f < x < f^q(x) < p_{n\ell+1}^f$. This, together with (4.13) implies (4.12).

Case 2b: $x \in (p_{n\ell+2}^f, p_{n\ell+3}^f)$. Then, using Lemma 4.2 the fact that $p_{n\ell}^f$ is a repulsive fixed point of f^q , $p_{n\ell+2}^f$ is also a repulsive fixed point of f^q . As in the previous case, Lemma 4.1 implies $p_{n\ell+2}^f < x < f^q(x) < p_{n\ell+3}^f$. And again using (4.13), this implies (4.12).

Case 2c: $x \in [p_{n\ell+1}^f, p_{n\ell+2}^f]$. Then $f(x) \in [p_{(n+1)\ell+1}^f, p_{(n+1)\ell+2}^f]$. Using that $(y_{n+1}, u_{n+1}) \supset [p_{(n+1)\ell+1}^f, p_{(n+1)\ell+2}^f]$ and $(u_{n+1}, v_{n+1}) \subset (p_{(n+1)\ell+2}^f, p_{(n+1)\ell+3}^f)$, and also the fact that h maps (y_{n+1}, u_{n+1}) to (u_{n+1}, v_{n+1}) , we have

$$(4.15) \quad hf(x) \in (p_{(n+1)\ell+2}^f, p_{(n+1)\ell+3}^f).$$

Applying (4.14) with $hf(x)$ in place of x and $n + 1$ in place of n and $m = q - 1$, we get

$$(4.16) \quad p_{(n+q)\ell}^f = p_{n\ell}^f < f^{q-1}(hf(x)) \leq (hf)^q(x) < p_{(n+q)\ell+3}^f = p_{n\ell+3}^f,$$

using again that $q\ell = Kp \equiv 0 \pmod{K}$. From (4.15), we have $f^{q-1}(hf(x)) \in (p_{(n+q)\ell+2}^f, p_{(n+q)\ell+3}^f) = (p_{n\ell+2}^f, p_{n\ell+3}^f)$, hence $(hf)^q(x) \in (p_{n\ell+2}^f, p_{n\ell+3}^f)$, using (4.16). Since in Case 2c, x was chosen from $[p_{n\ell+1}^f, p_{n\ell+2}^f]$, this implies (4.12) finishing the proof of the claim. \square

Therefore the proof of the theorem is also complete. \square

5. THE UNITARY GROUP

In this section we consider $\mathcal{U}(\ell^2)$, the group of unitary transformations of the Hilbert space ℓ^2 . We will use some well known facts from the spectral theory of unitary operators following the theory developed in [21, Appendix] (see also [19]) up to Theorem 5.2. Let $U \in \mathcal{U}(\ell^2)$ and $x \in \ell^2$, then the *cyclic subspace generated by x* is the closure of the linear span of $\{U^n(x) : n \in \mathbb{Z}\}$ and we denote this space by $Z(x)$. If x has norm one then by Bochner's theorem there exists a unique probability measure μ_x on \mathbb{S}^1 such that for every $n \in \mathbb{Z}$

$$\langle U^n(x), x \rangle = \int_{\mathbb{S}^1} z^n d\mu_x(z).$$

It can be shown that if $Z(x) \subset Z(y)$ then $\mu_x \ll \mu_y$. If $Z(x)$ is maximal, i. e. it is not a proper subspace of any other cyclic subspace, then μ_x is called a *spectral measure of U* . Moreover, if $Z(x)$ is maximal then for every x' we have $\mu_{x'} \ll \mu_x$. Thus, if x and x' generate maximal cyclic subspaces then $\mu_x \simeq \mu_{x'}$ (that is, $\mu_x \ll \mu_{x'}$ and $\mu_{x'} \ll \mu_x$) so the spectral measure is unique up to \simeq .

Moreover, we can decompose ℓ^2 into the orthogonal direct sum of cyclic subspaces as $\ell^2 = \bigoplus_{i \in \mathbb{N}} Z(x_i)$ such that $\mu_{x_0} \gg \mu_{x_1} \gg \dots$. Let $A_i \subset \mathbb{S}^1$ be the support of the function $\frac{d\mu_{x_i}}{d\mu_{x_1}}$ and define a function $n : \mathbb{S}^1 \rightarrow \mathbb{N} \cup \{\infty\}$ as $n = \sum_i \chi_{A_i}$. This function is called a *multiplicity function of U* and it is defined μ_{x_1} -almost everywhere. It can be shown that the multiplicity function is also unique a.e. with respect to a spectral measure, i. e. it does not depend on the choice of the vectors x_i . Thus, it makes sense to talk about *the spectral measure μ_U and the multiplicity function n_U* of an operator U .

From the description of the process we can derive the following simple corollary.

Corollary 5.1. *Let $U \in \mathcal{U}(\ell^2)$ be given with spectral measure μ_U and multiplicity function n_U , and let $q \in \mathbb{S}^1$. Then $qU \in \mathcal{U}(\ell^2)$, $\mu_{qU} = q_*\mu_U$ and $n_{qU} = n \circ q^{-1}$ where q and q^{-1} denote the multiplication on \mathbb{S}^1 by q and q^{-1} , and $q_*\mu_U$ denotes the push-forward measure of μ_U with respect to the multiplication by q .*

The following important theorem states that the spectral measure and the multiplicity function fully characterizes the conjugacy classes in $\mathcal{U}(\ell^2)$.

Theorem 5.2. *Let $U_1, U_2 \in \mathcal{U}(\ell^2)$ and let μ_1, μ_2 be spectral measures and n_1, n_2 multiplicity functions of U_1 and U_2 . Then U_1 and U_2 are conjugate iff $\mu_1 \simeq \mu_2$ and $n_1 = n_2$ holds μ_1 -almost everywhere.*

Now we are ready to prove our statement about the non-Haar null classes.

Theorem 5.3. *Let $U \in \mathcal{U}(\ell^2)$ be given with spectral measure μ_U and multiplicity function n_U . If the conjugacy class of U is non-Haar null then $\mu_U \simeq \lambda$ and n_U is constant λ -a. e., where λ denotes Lebesgue measure.*

Proof. Suppose that U has a non-Haar null conjugacy class C^U . Then by [7, Corollary 5.2] we have that for a small enough $\varepsilon > 0$ for every $q \in \mathbb{S}^1$ if $|q - 1| < \varepsilon$ then $qI \in (C^U)^{-1}C^U$. Hence for each $q \in \mathbb{S}^1$ with $|q - 1| < \varepsilon$ there exist $U', U'' \in C^U$ with $qU' = U''$. Thus by Corollary 5.1 and Theorem 5.2 we have $\mu_{qU} = q_*\mu_U \simeq q_*\mu_{U'} = \mu_{qU'} = \mu_{U''} \simeq \mu_U$ and similarly $n_{qU} = n_U \circ q^{-1} = n_{U'} \circ q^{-1} = n_{qU'} = n_{U''} = n_U$ holds μ_U -almost everywhere. Thus, if $r \in \mathbb{S}^1$ is arbitrary we can choose $k \in \mathbb{N}$ and

q with $|q - 1| < \varepsilon$ such that $q^k = r$, so pushing forward μ_U k times by multiplication by q we have that $\mu_{rU} = r_*\mu_U \simeq \mu_U$. This means that μ_U is quasi-invariant probability measure on \mathbb{S}^1 therefore $\mu_U \simeq \lambda$ (see e.g. [25, Theorem 3.1.5]).

Similarly, $n_{rU} = n_U \circ r^{-1} = n_U$ μ_U -almost everywhere. Therefore, n_U is μ_U -almost everywhere constant. \square

Let $k \in \mathbb{N} \cup \{\infty\} \setminus \{0\}$ and let $(e_{i,j})_{i \in k, j \in \mathbb{Z}}$ be an injective enumeration of an orthonormal basis of ℓ^2 and consider the unitary transformation U_k given by $e_{i,j} \mapsto e_{i,j+1}$. These operators are usually called *multishifts*. It is not hard to see that the spectral measure of such a transformation μ_{U_k} is equivalent to λ , while the multiplicity function $n_{U_k} \equiv k$ λ -almost everywhere. From this and Theorem 5.3 we obtain the following corollary.

Corollary 5.4. *In the unitary group every conjugacy class is Haar null possibly except for a countable set of classes, namely the conjugacy classes of the multishifts.*

Question 5.5. *Are all conjugacy classes in $\mathcal{U}(\ell^2)$ Haar null? Equivalently, are all the conjugacy classes of the multishifts in $\mathcal{U}(\ell^2)$ Haar null?*

6. AN APPLICATION: HAAR NULL-MEAGRE DECOMPOSITIONS

In this section we use our results to answer Question 1.5 in some important special cases. The following is an easy corollary of Theorem 3.2.

Corollary 6.1. *The group $\text{Homeo}^+([0, 1])$ can be partitioned into a Haar null and a meagre set.*

Proof. Let $\mathcal{N} = \{f \in \text{Homeo}^+([0, 1]) : |\text{Fix}(f)| > \aleph_0\}$ and let $\mathcal{M} = \{f \in \text{Homeo}^+([0, 1]) : |\text{Fix}(f)| \leq \aleph_0\}$. Clearly $\mathcal{M} \cup \mathcal{N}$ is a partition of $\text{Homeo}^+([0, 1])$, and using Theorem 3.2, \mathcal{N} is Haar null, while using the well-known fact that the generic homeomorphism has a Cantor set of fixed points (see e.g. [14] for an even stronger statement), \mathcal{M} is meagre. \square

Now we use Theorem 4.5 to show the following.

Corollary 6.2. *The group $\text{Homeo}^+(\mathbb{S}^1)$ can be partitioned into a Haar null and a meagre set.*

Proof. Similarly to the proof of the previous corollary, let

$$\mathcal{N} = \{f \in \text{Homeo}^+(\mathbb{S}^1) : \text{the number of periodic points of } f \text{ is uncountable}\}$$

and let

$$\mathcal{M} = \{f \in \text{Homeo}^+(\mathbb{S}^1) : \text{the number of periodic points of } f \text{ is countable}\}.$$

Clearly $\mathcal{M} \cup \mathcal{N}$ is a partition of $\text{Homeo}^+([0, 1])$, and using Theorem 4.5, \mathcal{N} is Haar null, while using the fact that for the generic $f \in \text{Homeo}^+(\mathbb{S}^1)$ the set of periodic points of f is homeomorphic to the Cantor set (see [6, Theorem 3]), \mathcal{M} is meagre. \square

The following proposition is an easy consequence of a result of Dougherty [8].

Proposition 6.3. *Let G and H be Polish groups and suppose that there exists a continuous, surjective homomorphism $\phi : G \rightarrow H$. If H can be partitioned into a Haar null and a meagre set then G can be partitioned in such a way as well.*

Proof. Let $H = N \cup M$, where N is Haar null and M is meagre. By [8, Proposition 8], the inverse image of a Haar null set under a continuous, surjective homomorphism is also Haar null, hence $\phi^{-1}(N)$ is Haar null in G . To see that $\phi^{-1}(M)$ is meagre, note that $M \subset \bigcup_{n \in \mathbb{N}} S_n$ where each S_n is closed, nowhere dense. Hence, for each n , $\phi^{-1}(S_n)$ is closed using that ϕ is continuous. Since ϕ is a continuous, surjective homomorphism, ϕ is open by [11, Theorem 2.3.3]. Therefore, $\phi^{-1}(S_n)$ is nowhere dense in G for each n . Thus $G = \phi^{-1}(N) \cup \phi^{-1}(M)$ is a Haar null-meagre decomposition of G . \square

Corollary 6.4. *Let G be a Polish group and $H \leq G$ a closed normal subgroup. Then G/H is a Polish group, and if G/H can be partitioned into a Haar null and a meagre set then G can be partitioned in such a way as well.*

Proof. For the well-known facts that G/H is a Polish group and the canonical projection from G to G/H satisfies the conditions of Proposition 6.3 see e.g. Proposition 1.2.3 of [3] and the preceding remarks. Then an application of Proposition 6.3 completes the proof. \square

We use Proposition 6.3 to obtain a corollary of Corollary 6.2.

Corollary 6.5. *The group $\text{Homeo}^+(\mathbb{D}^2)$ of orientation preserving homeomorphisms of the disc \mathbb{D}^2 can be partitioned into a Haar null and a meagre set.*

Proof. It is not hard to see that the restriction of every homeomorphism $f \in \text{Homeo}^+(\mathbb{D}^2)$ to the boundary of \mathbb{D}^2 is an element of $\text{Homeo}^+(\mathbb{S}^1)$. Then it is easy to see that this map is a continuous, open and surjective homomorphism, hence the proof is complete using Proposition 6.3. \square

The following questions, however, remain open.

Question 6.6. *Is it true that $\text{Homeo}^+(\mathbb{D}^n)$ (for $n \geq 3$) and $\text{Homeo}^+(\mathbb{S}^n)$ (for $n \geq 2$) can be partitioned into a Haar null and a meagre set?*

Question 6.7. *Is it true that the homeomorphism group of the Hilbert cube $[0, 1]^{\mathbb{N}}$ can be partitioned into a Haar null and a meagre set?*

7. OPEN PROBLEMS

In this section we collect the open problems of the paper. First of all, it would be interesting to obtain general results about homeomorphism groups.

Question 7.1. *What can we say about the size of conjugacy classes of the group of homeomorphisms of a compact metric space?*

The following special case is particularly important.

Question 7.2. *Which conjugacy classes are non-Haar null in the group of homeomorphisms of the Cantor set?*

Concerning the groups we investigated in this paper, the following two problems remain open.

Question 1.10. *Is the set $\{f \in \text{Homeo}^+(\mathbb{S}^1) : \tau(f) \notin \mathbb{Q}\}$ Haar null?*

Question 5.5. *Are all conjugacy classes in $\mathcal{U}(\ell^2)$ Haar null? Equivalently, are all the conjugacy classes of the multishifts in $\mathcal{U}(\ell^2)$ Haar null?*

As an application of our results, we answered special cases of the following question that remains open in general.

Question 1.5. *Suppose that G is an uncountable Polish group. Can it be written as the union of a meagre and a Haar null set?*

In particular, we are interested in the following special cases.

Question 6.6. *Is it true that $\text{Homeo}^+(\mathbb{D}^n)$ (for $n \geq 3$) and $\text{Homeo}^+(\mathbb{S}^n)$ (for $n \geq 2$) can be partitioned into a Haar null and a meagre set?*

Question 6.7. *Is it true that the homeomorphism group of the Hilbert cube $[0, 1]^{\mathbb{N}}$ can be partitioned into a Haar null and a meagre set?*

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